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Innovative Wind Machines, Executive Summary and Final Report

United States, Energy Research and Development Administration, Division of Solar Energy

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INNOVATIVE WIND MACHINES

Executive Summary and Final Report

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J. B. Fanucci
J. L. Loth
P. G. Migliore
N. Ness
G. M. Palmer
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June 1976

Work performed under Contract No. EY-76-C-05-5135

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Morgantown, West Virginia

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ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
Division of Solar Energy
INNOVATIVE WIND MACHINES

EXECUTIVE SUMMARY

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June 1976

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administered by
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION,
DIVISION OF SOLAR ENERGY
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PROJECT SUMMARY

This project has been funded by the U. S. Energy Research and Development Administration (ERDA) with funds administered through the National Science Foundation. The project began in March, 1975, and this report contains the results of approximately the first year's efforts.

The research conducted was concerned with the initial investigation of two innovative concepts in wind energy conversion turbines, or wind machines. The first concept investigated was that of a vortex concentrator, a device which creates a strong vortex in the ambient wind. The energy per unit area in the vortex region is then much higher than for the undisturbed wind, a fact which suggests that the energy might be more efficiently harnessed and converted to more useful forms. The second concept investigated was that of a vertical axis wind turbine which would use straight blades composed of airfoil shapes having high efficiency, i.e., high lift to drag ratios. This would be attained by using circulation controlled (c.c.) airfoils for the blades; these airfoils contain slots near the rounded trailing edges through which a small amount of compressed air is blown to obtain high lift forces. Straight blades allow cyclic pitch control, as well as locating each blade element at a large radius from the shaft so that the maximum rotor torque is produced.

In order to become useful devices, each concept must be able to be built on a basis economically competitive with power produced from other energy sources and with other wind turbines. In order to evaluate the economic feasibility, the basic technical reasons for the proper operation of each device must be known, as well as the best designs for each and the amount of the wind's energy which can be captured. Then, the cost of the power produced can be evaluated. The project described herein addresses all of these subjects, with emphasis on evaluating the machines' aerodynamic performance characteristics.

To date, an extensive study of the analysis of vortices has been performed, and an outline developed which will allow extension of the analysis to turbulent flows. The available theory has been used to predict the amount of energy concentration produced by vertically oriented wings. This analysis has shown that a five-fold concentration might be attained, which would allow energy conversion by a relatively small high-speed turbine instead of a large rotor. Tests in a wind tunnel using flow visualization have been initiated, and have been used to verify some of the theoretical results obtained. More extensive testing on a larger scale has been planned, and has resulted in the design of a outdoor test device, a 4.57 m (15 ft) high wing with a 1.83 m (6 ft) chord.

The analysis of the vertical axis turbine (VAT) has included two theoretical approaches to the problem. The first, a modified strip theory, was initially used to compute the performance of straight-bladed VAT's, both with conventional and circulation controlled (c.c.) airfoil sections used for the blades. This
analysis involved several necessary approximations which resulted in some doubt as to the absolute accuracy of the findings; however, it showed that relatively, the VAT with the c.c. blades was more efficient than the VAT with conventional symmetrical airfoils used for the blades. Also, the c.c. VAT operated at peak efficiencies at lower rotational speeds and thus centrifugal forces on the blades, a critical design condition, were smaller. The analysis showed that the efficiencies of the VATs were on the order of those of horizontal axis turbines, and were very dependent on the blade drag coefficient. Modulation of the blade pitch angle as the blade travels around the VAT's circular path was shown to increase the performance considerably as compared to fixed pitch operation.

Advanced theoretical work, utilizing a non-steady lifting surface theory, has been developed for the VAT. A test model for the VAT was designed and constructed, in order to confirm the theoretical results, and in particular to verify the proper drag coefficients of the blades in the non-steady flow. This test model is two-bladed and uses a rotor 3.05 m (10 ft) in diameter with 3.25 m (10 ft 8 in) long blades.

Theoretical analyses, experiments, and system and economic analyses of both types of innovative wind machines are continuing at West Virginia University with funding supplied by ERDA.

1. INTRODUCTION

The work reported herein was supported by the U.S. Energy Research and Development Administration through the National Science Foundation, and represents an initial evaluation of two innovative wind machines. The first is a vortex concentrator, which is a vertical wing placed in the natural wind in a way that a strong wing tip vortex occurs; a turbine is then placed behind the wing tip in the concentrated energy region of the vortex to convert the energy to more directly-useful forms. The second concept involves a vertical axis turbine with straight blades; the blades are composed of special circulation controlled airfoils which show promise of increasing the overall efficiency of the turbine. Results of approximately the first year's theoretical and experimental investigation are reported, along with conclusions and recommendations for future research efforts.

2. PROGRAM STUDIES AND RESULTS

2.1. Wing Trailing Vortices

A thorough review of the literature has been performed in order to establish the known theoretical and experimental results concerning wing tip vortices. Most of the work was concerned with
the vortices behind aircraft wings (Fig. 2.1.1). Theoretical results were found to be available for laminar flow models. It was realized that laminar free-stream flow might not be realistic for certain atmospheric conditions and a mathematical model was developed to extend the analysis to the turbulent flow case. The solutions for this model are quite complicated, and continuing work will be directed toward obtaining them and comparing the results with the laminar flow solutions. This will determine the importance of including the complex turbulent flow phenomena when making calculations of the energies in various parts of the vortex.

