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A Systematic Construction of Curved Phase Space: A gravitational gauge theory with symplectic form

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A Systematic Construction of Curved Phase Space

A gravitational gauge theory with symplectic form

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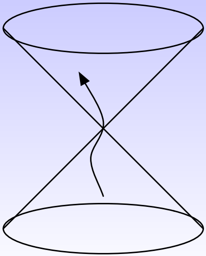
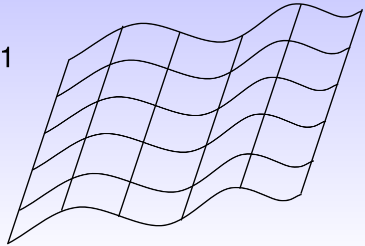
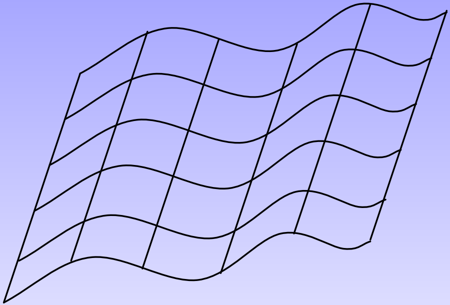




Summary

GR as a special case of Biconformal Space

1. Curved Phase Space
   - Why *Curved* Phase Space?
   - Systematizing the Approach
2. Systematic Approach to Curved Phase Space
   - Gravitational Gauge Theory
   - Biconformal Space
   - General Relativity
   - Reciprocal GR
   - The Intriguing List (Quantum Implications)



Curved Phase Space

- One obvious generalization of general relativistic phase space.
- Born's reciprocal balance of coordinates and momentum in quantum mechanics.
  - Heisenberg Uncertainty Principle
  - All equations of quantum mechanics can be Fourier transformed between configuration space and momentum space.

It seems to me unjustified to assume that these two reciprocal ... cases should be subject to the same metric, based on the line element in the x-space. I suggest that the conception of a metric is inapplicable for those phenomena in which x-space and p-space are involved simultaneously with about equal weight...

-Max Born, 1938

Curved Momentum Space

In the Literature

- First known introduction by Born (Born, 1938)
- In relation to non-commutative spacetime (Snyder, 1947)
- Looked at to try and alleviate UV divergence in QFT (Gol'fand, 1959)
- Regularization of 2+1 gravity (Friedel, 2004)
- Principle of Relative-Locality (Amelino-Camelia, et.al 2010)

Systematic Approach to Curved Phase Space

Requirements/ Goals

- Metric space
- Lorentz Signature
- Symplectic form

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How do we add the phase space curvatures while keeping the known symmetries of spacetime?

Quotient Manifold Method

- Ne'emann and Regge, 1978
- The quotient of a Lie group by one of its Lie subgroups is a manifold.

$$\frac{\mathcal{G}}{\mathcal{H}} = \mathcal{M}$$
$$dim(\mathcal{M}) = dim(\mathcal{G}) - dim(\mathcal{H})$$

- The Maurer-Cartan structure equations allow us to write the Lie algebra in terms of a connection on $\mathcal{M}$.
- Generalize the connection to allow curvature on $\mathcal{M}$.
- The subgroup, $\mathcal{H}$, becomes the local symmetry of the final geometry.

Gravitational Gauge Theory

Overview

Smallest Groups that contain Poincaré

$ISO(p, q)$	Poincaré	Kibble
$SO(p+1, q)$	de Sitter	MacDowell & Mansouri
$SO(p, q+1)$	anti-de Sitter	MacDowell & Mansouri
$ISO(p, q) \otimes \mathbb{R}^+$	Inh Weyl	Ivanov & Niederle
$SO(p+1, q+1)$	Conformal	Ivanov & Niederle

In their 1982 paper Ivanov and Niederle considered applications of the quotient manifold method to various groups containing the Lorentz group.

Structure Equations

$ISO(p,\ q)$

$$\mathbf{R}^a_{\ b} = \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b}\omega^a_{\ c}$$
$$\mathbf{T}^a = \mathbf{d}\mathbf{e}^a - \mathbf{e}^b\omega^a_{\ b}$$

Structure Equations

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$$\mathbf{\Omega} = \mathbf{d}\omega$$

Structure Equations

$$\mathbf{R}^a_{\ b} = \mathbf{d}\omega^a_{\ b} - \omega^c_{\ b}\omega^a_{\ c} - \left(\delta^a_d\delta^c_b - \eta^{ac}\eta_{db}\right)\mathbf{f}_c\mathbf{e}^d$$

$$\mathbf{T}^a = \mathbf{d}\mathbf{e}^a - \mathbf{e}^b\omega^a_{\ b} - \omega\mathbf{e}^a$$

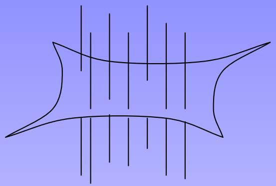
$$\mathbf{S}_a = \mathbf{d}\mathbf{f}_a + \mathbf{f}_b\omega^b_{\ a} + \omega\mathbf{f}_a$$

$$\boldsymbol{\Omega} = \mathbf{d}\omega - \mathbf{e}^a\mathbf{f}_a$$

$$SO(p+1,\ q+1)$$

Biconformal Space

$$\frac{SO(p+1,\ q+1)}{SO(p,\ q)\otimes\mathbb{R}^+}$$



Properties

1. **Symplectic Form.** The only Lorentz covariant quotient of $SO(p+1, q+1)$ for which the dilatation equation describes a symplectic form.

