An Introduction to DifferentialGeometry with Maple
PCMI - July 5, 2013

Einstein Metrics On Homogeneous Spaces

Synopsis

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. In this worksheet we present an example of a G invariant Einstein metric on the homogeneous space G/H. These are purely algebraic calculations which can be done with the Lie algebra $\mathfrak{g}$, without the need to introduce local coordinates on G/H. In this setting the Einstein equations are purely algebraic equations.

The centralizer K of H in G plays a special role. Indeed, the action of K on G descends to an action on G/H which preserves the space of G-invariant metrics G/H. It is essential to use these diffeomorphisms to simplify the general form of the G invariant metrics and thereby simplify the algebraic equations defining the G invariant Einstein metrics.

Jensen showed there are 2 symplectically invariant Einstein metrics on the spheres S^{4k+1}. Shortly thereafter Ziller showed that these are the only possibilities. We check these results for the special case $k = 1$.

The Sympletic Algebra g = sp(4) , the Subalgebra h = sp(2, R), and the Reductive Decomposition g = m + h

```
> with(DifferentialGeometry): with(LieAlgebras): with(Tensor): with
  (GroupActions):
```

Step 1. We begin by initializing the 10-dimensional Lie algebra $sp(4)$ -- the algebra of 4×4 symplectic matrices. This is done with the commands SimpleLieAlgebraData and DGsetup.

```
> LD1 := SimpleLieAlgebraData("sp(4, 0)", sp4);
```

$$LD1 := [[e1, e2] = e3, [e1, e3] = -2\,e2 + 2\,e4, [e1, e4] = -e3, [e1, e5] = e6, [e1, e6] = \qquad (1.1)$$

$$-2\,e5 + 2\,e7,\ [e1, e7] = -e6,\ [e1, e8] = e9,\ [e1, e9] = -2\,e8 + 2\,e10,\ [e1, e10] = -e9,$$
$$[e2, e3] = e1,\ [e2, e5] = 2\,e8,\ [e2, e6] = e9,\ [e2, e8] = -2\,e5,\ [e2, e9] = -e6,\ [e3, e4]$$
$$= e1,\ [e3, e5] = e9,\ [e3, e6] = 2\,e8 + 2\,e10,\ [e3, e7] = e9,\ [e3, e8] = -e6,\ [e3, e9] =$$
$$-2\,e5 - 2\,e7,\ [e3, e10] = -e6,\ [e4, e6] = e9,\ [e4, e7] = 2\,e10,\ [e4, e9] = -e6,\ [e4, e10]$$
$$= -2\,e7,\ [e5, e6] = e1,\ [e5, e8] = 2\,e2,\ [e5, e9] = e3,\ [e6, e7] = e1,\ [e6, e8] = e3,\ [e6,$$
$$e9] = 2\,e2 + 2\,e4,\ [e6, e10] = e3,\ [e7, e9] = e3,\ [e7, e10] = 2\,e4,\ [e8, e9] = e1,\ [e9,$$
$$e10] = e1]$$

```
> DGsetup(LD1);
```

Lie algebra: sp4 **(1.2)**

Step 2. We now identify the sub-algebra $sp(2) \subset sp(4)$. The DifferentialGeometry software provides a general mechanism for creating such Lie algebras. Let $\rho : \mathfrak{g} \to aut(V)$ be the standard representation of $\mathfrak{g}$. From this representation one can define representations of $\mathfrak{g}$ on the various tensor spaces of V. Given a list of tensors on V, the command MatrixSubalgebra will find the subalgebra of $\mathfrak{g}$ which leaves invariant the given tensors.

Since $sp(2, R) \subset sp(4, R)$ is the set of matrices which leaves invariant the 2nd and 4th vectors in the standard representation of $sp(4, R)$, we can proceed as follows. Define a 4-dimensional space to serve as the representation space for the standard representation.

```
sp4 > DGsetup([x1, x2, x3, x4], V);
```

frame name: V **(1.3)**

Create the standard representation.

```
V > rho := StandardRepresentation(sp4, representationspace = V);
```

$$\rho := \left[\left[e1, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right], \left[e2, \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right], \left[e3, \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & -I & 0 \end{bmatrix}\right], \left[e4, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}\right], \left[e5, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right], \left[e6, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\right], \left[e7, \right.$$ **(1.4)**

$$\left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\right], \left[e8, \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right], \left[e9, \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}\right], \left[e10, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}\right]$$