2.2. Vortex Type Concentrators

Results of the available vortex analyses and experimental work reported in the literature were used to predict the performance of the vortex type energy concentrator. An experimental program emphasizing visualization of the flow around lifting wing surfaces was performed in two of the West Virginia University wind tunnels. Fig. 2.2.1 shows the flow around the tip of a wing constructed of high-lift Liebeck-type airfoil sections; the flow is visualized by photographing neutrally buoyant helium soap bubbles with steady and pulsed light sources. Thus, the trajectories and velocities of the bubbles can be determined, as well as the areas containing the most concentrated energy. Results of this type are necessary to assure optimum location of the turbine behind the wing tip, as well as allowing optimum design of the turbine blades. Smoke studies were also performed, as well as tests with obstructions placed behind the models to simulate the turbine.
Fig. 2.2.2 shows a model of a wind machine utilizing the vortex concentrator concept. The tower of a more "conventional" horizontal axis turbine has been replaced by a vertical wing which is free to rotate with the wind. Suspended behind the wing is a shrouded turbine. Since the wing tip vortex concentrates the energy per unit area of the airflow, the turbine can be much smaller than an unaugmented rotor with the same power production. Thus, larger systems might be built than is possible using unaugmented rotors, or, from another viewpoint, a vortex concentrator system may produce more power for the same initial installed capital cost.

Some results of the theoretical study are shown in Fig. 2.2.3, where the energy concentration ratio (R) is plotted as a function of the turbine diameter (d) from the vortex core which is normalized by the wing chord (c). Curves are presented for several values of $C_L^2/e$, the square of the wing lift coefficient divided by the wing efficiency factor. The importance of having a high value of lift coefficient is easily seen. Although the peak energies are near the vortex core, it is seen that high levels of energy concentration
exist far outward. Determination of the optimum-size turbine must be made on an economic basis. The area ratio (A) shown in the figure is that of the wing area plus the turbine area, normalized by the turbine area. This value (A) is seen to increase as the wing aspect ratio (W), or wing slenderness, increases. High wing aspect ratios are needed to produce the high lift coefficients required for high energy concentrations in the vortex. Assuming that the cost of the machine will be partly determined by the size, the optimum values of wing aspect ratio and turbine diameter must be determined by a thorough structural analysis and cost study, once the machine performance is adequately predicted.

Work which is continuing on the vortex concentrator concept is concerned with testing an outdoor 4.6 m (15 ft) high wing, and performing flow surveys to determine the effect of using a larger sized wing. A theoretical turbine design analysis will also be performed. The possible disruption of the vortex when the turbine is placed in the flow is also a major item to be investigated; the necessity of having a shrouded turbine, which might stabilize the vortex, must also be determined. As more information on the optimum design is obtained, the structural and economic analyses can be performed.

2.3. Vertical Axis Turbine Strip Theory

Initial analysis of the vertical axis turbine (VAT) was performed using a two dimensional strip theory. This theory is relatively simple, and can be used to predict turbine performance coefficients, as shown in Fig. 2.3.1. The results clearly exhibit the expected trends, with a high performance indicated by having the blade attitude optimized, i.e., the blade pitch continuously changes to provide maximum torque at each rotational position. Operation at fixed blade angles (0°) or at two-... Fig. 2.3.1. VAT calculated performance. position per revolution values (1°, 6°) shows the expected reduced efficiencies. It was found that the results of this type of calculation were very dependent on the "blockage factor" which was required to be selected for the momentum type flow equations. This blockage factor is an attempt to allow for the interference which exists among the blades of the turbine, and which reduces the turbine efficiency. The results were also relatively sensitive to the drag coefficients used for the blades of the machine. For these reasons the absolute values of the predicted performance coefficients could not be taken as being necessarily of high accuracy, especially with a lack of experimental data. However, the relative results indicated that a turbine using circulation controlled (c.c.) airfoils on the blades should perform at somewhat
higher efficiencies than a similar turbine with "conventional" symmetrical airfoils. Also, the c.c. turbine peak performance occurred at lower values of the rotational speed than that for the symmetrical blades.

Although this theory has not produced absolute results, it still has potential for use as a design tool, once proper blockage factors and drag coefficients are known. Its chief value is that it is rapidly solved with a computer, thus giving low-cost outputs. Future efforts will be made to determine if the proper inputs can allow the theory to adequately predict VAT performance.

2.4. Vertical Axis Turbine Vortex Theory

A more exact theory has been developed for the VAT which determines the non-steady lift and moment characteristics of cross-flow wind turbines. The method employs a force-free wake model, accounts for wake/blade interactions, reflects transient aerodynamics, and accommodates time varying winds. Introduction of experimental blade drag data permits calculation of the power produced. The analysis is not dependent upon the selection of an artificial blockage factor, as is necessary for the previously discussed strip theory. A numerical solution of the equations is necessary, and solutions have thus far been obtained for a single bladed turbine.