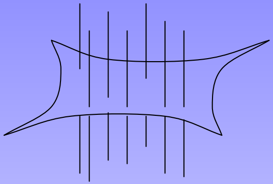
$$\mathbf{\Omega} = \ \mathbf{d}\omega - \mathbf{e}^a\mathbf{f}_a$$

2. **Metric Space.** The only Lorentz covariant quotient of $SO(p+1, q+1)$ that has a non-degenerate Killing metric on $\mathcal{M}$.

$$K_{AB} = \ \lambda\ c^C{}_{DA}c^D{}_{CB}$$

Biconformal Space

$$\frac{SO(p+1,\ q+1)}{SO(p,\ q)\otimes\mathbb{R}^{+}}$$



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3. Completely Curved Phase Space.
4. An action **linear** in curvatures.

Biconformal Space

Flat Case

Lorentzian Signature Theorem[1]

Flat 8-dim biconformal space is a phase space with canonically conjugate, orthogonal, metric submanifolds if and only if the initial 4-dim space we gauge is Euclidean or signature zero. In either of these cases the resulting configuration sub-manifold is necessarily Lorentzian.

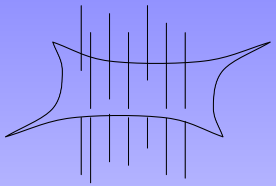
$$h_{ab} = \frac{1}{\left(W^2\right)^2}\left(2W_a W_b - W^2 \delta_{ab}\right)$$

Where W_a turns out to be the momentum-like coordinate, y_a.

[1] Spencer, Wheeler.Int.Jour.Geom.Meth.Mod.Ph. Vol.8, 2(2011), pg 273-301

Biconformal Space

$$\frac{SO(p+1,\ q+1)}{SO(p,\ q)\otimes\mathbb{R}^{+}}$$



Properties

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2. **Metric Space.** The only Lorentz covariant quotient of $SO(p+1, q+1)$ that has a non-degenerate Killing metric on $\mathcal{M}$.
3. Completely Curved Phase Space.
4. An action **linear** in curvatures.
5. **Time.** There exists a natural notion of time, which gives rise to familiar, canonical structures, such as the ADM metric. The Lorentz Signature Thm reveals that time is related to the Weyl vector.

Biconformal Gauging

General Relativity

1. Vary the action.
2. $\mathbf{T}^a = 0$
3. Gauge Choice

Co-tangent bundle of spacetime $\Rightarrow$(Wehner/Wheeler, 1999)[2]

- Configuration and momentum submanifolds are not orthogonal
- The wrong torsion vanishes

[2] Wehner, Wheeler, Nucl.Phys.B 557(1999)380-406, arXiv:hep-th/9812099.

Biconformal Gauging

General Relativity

1. Vary the action.
2. $\mathbf{T}^a = 0$
3. Gauge Choice
4. Impose flat momentum sector
5. Orthogonal Basis

$\Rightarrow$ **General Relativity**

Metric compatible connection.

$R_{\alpha\beta} = 0$ on both submanifolds.

[2] Wehner, Wheeler, Nucl.Phys.B 557(1999)380-406, arXiv:hep-th/9812099.

Spacetime as the Tangent Space

Limit in the other direction

In order to see a curved momentum space with spacetime as its tangent space we set the curvature to zero on spacetime

Spacetime as the Tangent Space

Limit in the other direction

In order to see a curved momentum space with spacetime as its tangent space we set the curvature to zero on spacetime

- We get a Ricci flat space on momentum space.
- The flat spacetime completes the tangent bundle.
- Possible to keep the torsion, but the co-torsion is still present in either case.
  $\rightarrow$Possibly points to reason why momentum space of relative locality has torsion
  Simplified Non-metricity

Curved Phase Space

Reprise

Requirements/ Goals

- Symplectic form
- Lorentz Signature
- Metric
- Unified action over phase space
- 8-dimensional fully curved arena
- Some notion of a limit to *either* the co-tangent bundle of spacetime, *or* the the tangent bundle of momentum space General Relativity

Exciting Other Features

- Time is shown to be a consequence of one of the gauge theories of general relativity.
- Hamilton mechanics arises directly in flat biconformal space if we take the particle action to be the integral of the Weyl vector. The classical paths are exactly the subset of curves on which the Weyl vector can be gauged away.
- Quantum mechanics arises in the same way as Hamiltonian mechanics simply by dropping the assumption that particles follow preferred (i.e., extremal) paths. (à la Feynmann)
- When the coordinates on the momentum-like sub-manifold are redefined to match our normal notion of signature, the i automatically comes into the Poissón brackets.

Thank You