Find the sub-algebra which leaves the vectors D_{x^2} and D_{x^4} invariant. This is the sub-algebra $sp(2)$.

```
sp4 > h := MatrixSubalgebra(rho, [D_x2, D_x4]);
```

$$h := [e2, e5, e8] \tag{1.5}$$

Check that h = sp(2) = so(3).

```
sp4 > LieAlgebraData(h);
```

$$[[e1, e2] = 2\,e3, [e1, e3] = -2\,e2, [e2, e3] = 2\,e1] \tag{1.6}$$

Step 3. Choose a complementary subspace $\mathfrak{m}$ to the sub-algebra $\mathfrak{h}$.

```
sp4 > m := [e1, e3, e6, e9, e4, e7, e10];
```

$$m := [e1, e3, e6, e9, e4, e7, e10] \tag{1.7}$$

We note (using the command Query) that the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is reductive but not symmetric.

```
sp4 > Query(h, m, "ReductivePair");
```

$$true \tag{1.8}$$

```
sp4n > Query(h, m, "SymmetricPair");
```

$$false \tag{1.9}$$

The Centralizer k of h and the Reductive Decomposition m = l + k

The next step is to determine the centralizer of h in g.

```
sp4 > k := Centralizer(h);
```

$$k := [e10, e7, e4] \tag{2.1}$$

Find a complement of k in m.

```
sp4 > ell := ComplementaryBasis(k, m);
```

$$ell := [e1, e3, e6, e9] \tag{2.2}$$

Create a new basis for sp(4) adapted to the decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{k} + \mathfrak{h}$.

```
sp4 > newBasis := [op(ell), op(k), op(h)];
```

$$newBasis := [e1, e3, e6, e9, e10, e7, e4, e2, e5, e8] \tag{2.3}$$

Re-initialize sp(4) in this new basis.

```
sp4 > LD2 := LieAlgebraData(newBasis, sp4n):
sp4 > DGsetup(LD2, [l1,l2, l3, l4, k1, k2, k3, h1, h2 ,h3],
      [omega]);
```

$$Lie\ algebra: sp4n \tag{2.4}$$

```
sp4n > H := [h1, h2, h3];
```

$$H := [h1, h2, h3] \tag{2.5}$$

```
sp4n > M := [l1, l2, l3, l4, k1, k2, k3];
```

$$M := [l1, l2, l3, l4, k1, k2, k3] \tag{2.6}$$

```
sp4n > K := [k1, k2, k3];
```

$$K := [k1, k2, k3] \tag{2.7}$$

The multiplication table shows the properties of this decomposition.

```
sp4n > interface(rtablesize = 15);
```

$$10 \tag{2.8}$$

```
sp4n > MultiplicationTable("LieTable");
```

$$\begin{bmatrix}
[,|, l1, l2, l3, l4, k1, k2, k3, h1, h2, h3], \\
[,----,----,----,----,----,----,----,----,----,----,----], \\
[l1,|,0,2\,k3-2\,h1,2\,k2-2\,h2,2\,k1-2\,h3,-l4,-l3,-l2,l2,l3,l4], \\
[l2,|,-2\,k3+2\,h1,0,2\,k1+2\,h3,-2\,k2-2\,h2,-l3,l4,l1,-l1,l4,-l3], \\
[l3,|,-2\,k2+2\,h2,-2\,k1-2\,h3,0,2\,k3+2\,h1,l2,l1,-l4,-l4,-l1,l2], \\
[l4,|,-2\,k1+2\,h3,2\,k2+2\,h2,-2\,k3-2\,h1,0,l1,-l2,l3,l3,-l2,-l1], \\
[k1,|,l4,l3,-l2,-l1,0,-2\,k3,2\,k2,0,0,0], \\
[k2,|,l3,-l4,-l1,l2,2\,k3,0,-2\,k1,0,0,0], \\
[k3,|,l2,-l1,l4,-l3,-2\,k2,2\,k1,0,0,0,0], \\
[h1,|,-l2,l1,l4,-l3,0,0,0,0,2\,h3,-2\,h2], \\
[h2,|,-l3,-l4,l1,l2,0,0,0,-2\,h3,0,2\,h1], \\
[h3,|,-l4,l3,-l2,l1,0,0,0,2\,h2,-2\,h1,0]
\end{bmatrix} \tag{2.9}$$

The sp(2) Invariant Metrics on m

We use the command InvariantGeometricObjectFields to define the general invariant symmetric quadratic form. There are 7 invariant symmetric quadratic forms.