Fig. 2.4.1 shows the details obtainable with the procedure. A wake-blade interaction is depicted in the transient wake. As shown by the shed vortex trajectories, the results depict actual vortex positions as functions of time; this allows the blade forces and turbine torques to be calculated. Fig. 2.4.2 shows the calculated power coefficients ($C_p$)
for a one bladed turbine utilizing two different blade drag coefficients, plotted vs. the machine rotational speed ($\omega R/V_m$). The comparison with the strip theory is seen to be poor, with the particular blockage factor which was assumed. Rather high power coefficients are obtained, with a strong dependence on the drag coefficient.

Work in progress includes extension of the analysis to the multi-bladed configuration, and to turbines with circulation controlled blades. Parametric studies will then be performed, and compared with available test data.

2.5. Vertical Axis Turbine Test Model

As previously discussed, there are several factors of unknown accuracy in the theoretical developments for the VAT performance. Experimental data is necessary for the determination of the theory accuracy and the limits of its validity. An experimental program is underway at West Virginia University to provide this information. Fig. 2.5.1 shows the VAT test model which has been designed and constructed.

The turbine rotor is 3.05 m (10 ft) in diameter and its blades are 3.25 m (10 ft 8 in) long. The expected power output is on the order of 0.5 kW with winds of 6 m/s (13.4 mph).
This turbine will initially be tested with symmetrical "conventional" airfoils, and then will be equipped with a circulation controlled airfoil system. One of the primary advantages of utilizing a relatively large turbine is that more accurate blade and rotor drag data is expected to be obtained than would be possible with small-scale wind tunnel test models. Also, operational problems can be evaluated, and costs can be determined. Thus, a first step at determining the economic feasibility of this design can be made. The approximate cost of the test model was $5000. Future analysis will be directed at various cost-cutting techniques for the turbine, and optimizing the design on an economic basis.

3. CONCLUSIONS

Based on the results obtained thus far, both of the innovative concepts investigated show promise of being efficient wind energy conversion devices. Initial theoretical and experimental analyses have been conducted, and a firm base established to allow more accurate theoretical methods to be developed to allow parametric analysis of the machines. A continuing test program is well underway to allow confirmation of theoretical results. A test model of the VAT has been built and is in use in the natural wind; a similar type model has been designed for the vortex concentrator. Additional work involving the costs and economics of the two designs must be further investigated.

4. RECOMMENDATIONS

Continuing effort should be expended to allow more thorough evaluations of both circulation controlled vertical axis turbines and vortex concentrator devices. Increasing emphasis should be placed on optimization of designs and providing more detailed estimates of machine capital and operational costs.
FOREWORD

The research described in this report was performed for the National Science Foundation under Grant AER 7500367-000. The report covers the period of work from March 1, 1975 to June 30, 1976. Mr. Marcel Harper was the NSF program manager for the project. The research is continuing under sponsorship of the U. S. Energy Research and Development Administration, Division of Solar Energy.

The research is concerned with investigation of concepts for two innovative wind machines which can convert wind energy to more useful forms: a vortex concentrator device, and a vertical axis wind turbine operating with circulation controlled airfoils. The purpose of the work is to investigate the technical feasibility of these concepts, both theoretically and experimentally, and to evaluate the structural and economic aspects of the designs.

Work was accomplished by several members of the staff and graduate students of the Department of Aerospace Engineering of West Virginia University. Mrs. Cheryl Fries and Mrs. Carolyn Swecker typed the manuscript. The efforts of individuals in the preparation of sections of this report is acknowledged in the CONTENTS section; the Project Manager, Richard E. Walters, was the report editor.
SUMMARY

This report describes theoretical and experimental research accomplished which concerns the evaluation of two concepts for wind (energy conversion) machines. Plans for future work are also discussed.

The first concept is that of a vortex concentrator: a high-lift vertical airfoil in the ambient wind generates a trailing vortex which has its energy harnessed by a relatively small high-speed turbine located just downstream of the wing tip. The device concentrates wind energy so that for a given-size turbine the potential power output is greatly increased. Work summarized in this report includes a partial review of available papers on the subject of wing trailing vortices, and some calculations concerning the anticipated output of such a device. Results of wind tunnel flow visualization tests using smoke and helium bubbles are presented. Future theoretical and experimental efforts on this project are also discussed.

The second concept described is that of a vertical axis wind turbine with circulation controlled airfoils used for the blades. A preliminary theoretical analysis utilizing strip theory has shown the effect of different design features on its operation, and has led to the design of a test model. The configuration, instrumentation, and control systems of the test model are discussed. Also contained herein is a more exact flow theory, which properly takes into account the unsteady aerodynamics involved. Solutions for a one bladed turbine are presented and compared with results from other theoretical work.
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1. INTRODUCTION

Although wind power machines have been in existence for about 2000 years, such devices provide only an extremely small amount of the world's power today. Until recently, fossil fuels were so inexpensive and widely available that the large capital investment required for a wind energy conversion system was not economically justifiable, even though the "fuel", the wind, was free. Today, high world-wide fuel costs and limits of the available fossil-derived energy to certain areas of the world have renewed interest in wind machines. The research described in this report is one of several active projects designed to investigate the current potential of wind machines using modern technology. In this research, two innovative designs are being considered: one, the concept of a vortex generator or energy concentrator, and two, a vertical axis turbine utilizing circulation controlled airfoils for the blades.

