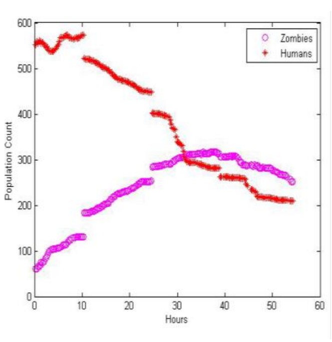
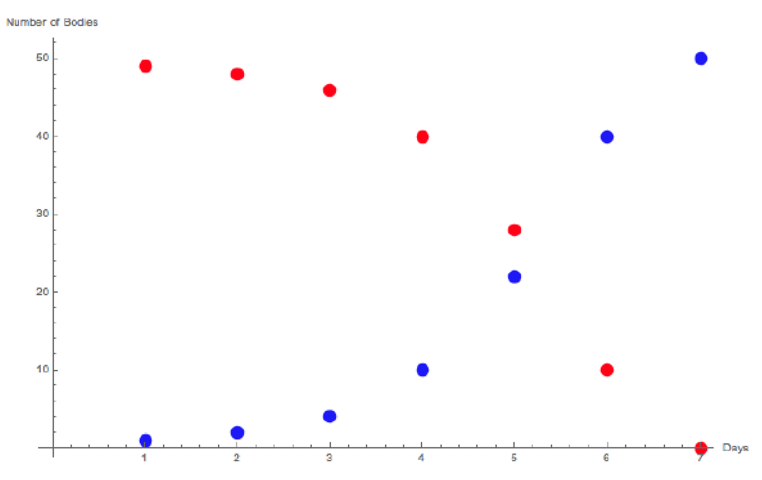
Disease Lab

Laboratory Experiences in Mathematical Biology



Data and Examples: Data along with some student approaches from an introductory mathematical biology course and an ODE course are presented to illustrate some common student approaches and to help prepare teachers to scaffold student thinking.

Here is an example of one type of data set and the associated plot for the basic zombie disease.



Day	1	2	3	4	5	6	7
Humans	49	48	46	40	28	10	0
Zombies	1	2	4	10	22	40	50

In Utah State University's Applied Mathematics in Biology class the Disease Lab is used to introduce discrete modeling.

For most groups the model progression is as follows:

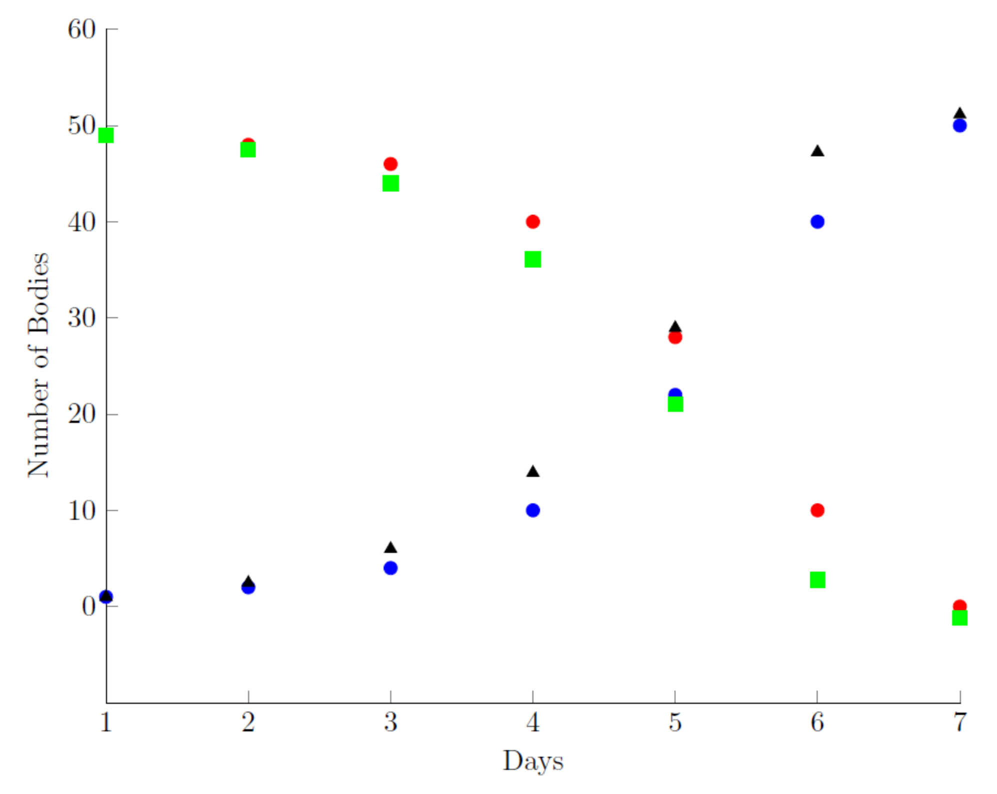
- Suppose n is the number of turns which have been played in the disease game, and Z_n is the number of zombies in the n^{th} turn of the game. Then one may write
$$Z_{n+1} = Z_n + Z_{new}$$
where Z_{new} is the number of individuals which are newly infected during turn n.

- A beginning model can be put together by assuming that the distribution of infective hexes and zombies creating them is random. If each infected individual occupies 3hexes (one for the hex they stand in and one for each of their two arms), and the board contains 100 hexes, then an approximation for the total number of hexes which are infectious at turn n is
$$Z_n \times \frac{3}{100}.$$

- The number of humans on turn n, H_n, is the total (T) less the number of current zombies, that is
$$H_n = T - Z_n.$$

- Then the number of new zombies can be approximated
$$Z_{new} = H_n \times Z_n \times \frac{3}{100}.$$

- Putting this all together gives an initial, discrete logistic model for the propagation of disease:
$$Z_{new} = Z_n + Z_n \times \frac{3}{100} \times (T - Z_n).$$



Data from the basic zombie game along with predicted zombie (▲) and human (■) populations from the discrete logistic model.