First we generate all the rank 2 symmetric tensors on M.

```
S7 > Omega := [omega1, omega2, omega3, omega4, omega5, omega6 ,
     omega7];
```

$$\Omega := [\omega 1, \omega 2, \omega 3, \omega 4, \omega 5, \omega 6, \omega 7] \tag{3.1}$$

```
S7 > Q := GenerateSymmetricTensors(Omega, 2);
```

$$Q := \left[\omega 1 \otimes \omega 1, \frac{1}{2}\omega 1 \otimes \omega 2 + \frac{1}{2}\omega 2 \otimes \omega 1, \frac{1}{2}\omega 1 \otimes \omega 3 + \frac{1}{2}\omega 3 \otimes \omega 1, \frac{1}{2}\omega 1 \otimes \omega 4 + \frac{1}{2}\omega 4 \otimes \omega 1, \frac{1}{2}\omega 1 \otimes \omega 5 + \frac{1}{2}\omega 5 \otimes \omega 1, \frac{1}{2}\omega 1 \otimes \omega 6 + \frac{1}{2}\omega 6 \otimes \omega 1, \frac{1}{2}\omega 1 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 1, \omega 2 \otimes \omega 2, \frac{1}{2}\omega 2 \otimes \omega 3 + \frac{1}{2}\omega 3 \otimes \omega 2, \frac{1}{2}\omega 2 \otimes \omega 4 + \frac{1}{2}\omega 4 \otimes \omega 2, \frac{1}{2}\omega 2 \otimes \omega 5 + \frac{1}{2}\omega 5 \otimes \omega 2, \frac{1}{2}\omega 2 \otimes \omega 6 + \frac{1}{2}\omega 6 \otimes \omega 2, \frac{1}{2}\omega 2 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 2, \omega 3 \otimes \omega 3, \frac{1}{2}\omega 3 \otimes \omega 4 + \frac{1}{2}\omega 4 \otimes \omega 3, \frac{1}{2}\omega 3 \otimes \omega 5 + \frac{1}{2}\omega 5 \otimes \omega 3, \frac{1}{2}\omega 3 \otimes \omega 6 + \frac{1}{2}\omega 6 \otimes \omega 3, \frac{1}{2}\omega 3 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 3, \omega 4 \otimes \omega 4, \frac{1}{2}\omega 4 \otimes \omega 5 + \frac{1}{2}\omega 5 \otimes \omega 4, \frac{1}{2}\omega 4 \otimes \omega 6 + \frac{1}{2}\omega 6 \otimes \omega 4, \frac{1}{2}\omega 4 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 4, \omega 5 \otimes \omega 5, \frac{1}{2}\omega 5 \otimes \omega 6 + \frac{1}{2}\omega 6 \otimes \omega 5, \frac{1}{2}\omega 5 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 5, \omega 6 \otimes \omega 6, \frac{1}{2}\omega 6 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 6, \omega 7 \otimes \omega 7\right] \tag{3.2}$$

We find that the space of h-invariant quadratic forms is 7-dimensional.

```
S7 > InvQ := InvariantGeometricObjectFields(H, Q,
     coefficientvariables = [], output = "list");
```

$$InvQ := \left[\omega 1 \otimes \omega 1 + \omega 2 \otimes \omega 2 + \omega 3 \otimes \omega 3 + \omega 4 \otimes \omega 4, \omega 5 \otimes \omega 5, \frac{1}{2}\omega 5 \otimes \omega 6 + \frac{1}{2}\omega 6 \otimes \omega 5, \frac{1}{2}\omega 5 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 5, \omega 6 \otimes \omega 6, \frac{1}{2}\omega 6 \otimes \omega 7 + \frac{1}{2}\omega 7 \otimes \omega 6, \omega 7 \otimes \omega 7\right] \tag{3.3}$$

```
sp4 > nops(InvQ);
```

$$7 \tag{3.4}$$

It is helpful to write the general invariant metric in matrix form.

```
G > Q0 := DGzip([a1, a2, a3, a4, a5, a6, a7], InvQ, "plus");
```