Fig. 1.1 shows some parameters of importance in determining the cost of the power generated by a wind machine. As indicated by the arrows around the circle of parameters, each interacts with the other, so that the final system cost per unit of power delivered (shown at the center) is a resultant of all of the interrelated parameters. For example, the size of any machine affects the initial and operating costs, and for minimum cost of the power delivered, is directly related to all the parameters shown. For this reason some designs may be efficient for small sizes and low power outputs, and other designs might be better suited for large
power generating applications. With the present status of development of wind energy conversion systems, it cannot be determined which design concepts are optimum for different applications, and continuing research and development is necessary to make this determination.

It is well known that power can be derived from the wind; thus, the real question is how much will it cost? Some capital costs of various machines which have been built, proposed, or are now available commercially are presented in Fig. 1.2 and Table 1-I. A meaningful comparison of these figures is difficult. The rated wind speeds of the machines, as well as their sizes, vary. Some are only proposed and therefore represent estimated costs from a preliminary design, others were built years ago, and yet others have small output and are semi-mass produced. With the exception of the Putnam turbine, no data is available on any of these designs giving the operating cost over a long period of time. However, the graph does show a considerable reduction in cost per KW_e of output as the machine rated powers increase. Most of these results are for the "conventional" horizontal-axis propeller type machines.

An interesting anomaly is seen in the results plotted for the farm-type machines, nos. 10 through 18. As the rated power increases, the cost per KW_e also increases, which is a contrary trend to the majority of the other data. However, the Amerenalt machines (nos. 3, 4) which are also multi-bladed designs show the opposite trend. This result could be influenced by the rather arbitrary assumption of the same overall efficiency for all the farm type shown, where in reality the larger machines should have somewhat higher efficiencies. Also, the optimum wind speed assumption of approximately 10 m/s (23 mph) for all of these machines is probably not correct.

Site preparation and the costs of accessories such as towers, storage batteries, and inverters would add to the costs shown. Such factors will be considered in the continuing effort to evaluate the costs of various wind energy conversion systems.

Cost data for machines based on innovative concepts should be compared with these results to evaluate their practicality. The present research project does include initial analysis of the structural and manufacturing aspects of the proposed innovative designs, which, in turn, will provide a basis for estimating the cost of the machines. More emphasis will be placed on the cost analysis as the project continues, and as various design concepts are evaluated.

The remainder of this report contains the following: section 2 deals with the theoretical analysis of wing trailing vortices, while section 3 contains some calculations on the anticipated output of the vortex concentrator wind machine. Section 4 develops the theory and provides results for the vertical axis turbine using a rather simplified strip-type blade theory. Section 5 outlines a more realistic (and more complex) theoretical approach to the cross-flow (vertical axis) turbine analysis which involves a numerical technique employing a nonsteady lifting-surface approach. In order to evaluate the results of the theories developed in sections 4 and 5, a test model will be used; its design is discussed in section 6 of
this report. Construction of this model is presently underway; some of its structural aspects as well as the instrumentation and control systems to be used are also discussed in section 6. An evaluation of the test model costs is also given.
Fig. 1.1. Wind machine parameters.
# TABLE 1-1
## WIND MACHINE COST SUMMARY

<table>
<thead>
<tr>
<th>NO.</th>
<th>MANUFACTURER</th>
<th>RATED WIND SPEED</th>
<th>RATED POWER</th>
<th>$/KW</th>
<th>REMARKS</th>
</tr>
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<tr>
<td>1</td>
<td>Winco 1222H</td>
<td>10.3</td>
<td>23</td>
<td>0.20</td>
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<td>2</td>
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<td>21</td>
<td>1.22</td>
<td>1385</td>
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<td>28</td>
<td>1.50</td>
<td>1767 1975 Price</td>
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<td>5</td>
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<td>30</td>
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<td>1828 1975 Price, Note 5</td>
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<td>7</td>
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<td>26</td>
<td>15.00</td>
<td>1000 Quoted to 2000</td>
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<tr>
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<td>20</td>
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<tr>
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<tr>
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<td>8.1</td>
<td>18</td>
<td>500</td>
<td>881 VAT, Note 2</td>
</tr>
</tbody>
</table>

**NOTES:**

1. At rated wind speed; rotor plus generator and 1976 prices unless noted.
2. Not available commercially; ERDA study estimate for production units; includes tower, foundation, and site preparation.
3. Farm-type pumping mills; approximate KW calculated @23 mph assuming overall efficiency = 0.25; no generator-units supplied with gears and pump power transmission unit.
4. Includes small tower; not including import duty to U.S.
5. Includes tower and batteries.
6. Not available commercially; original machine costs updated.

