A Systematic Construction of Curved Phase Space: A gravitational gauge theory with symplectic form

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A gravitational gauge theory with symplectic form

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Summary
GR as a special case of Biconformal Space

1. Curved Phase Space
   - Why *Curved* Phase Space?
   - Systematizing the Approach

2. Systematic Approach to Curved Phase Space
   - Gravitational Gauge Theory
   - Biconformal Space
   - General Relativity
   - Reciprocal GR
   - The Intriguing List
     (Quantum Implications)
Curved Phase Space

- One obvious generalization of general relativistic phase space.
- Born’s reciprocal balance of coordinates and momentum in quantum mechanics.
  - Heisenberg Uncertainty Principle
  - All equations of quantum mechanics can be Fourier transformed between configuration space and momentum space.

*It seems to me unjustified to assume that these two reciprocal ... cases should be subject to the same metric, based on the line element in the x-space. I suggest that the conception of a metric is inapplicable for those phenomena in which x-space and p-space are involved simultaneously with about equal weight...*

- Max Born, 1938
Curved Momentum Space
In the Literature

- First known introduction by Born (Born, 1938)
- In relation to non-commutative spacetime (Snyder, 1947)
- Looked at to try and alleviate UV divergence in QFT (Gol’fand, 1959)
- Regularization of 2+1 gravity (Friedel, 2004)
- Principle of Relative-Localility (Amelino-Camelia, et.al 2010)
Systematic Approach to Curved Phase Space

Requirements/ Goals

- Metric space
- Lorentz Signature
- Symplectic form
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- Metric space
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- Unified action over phase space
- 8-dimensional fully curved arena
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- Some notion of a limit to *either* the co-tangent bundle of spacetime, *or* the tangent bundle of momentum space
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How do we add the phase space curvatures while keeping the known symmetries of spacetime?
Quotient Manifold Method

- Ne’emann and Regge, 1978
- The quotient of a Lie group by one of its Lie subgroups is a manifold.

\[ \frac{G}{H} = M \]

\[ \dim(M) = \dim(G) - \dim(H) \]

- The Maurer-Cartan structure equations allow us to write the Lie algebra in terms of a connection on \( M \).
- Generalize the connection to allow curvature on \( M \).
- The subgroup, \( H \), becomes the local symmetry of the final geometry.
## Gravitational Gauge Theory

### Overview

#### Smallest Groups that contain Poincaré

<table>
<thead>
<tr>
<th>Group</th>
<th>Poincaré</th>
<th>Kibble</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ISO(p, q)$</td>
<td>Poincaré</td>
<td>Kibble</td>
</tr>
<tr>
<td>$SO(p + 1, q)$</td>
<td>de Sitter</td>
<td>MacDowell &amp; Mansourí</td>
</tr>
<tr>
<td>$SO(p, q + 1)$</td>
<td>anti-de Sitter</td>
<td>MacDowell &amp; Mansourí</td>
</tr>
<tr>
<td>$ISO(p, q) \otimes \mathbb{R}^+$</td>
<td>Inh Weyl</td>
<td>Ivanov &amp; Niederle</td>
</tr>
<tr>
<td>$SO(p + 1, q + 1)$</td>
<td>Conformal</td>
<td>Ivanov &amp; Niederle</td>
</tr>
</tbody>
</table>

In their 1982 paper Ivanov and Niederle considered applications of the quotient manifold method to various groups containing the Lorentz group.
Structure Equations

ISO\((p, q)\)

\[
\begin{align*}
R^a_b &= d\omega^a_b - \omega^c_b \omega^a_c \\
T^a &= de^a - e^b \omega^a_b
\end{align*}
\]
Structure Equations

\[ ISO(p, q) \otimes \mathbb{R}^+ \]

\[ R^a_b = d\omega^a_b - \omega^c_b \omega^a_c \]

\[ T^a = de^a - e^b \omega^a_b - \omega e^a \]

\[ \Omega = d\omega \]
Structure Equations

\[ R^a_b = d\omega^a_b - \omega^c_b \omega^a_c - (\delta^a_d \delta^c_b - \eta^{ac} \eta_{db}) f_c e^d \]

\[ T^a = de^a - e^b \omega^a_b - \omega e^a \]

\[ S_a = df_a + f_b \omega^b_a + \omega f_a \]

\[ \Omega = d\omega - e^a f_a \]

\[ SO(p + 1, \ q + 1) \]
**Biconformal Space**

\[
\frac{SO(p + 1, q + 1)}{SO(p, q) \otimes \mathbb{R}^+}
\]