This model (discrete logistic equation) often serves as a foundation for students to build other, (sometimes) more advanced models, for data created from playing the game according to student created rules.

In calculus focused classes (e.g., calc 1,2 or ODE) we focus on continuous models. From our Basic Zombie Game perspective, the logistic population growth model is based on two concepts:

1. Individuals in the population are considered to have equal probability of being infected by a zombie a rate of α, the infection rate. For the Basic Zombie Game we expect $\alpha \approx 0.03$ since each zombie occupies 3 out of 100 hexes yielding an infection rate of 0.03. So, a zombie can attack and infect αN others per day, where N is the total population. The fraction of susceptible humans is thus $\frac{H}{N}$. Hence, the number of new zombies created in one day per zombie is then $\alpha N\left(\frac{H}{N}\right)$, giving the rate of new zombie creation as (1)
$$\frac{dZ}{dt} = \alpha N\left(\frac{H}{N}\right)Z = \alpha H\, Z.$$

2. When a human is attacked, it must become an zombie the next day. Let β represent the conversion rate of humans to zombies. Then, because every gain of the zombie population is an equivalent loss from the human population (i.e., $\beta = 1$), the change in the human population can be characterized with
$$\frac{dH}{dt} = -\beta\alpha H\, Z.$$

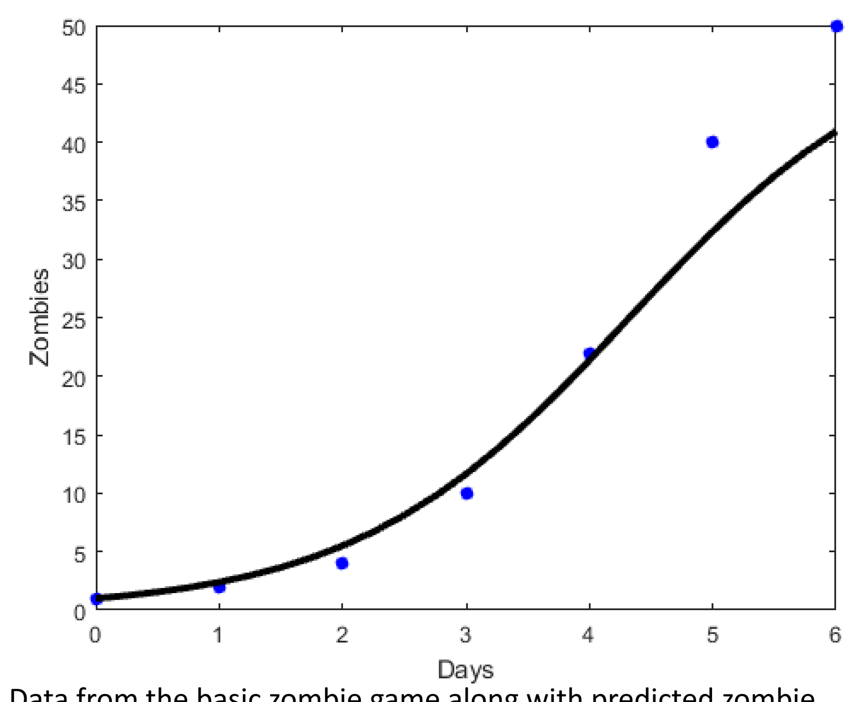
Now, note that
$$\beta\frac{dZ}{dt} = -\frac{dH}{dt},$$
relates only derivatives and can therefore be integrated. Integration yields
$$H(t) = H_0 - \beta(Z(t) - Z_0).$$

Finally, substituting into (1) leads to
$$\begin{aligned}\frac{dZ}{dt} &= \alpha Z(H_0 - \beta(Z(t) - Z_0)),\\ &= \alpha Z(H_0 - \beta Z_0 - \beta Z(t)),\\ &= \frac{\alpha}{\beta Z}\left(\frac{H_0 - \beta Z_0}{\beta} - Z(t)\right).\end{aligned}$$

Now, let $\lambda = \frac{\alpha}{\beta}$ and $K = \frac{H_0 - \beta Z_0}{\beta}$ to get
$$\frac{dZ}{dt} = \lambda Z(K - Z),$$
a common form of the Logistic Model where K represents the carrying capacity and λK is the rate of maximum population growth. However, the logistic equation was originally published by Verhulst as
$$\frac{dZ}{dt} = r\, Z\left(1 - \frac{Z}{K}\right),$$
where r is precisely the growth rate λK.

Solving for the population $Z(t)$ we get
$$Z(t) = \frac{KZ_0}{Z_0 + (K - Z_0)e^{-K\lambda t}}.$$

In most calculus and differential equations classes λ is approximated by using a data from the set; i.e., substituting $t = 4, Z = 22, Z_0 = 1$ and $K = 50$ into our solution yields $\lambda \approx 0.018$. Alternatively students can get a better feel for fitting by simply "eyeballing" λ.



Data from the basic zombie game along with predicted zombie population from the continuous logistic model. Here, $\lambda \approx 0.018$.

Since the logistic model does not perform well students typically adjust the assumptions of the models (e.g., per capita growth is not linear) to build their own, more descriptive models.