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**Fig. 1.2.** Cost per unit output vs. turbine capacity of wind machines.
2. WING TRAILING VORTICES
(A Partial Review of the Literature and Proposed Extension)

Abstract
Wing trailing vortices have previously been studied mainly because of their effect on following aircraft. However, their usage as wind-energy converters has recently been recognized. It is for the latter reason that this review has been undertaken. A proposed extension to the previous work is also included.

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Nomenclature
English Symbols

\( a \) radial distance to maximum value of \( U_o \)
\( R = \frac{(2b)^3}{S} \) aspect ratio
\( b \) semi span of actual wing
\( b' \) semi span of simple horseshoe vortex
\( \alpha \) sectional lift coefficient
\( \alpha_o \) sectional lift coefficient at midspan of actual wing
\( C_o \) chord at midspan of actual wing
\( \beta \) pressure coefficient
\( \text{CD} \) induced drag coefficient of actual wing
\( \text{CL} \) lift coefficient of actual wing
\( D \) total wing drag
\( D_o \) induced drag of actual wing
\( D_p \) profile drag of actual wing
\( e^i(\eta) \) exponential integral; Eqn. (2.8-3)
\( h \) distance to circle of radius \( a \); Fig. 2.2
\( I(\eta) \) defined by Eqn. (2.8-1); evaluated by Eqn. (2.8-4)
\( K_{irrot} \) kinetic energy per unit \( \pi \) length in irrotational flow field
\( K_{rot} \) kinetic energy per unit \( \pi \) length in rotational flow field
\( L \) total lift
\( L' \) lift per unit span
\( m \) slope of theoretical \( \alpha \) versus \( \text{CL} \) curve
\( p \) static pressure
\( P \) total pressure
\( P = p + \frac{\rho U^2}{2} \) total pressure
\( Q_1(\eta) \) Eqn. (2.4-23)
\( Q_2(\eta) \) Eqn. (2.4-27)
\( r \) radial coordinate in cylindrical coordinates
\( S \) wing planform area
\( t \) time
\( \mu \) perturbation velocity in \( \pi \) direction
velocity vector

\( \vec{v} \)

velocity component in \( x \) direction

\( v_x \)

velocity component in \( \theta \) direction

\( v_\theta \)

maximum velocity in \( \theta \) direction; occurs at \( r = a \)

\( v_z \)

velocity component in \( z \) direction

resultant velocity

\( \omega (z) \)

complex potential

\( x \)

cartesian coordinate along span

\( y \)

cartesian coordinate normal to \( xz \) plane

\( z \)

coordinate in axial direction

Greek Symbols

\( \alpha \)

angle of attack

\( \beta \)

a constant; Eqn. (2.2-17)

\( \gamma = 0.577 \ldots \)

Euler's constant

\( \gamma \)

circulation

\( \varepsilon \)

one order of magnitude less than unity

\( \gamma^* \)

axial component of the vorticity

\( \gamma \)

similarity variable; Eqn. (2.3-6)

\( \phi \)

angular coordinate in cylindrical coordinates

\( \rho \)

a constant

\( \mu \)

coefficient of viscosity

\( \nu \)

molecular kinematic viscosity

\( \nu_t \)

turbulent kinematic viscosity

\( \rho_f \)

fluid density

\( \sigma \)

a constant; Eqn. (2.4-19)

\( \tau \)

turbulent shear stress; Eqn. (2.5-2)

\( \phi \)

velocity potential

\( \psi \)

stream function

\( \omega \)

solid body angular velocity

Subscripts

\( \gamma_0 \)

evaluated at \( r = a \)

\( \gamma_0 \)

evaluated at \( r = \infty \)

\( \gamma_f \)

free stream value

Superscripts

\( \gamma \)

time averaged

\( \gamma' \)

perturbation

\( \gamma \)

unit vector

2.1. Introduction

A wing in level steady flight generates a vortex sheet which, a short distance downstream, rolls up into two trailing vortices. An excellent qualitative discussion of vortexes in aircraft wakes has been provided by Chigier\(^{2,1}\) and Fig. 2.1 in this report has been reproduced from Chigier's article. (Fig. 2.1 in its original form is more dramatic as it is in color). A theoretical description of the flow in the trailing vortices is the concern of this paper.

To that end a review of some of the relevant literature is presented.

McCormick\(^{2,2}\) replaced the vortex sheet and the trailing vortices by a single horseshoe vortex of the Rankine type, i.e., a core of
fluid of radius $a$ rotating like a solid body surrounded by a free irrotational vortex. In this model the flow is unchanged in all planes normal to the downstream $\mathbf{z}$ direction even as $\mathbf{z} \to \infty$. In reality it is anticipated that far downstream the velocity in the cross plane becomes zero and the vortex core has completely decayed. Furthermore, in the McCormick model it is assumed that the axial velocity within the core is constant. In general this does not appear to be true. McCormick's analysis is discussed in section 2.2.