$$Q0 := a1\,\omega1 \otimes \omega1 + a1\,\omega2 \otimes \omega2 + a1\,\omega3 \otimes \omega3 + a1\,\omega4 \otimes \omega4 + a2\,\omega5 \otimes \omega5 + \frac{a3}{2}\,\omega5 \otimes \omega6 + \frac{a4}{2}\,\omega5 \otimes \omega7 + \frac{a3}{2}\,\omega6 \otimes \omega5 + a5\,\omega6 \otimes \omega6 + \frac{a6}{2}\,\omega6 \otimes \omega7 + \frac{a4}{2}\,\omega7 \otimes \omega5 + \frac{a6}{2}\,\omega7 \otimes \omega6 + a7\,\omega7 \otimes \omega7 \tag{3.5}$$

```
G > A := Matrix(7, 7, (i,j)-> Hook([M[i], M[j]], Q0));
```

$$A := \begin{bmatrix} a1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a2 & \frac{a3}{2} & \frac{a4}{2} \\ 0 & 0 & 0 & 0 & \frac{a3}{2} & a5 & \frac{a6}{2} \\ 0 & 0 & 0 & 0 & \frac{a4}{2} & \frac{a6}{2} & a7 \end{bmatrix} \tag{3.6}$$

Since K acts on $\{\omega_5, \omega_6, \omega_7\}$ by infinitesimal rotations, the action of the group $\exp(K)$ on the matrix A is by conjugation by rotation matrices on the lower 3 x 3 block. Since every symmetric matrix can be diagonalized by a rotation matrix, we can use a diffeomorphism $\exp(K)$ to transform the off-diagonal metric components to zero.

For the purposes of finding homogeneous Einstein metrics we can also normalize a_1 to 1.

```
sp4n > Qreduced := Tools:-DGsimplify(subs({a1 = 1, a3 = 0, a4
       = 0 , a6 = 0}, Q0));
```

$$Qreduced := \omega1 \otimes \omega1 + \omega2 \otimes \omega2 + \omega3 \otimes \omega3 + \omega4 \otimes \omega4 + a2\,\omega5 \otimes \omega5 + a5\,\omega6 \otimes \omega6 + a7\,\omega7 \otimes \omega7 \tag{3.7}$$

The Einstein Metrics

Use FrameData and to create the computation environment for homogeneous spaces.

```
sp4n > FD := FrameData([], M, H, S7);
```

$$FD := [d\Theta 1 = \Theta 4 \wedge \Theta 10 - \Theta 2 \wedge \Theta 7 + \Theta 2 \wedge \Theta 8 - \Theta 3 \wedge \Theta 6 + \Theta 3 \wedge \Theta 9 - \Theta 4 \wedge \Theta 5, d\Theta 2 = \Theta 1 \wedge \Theta 7 - \Theta 1 \wedge \Theta 8 - \Theta 3 \wedge \Theta 10 - \Theta 3 \wedge \Theta 5 + \Theta 4 \wedge \Theta 6 + \Theta 4 \wedge \Theta 9, d\Theta 3 = \Theta 1 \wedge \Theta 6 - \Theta 1 \wedge \Theta 9 + \Theta 2 \wedge \Theta 10 + \Theta 2 \wedge \Theta 5 - \Theta 4 \wedge \Theta 7 - \Theta 4 \wedge \Theta 8, d\Theta 4 = -\Theta 1 \wedge \Theta 10 + \Theta 1 \wedge \Theta 5 - \Theta 2 \wedge \Theta 6 - \Theta 2 \wedge \Theta 9 + \Theta 3 \wedge \Theta 7 + \Theta 3 \wedge \Theta 8, d\Theta 5 = -2\,\Theta 1 \wedge \Theta 4 - 2\,\Theta 2 \wedge \Theta 3 + 2\,\Theta 6 \wedge \Theta 7, d\Theta 6 = -2\,\Theta 1 \wedge \Theta 3 + 2\,\Theta 2 \wedge \Theta 4 - 2\,\Theta 5 \wedge \Theta 7, d\Theta 7 = -2\,\Theta 1 \wedge \Theta 2 - 2\,\Theta 3 \wedge \Theta 4 + 2\,\Theta 5 \wedge \Theta 6, d\Theta 8 = 2\,\Theta 1 \wedge \Theta 2 - 2\,\Theta 9 \wedge \Theta 10 - 2\,\Theta 3 \wedge \Theta 4, d\Theta 9 = 2\,\Theta 1 \wedge \Theta 3 + 2\,\Theta 8 \wedge \Theta 10 + 2\,\Theta 2 \wedge \Theta 4, d\Theta 10 = 2\,\Theta 1 \wedge \Theta 4 - 2\,\Theta 2 \wedge \Theta 3 - 2\,\Theta 8 \wedge \Theta 9] \quad (4.1)$$

```
sp4n > DGsetup(FD, [E], [sigma]);
```

frame name: S7 (4.2)