**Properties**

1. **Symplectic Form.** The only Lorentz covariant quotient of \(SO(p + 1, q + 1)\) for which the dilatation equation describes a symplectic form.

\[
\Omega = d\omega - e^a f_a
\]

2. **Metric Space.** The only Lorentz covariant quotient of \(SO(p + 1, q + 1)\) that has a non-degenerate Killing metric on \(M\).

\[
K_{AB} = \lambda c^C_{DA} c^D_{CB}
\]
Biconformal Space

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3. Completely Curved Phase Space.

4. An action **linear** in curvatures.
Biconformal Space

Flat Case

Lorentzian Signature Theorem

Flat 8-dim biconformal space is a phase space with canonically conjugate, orthogonal, metric submanifolds if and only if the initial 4-dim space we gauge is Euclidean or signature zero. In either of these cases the resulting configuration sub-manifold is necessarily Lorentzian.

\[ h_{ab} = \frac{1}{(W^2)^2} \left( 2W_a W_b - W^2 \delta_{ab} \right) \]

Where \( W_a \) turns out to be the momentum-like coordinate, \( y_a \).

---

Biconformal Space

\[ \frac{SO(p + 1, q + 1)}{SO(p, q) \otimes \mathbb{R}^+} \]

Properties

1. **Symplectic Form.** The only Lorentz covariant quotient of \( SO(p + 1, q + 1) \) for which the dilatation equation describes a symplectic form.

2. **Metric Space.** The only Lorentz covariant quotient of \( SO(p + 1, q + 1) \) that has a non-degenerate Killing metric on \( \mathcal{M} \).

3. **Completely Curved Phase Space.**

4. **An action **linear** in curvatures.**

5. **Time.** There exists a natural notion of time, which gives rise to familiar, canonical structures, such as the ADM metric. The Lorentz Signature Thm reveals that time is related to the Weyl vector.
Biconformal Gauging

1. Vary the action.
2. $T^a = 0$
3. Gauge Choice

Co-tangent bundle of spacetime
$\Rightarrow$ (Wehner/Wheeler, 1999)$^2$
- Configuration and momentum submanifolds are not orthogonal
- The wrong torsion vanishes

Biconformal Gauging

1. Vary the action.
2. $T^a = 0$
3. Gauge Choice
4. Impose flat momentum sector
5. Orthogonal Basis

$\Rightarrow$ General Relativity

Metric compatible connection.

$R_{\alpha\beta} = 0$ on both submanifolds.

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Spacetime as the Tangent Space
Limit in the other direction

In order to see a curved momentum space with spacetime as its tangent space we set the curvature to zero on spacetime
Spacetime as the Tangent Space
Limit in the other direction

In order to see a curved momentum space with spacetime as its tangent space we set the curvature to zero on spacetime

- We get a Ricci flat space on momentum space.
- The flat spacetime completes the tangent bundle.
- Possible to keep the torsion, but the co-torsion is still present in either case.
  → Possibly points to reason why momentum space of relative locality has torsion
  Simplified Non-metricity
Curved Phase Space
Reprise

Requirements/ Goals

- Symplectic form
- Lorentz Signature
- Metric
- Unified action over phase space
- 8-dimensional fully curved arena
- Some notion of a limit to either the co-tangent bundle of spacetime, or the tangent bundle of momentum space

General Relativity
Exciting Other Features

- Time is shown to be a consequence of one of the gauge theories of general relativity.
- Hamilton mechanics arises directly in flat biconformal space if we take the particle action to be the integral of the Weyl vector. The classical paths are exactly the subset of curves on which the Weyl vector can be gauged away.
- Quantum mechanics arises in the same way as Hamiltonian mechanics simply by dropping the assumption that particles follow preferred (i.e., extremal) paths. (à la Feynmann)
- When the coordinates on the momentum-like sub-manifold are redefined to match our normal notion of signature, the $i$ automatically comes into the Poissón brackets.

Thank You