Newman analyzed the flow in a single trailing vortex. The flow is assumed steady, laminar, viscous and incompressible. A boundary layer type order of magnitude analysis is applied to the flow equations expressed in cylindrical coordinates $\mathbf{r}, \theta, \mathbf{z}$ with $\mathbf{z}$ coincident with the vortex core axis; $\mathbf{z}$ is assumed of order one while $\mathbf{r}$ is assumed one order smaller of order $\epsilon$. In addition the small perturbation concept is applied to the axial velocity so that $v_\mathbf{z} = v_\mathbf{z}^0 + u$ where $u$ is of order $\epsilon$. In this manner the flow equations are linearized and the resulting equations are integrated. The nature of the approximations imply that the Newman analysis is valid only far downstream. Newman neglected the axial pressure gradient $\frac{2p}{r^2}$. This analysis gave an axial velocity and a velocity in the cross plane which became negligible as $\mathbf{z} \to \infty$. This paper is discussed in section 2.3.

Batchelor redid the Newman analysis with the same assumptions but included the axial pressure gradient term $\frac{2p}{r^2}$. Batchelor argued that the link between the motion in the cross plane and the axial motion in a steady line vortex is provided by the pressure, and that it is necessary to include the axial pressure gradient to obtain the correct axial velocity. The Batchelor analysis is contained in section 2.4.

Hoffman and Joubert attempted to establish the laws governing the flow in a turbulent line vortex. An approach similar to that employed in turbulent boundary layer theory is used to express the Reynolds turbulent shear stress in terms of the circumferential velocity $U_\theta$. The radial flow equation is then integrated, neglecting variations with respect to $\mathbf{z}$ to determine $U_\theta \left( \mathbf{r} \right)$. It is predicted by this theory, and confirmed by experiment, that the circulation in the vortex $\int \mathbf{r} \times \mathbf{u} \mathbf{r} \mathbf{u}_\mathbf{r} \mathbf{r}$ is proportional to the logarithm of the radius under certain conditions. This analysis is discussed in section 2.5.

Below are given the general equations for turbulent flow in cylindrical coordinates $\mathbf{r}, \theta, \mathbf{z}$ assuming the flow steady, axisymmetric and incompressible. Newman and Batchelor solved these equations for laminar flow applying an order of magnitude analysis. Hoffman and Joubert obtained a highly restrictive solution for turbulent flow. In these equations a barred quantity ($\bar{\cdot}$) denotes time averaged while a primed quantity ($'\cdot$) denotes a perturbation. A hatted quantity ($\hat{\cdot}$) represents a unit vector in the direction indicated so that, for example, the notation $\mathbf{r} \mathbf{r}$ means the Navier-Stokes momentum equation in the $\mathbf{r}$ direction. For turbulent flow the equations are:
Continuity:
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  
(2.1-1)

Momentum \( \hat{z} \):
\[ \rho \frac{\partial}{\partial t} (\rho u \hat{u}) + \rho \frac{\partial}{\partial x} (v \hat{u}) + \rho \frac{\partial}{\partial y} (w \hat{u}) = 0 \]
(2.1-2)

Momentum \( \hat{y} \):
\[ \rho \frac{\partial}{\partial t} (\rho u \hat{v}) + \rho \frac{\partial}{\partial x} (v \hat{v}) + \rho \frac{\partial}{\partial y} (w \hat{v}) = 0 \]
(2.1-3)

Momentum \( \hat{x} \):
\[ \rho \frac{\partial}{\partial t} (\rho u \hat{w}) + \rho \frac{\partial}{\partial x} (v \hat{w}) + \rho \frac{\partial}{\partial y} (w \hat{w}) = 0 \]
(2.1-4)

2.2. McCormick's Analysis (Ref. 2.2)

McCormick\(^2\) assumed that the trailing vortex sheet of the actual wing of span \( 2b \) is replaced by a single horseshoe vortex of constant strength \( \Gamma_\infty \) and constant width \( 2b' \) (Fig. 2.2a). It is assumed that the lift on the actual wing is equal to the lift on the bound portion of the simple horseshoe vortex. Thus

\[ L = C_L \frac{V_\infty^2}{2} = L'(2b') \]  
(2.2-1)

where \( C_L \) is the total lift coefficient of the actual wing and \( L' \) is the lift per unit span on the bound vortex. \( L' \) is set equal to the lift per unit span at the midspan of the actual wing. Thus,

\[ L' = \rho n_0 \frac{\Gamma_\infty^2}{2} = C_{L0} C_s \frac{V_\infty^2}{2} \]  
(2.2-2)

where \( C_{L0} \) and \( C_s \) are, respectively, the sectional lift coefficient and the chord at the midspan of the actual wing. Since the aspect ratio of the actual wing is \( R = \frac{2b}{b'} \), Eqs. (2.2-1) and (2.2-2) provide the relations

\[ \frac{b'}{b} = \frac{C_{L0}}{C_L} \frac{2b}{b'} \]  
(2.2-3)

\[ \frac{\Gamma_\infty^2}{4R} = \frac{C_{L0}^2 V_\infty^2}{4R} \left( \frac{b}{b'} \right)^2 \]  
(2.2-4)

Eqn. (2.2-3) is an expression for \( b' \), the semispan of the single horseshoe vortex and Eqn. (2.2-4) gives its strength.