Re-define the invariant metric on the Lie algebra as a metric on S7.

```
sp4n > g := subs(sp4n = S7, Qreduced);
```

$$g := \sigma 1 \otimes \sigma 1 + \sigma 2 \otimes \sigma 2 + \sigma 3 \otimes \sigma 3 + \sigma 4 \otimes \sigma 4 + a2\,\sigma 5 \otimes \sigma 5 + a5\,\sigma 6 \otimes \sigma 6 + a7\,\sigma 7 \otimes \sigma 7 \quad (4.3)$$

Solve the Einstein equations with DGsolve.

```
G > h := InverseMetric(g);
```

$$h := E1 \otimes E1 + E2 \otimes E2 + E3 \otimes E3 + E4 \otimes E4 + \frac{1}{a2} E5 \otimes E5 + \frac{1}{a5} E6 \otimes E6 + \frac{1}{a7} E7 \otimes E7 \quad (4.4)$$

```
S7 > _EnvExplicit :=true;
```

$$_EnvExplicit := true \quad (4.5)$$

```
G > EinsteinMetrics := DGsolve (EinsteinTensor(g) - Lambda*h, g,
    {a2 ,a5, a7, Lambda}, auxiliaryequations = {a2>-infinity});
```

$$EinsteinMetrics := \left\{\sigma 1 \otimes \sigma 1 + \sigma 2 \otimes \sigma 2 + \sigma 3 \otimes \sigma 3 + \sigma 4 \otimes \sigma 4 + \sigma 5 \otimes \sigma 5 + \sigma 6 \otimes \sigma 6 + \sigma 7 \otimes \sigma 7, \sigma 1 \otimes \sigma 1 + \sigma 2 \otimes \sigma 2 + \sigma 3 \otimes \sigma 3 + \sigma 4 \otimes \sigma 4 + \frac{1}{5}\sigma 5 \otimes \sigma 5 \right. \quad (4.6)$$

$$+\frac{1}{5}\,\sigma 6\otimes\sigma 6+\frac{1}{5}\,\sigma 7\otimes\sigma 7\Big\}$$

We find there are 2 metrics.

```
S7 > g1 := EinsteinMetrics[1];
```

$$g1 := \sigma 1\otimes\sigma 1+\sigma 2\otimes\sigma 2+\sigma 3\otimes\sigma 3+\sigma 4\otimes\sigma 4+\sigma 5\otimes\sigma 5+\sigma 6\otimes\sigma 6+\sigma 7\otimes\sigma 7 \quad (4.7)$$

```
S7 > g2 := EinsteinMetrics[2];
```

$$g2 := \sigma 1\otimes\sigma 1+\sigma 2\otimes\sigma 2+\sigma 3\otimes\sigma 3+\sigma 4\otimes\sigma 4+\frac{1}{5}\,\sigma 5\otimes\sigma 5+\frac{1}{5}\,\sigma 6\otimes\sigma 6+\frac{1}{5}\,\sigma 7\otimes\sigma 7 \quad (4.8)$$

Properties of the Einstein Metrics

The first metric has covariantly constant curvature and 28 dimensional isometry algebra. It is the standard metric on S^7.

```
S7 > C1 := Christoffel(g1):
S7 > R1 := CurvatureTensor(g1):
S7 > CovariantDerivative(R1, C1);
```

$$0\,E1\otimes\sigma 1\otimes\sigma 1\otimes\sigma 1\otimes\sigma 1 \quad (5.1)$$

```
M > d1, iso1, isotropy1 := IsometryAlgebra(g1, alg, []):
S7 > d1;
```

$$28 \quad (5.2)$$

```
S7 > Hol1 := InfinitesimalHolonomy(g1, []):
S7 > nops(Hol1);
```

$$21 \quad (5.3)$$

```
S7 > infolevel[IsometryAlgebra] := 3;
```

$$infolevel_{IsometryAlgebra} := 3 \quad (5.4)$$

```
S7 > d2, iso2, isotropy2 := IsometryAlgebra(g2, alg, []):
Computing the 1st derivative of curvature.
The upper bound on the dimension of the isometry algebra is 13.
Computing the 2 derivative of curvature
   Curvature calculation complete
The upper bound on the dimension of the isometry algebra is 13.
Calculating Lie algebra data.
Calculating isotropy subalgebra data
```

```
S7 > nops(isotropy2);
```

(5.5)