At section A-A the vortex system is as appears in Fig. 2.2b. The flow system consists of two line vortices of equal but opposite strength \( \Gamma_\infty \) situated at \( x = \pm b' \). McCormick assumed the classical Rankine vortex for the trailing vortices, i.e., a solid rotating core with an irrotational outer field. The stream function for an irrotational positive line vortex (positive counterclockwise)
at the origin of a coordinate system is \( \psi = -\frac{7}{2\pi} \ln r \). Hence

for the two line vortices shown in Fig. 2.2b

\[
\psi = \frac{7}{2\pi} \ln \sqrt{(x-h')^2 - y^2} \quad (2.2-5)
\]

where \( \psi \) = constant represents the value of the stream function for the streamlines of the combined flow. The question is posed whether one of these streamlines is a circle of radius \( a \) centered at \( (h, 0) \) (Fig. 2.2a). Designating this particular streamline by \( \psi_a \), and defining a constant \( \alpha = e^{2\pi \psi_a/\psi} \),

Eqn. (2.2-5) provides

\[
x^2 - 2b \frac{r_n^2}{(a_n^2 - r_n^2)} x + b^2 + y^2 = 0 \quad (2.2-6)
\]

From Fig. 2.2b, the equation of the circle at \( h, 0 \) is

\[
a^2 = (x_h - h)^2 + y_a^2 \quad \text{or} \quad x_a^2 - 2h x_a + (h^2 - a^2) + y_a^2 = 0 \quad (2.2-7)
\]

Equating (2.2-6) to (2.2-7) give the results

\[
h = \pm \sqrt{b^2 + a^2} \quad (2.2-8)
\]

\[
\alpha = \frac{\sqrt{b^2 + a^2} + b'}{\sqrt{b^2 + a^2} - b'} \quad (2.2-9)
\]

Thus the combined flow of the two irrotational line vortices provides two circles of radius \( a \) situated at \( h = h = \pm \sqrt{b^2 + a^2} \) and \( y = 0 \). Using the definition for \( \alpha \), Eqn. (2.2-9) gives the stream function for the circle at \( \pm h, 0 \) as

\[
\psi_a = \frac{7}{2\pi} \ln \frac{\sqrt{b^2 + a^2} + b'}{\sqrt{b^2 + a^2} - b'} \quad (2.2-10)
\]

The line vortices at \( x = \pm b' \) represent singularities in the flow and the streamlines \( \psi_a \) enclose these singularities.

The fluid within the circles of radius \( a \) is rotational undergoing solid body rotation with constant angular velocity \( \omega \) about \( \pm h, 0 \). The flow outside the circles is irrotational.

Another relation available for the solution of the vortex system is the condition that the induced drag \( D_i \) of the wing is equal to the kinetic energy of the fluid in the plane of section \( A-A \) per unit length in the \( Z \) direction. Since the flow field consists of a rotational and an irrotational domain it follow that

\[
D_i = K_{net} + K_{irrot} \quad (2.2-11)
\]

where \( K \) represents kinetic energy per unit \( Z \) length; \( K_{net} \) is the kinetic energy of the two rotational cores of radius \( a \) and \( K_{irrot} \) is the kinetic energy in the irrotational domain from the rotational cores to infinity. If \( v \) represents the velocity of the flow in the section \( A-A \), then

\[
K_{net} = 2 \int_0^a \int_0^{\alpha} \frac{\rho}{2} d\theta d\rho = 2 \int_0^a \int_0^{\alpha} \frac{\rho}{2} \rho (2\pi \rho sin \theta) = \frac{\rho a^2}{2\pi} \quad (2.2-12)
\]

since \( v = \frac{\rho a^2}{2\pi} \). Similarly \( K_{irrot} = \frac{\rho a^2}{2\pi} \rho (2\pi \rho sin \theta) \) where \( \alpha \) encompasses all the area to infinity outside the rotational cores. With the aid of Stokes' divergence theorem and the fact that
the flow is irrotational, the area integral can be converted to a line integral enclosing the rotational cores and enclosing the fluid to infinity (Appendix 2.A). It is derived therein that

$$K_{\text{net}} = \frac{\rho}{2} \int_\gamma \mathbf{v} \cdot \mathbf{n} \, d\Gamma = \frac{\rho}{2} \int_\gamma \mathbf{v} \cdot \mathbf{n} \, d\Gamma$$

Evaluation of this line integral (Appendix 2.A) then provides the result

$$K_{\text{net}} = \int_{r_a}^{r} \mathbf{v} \cdot \mathbf{n} \, d\Gamma$$

where $\mathbf{v}$ is given by

Equation (2.2-10). Thus

$$K_{\text{net}} = \frac{\rho \gamma}{2\pi} \int_{r_a}^{r} \frac{1}{\sqrt{\frac{b^3}{a^2} - b}}$$

and the induced drag from Eqs. (2.2-11), (2.2-12), (2.2-14) is

$$D_i = \frac{\rho \gamma^2}{8\pi} \left[1 + \frac{2}{\Delta} \sin \left(\frac{\theta}{2} - \frac{\Delta}{2}\right) \right]$$

Since $D_i = C_{D_i} \rho \gamma^2$ where $C_{D_i}$ is the induced drag coefficient of the actual wing and using Eqn. (2.2-4), Eqn. (2.2-15) can be expressed in the form

$$a = \frac{b}{b'} \left[ \left( e^{\alpha - \frac{b}{b'}} - \frac{e^{\alpha - \frac{b^2}{b}}}{e^{\alpha - \frac{b^2}{b}}} \right) \right]$$

which is an expression for the vortex core radius $a$ : $\beta$ is defined by

$$\beta = \frac{8\pi R c_{D_i} \rho \gamma}{c_{D_i}} \left( \frac{b'}{b} \right)^2$$

The static pressure results from a balance of the radial pressure gradient and the centrifugal force:

$$\frac{\partial P}{\partial r} = \frac{\rho \gamma^2}{r^2}$$

Integrating this expression from the vortex origin to infinity gives the pressure coefficient at $r = 0$. Since $\mathbf{u} = \frac{\tau r}{2\pi a^2}$, $0 \leq r \leq a$; $C_p = \frac{\tau}{2\pi a^2}$, $a \leq r \rightarrow \infty$, the minimum pressure coefficient which exists at $r = 0$ is:

$$C_{p_m} = \frac{\tau - P_m}{\frac{1}{2} \rho \gamma^2} = -2 \left( \frac{\tau}{2\pi a \rho \gamma} \right)^2$$

Equation (2.2-16) was used to check this theory with experimental pressure data along the axis of symmetry in the trailing vortex of a wing. To check the experimental data, Eqn. (2.2-12) was rewritten in a different form. From Eqn. (2.2-2),

$$\gamma = \frac{c_{\text{vortex}}}{c_{\text{vortex}}} \frac{\gamma}{\gamma}$$

hence

$$C_{p_m} = -\frac{1}{8\pi^2} \left( \frac{c_{D_i} c_{D_i}}{\alpha} \right)^2$$

Assuming $C_{p_m} = (\text{constant}) \alpha$, then

$$\frac{dc_{p_m}}{d\alpha} = \text{constant} \frac{c_{D_i}}{c_{D_i}} \quad \text{or} \quad \frac{dc_{p_m}}{d\alpha} = 1$$

so the axis of symmetry pressure coefficient can be written

$$-\frac{c_{p_m}}{\alpha^2} = \frac{1}{8\pi^2} \left( \frac{c_{D_i}}{c_{D_i}} \frac{c_{D_i}/b}{a/b} \right)^2$$

This expression was theoretically evaluated for elliptic, rectangular, and delta planforms for a range of aspect ratio; it is shown plotted in Fig. 3 of 'Tectonics'. A sample calculation is as follows. Assuming an elliptical wing $S = \pi b c$, $A = (2b)^2$ therefore $\frac{c_{D_i}}{b} = \frac{a}{bR}$. Assuming $R = 6$ gives $c_{D_i} = 0.4744$. For an elliptical wing $c_{D_i} = c_{D_i}$ i.e. the wing lift coefficient is
equal to the sectional lift coefficient; therefore \( C_l = 1 \). Also
the lift curve slope \( \frac{dC_l}{C_l} = \frac{m_1}{C_l(1 + \frac{m_1}{\gamma R})} = 2\pi \left(1 + \frac{z}{\gamma R}\right) \)
\( = 4.7/4 \). From Eqn. (2.2-3), \( \frac{dC_l}{C_l} = 0.7853 \) while Eqn.
(2.2-17) gives \( \delta = 4.934 \). Hence, from Eqn. (2.2-16),
\( \frac{dC_l}{C_l} = 0.1731 \) and Eqn. (2.2-19) gives \( \frac{dC_l}{C_l} = 0.000515 \)
with \( \alpha \) in degrees.

The theoretical predictions for \( \frac{dC_l}{C_l} \) in Fig. 3 of
"McLarnick" did not agree with experiments. For example, a
rectangular wing of \( R = 4 \) experimentally gave \( \frac{dC_l}{C_l} = 0.026 \)
which was 27 times greater than the theoretical prediction.

2.3. Newman's Analysis (Ref. 2.3)
Newman \( \text{analyzed the flow in a single trailing vortex. He}
\text{solved Eqs. (2.1-1) through (2.1-4) for laminar flow after imposing}
a boundary layer type order of magnitude analysis to them. Assuming}
increasing orders of magnitude by the sequence \( \ldots, \varepsilon^2, \varepsilon, 1 \),
\( \varepsilon^3, \varepsilon^4, \ldots \) Newman assumed \( R = o(\varepsilon^2), \varepsilon = o(1) \). In
addition the axial velocity component \( u_2 \) was replaced by \( u_2 = u_2 + u \)
with \( u_2 = o(1), v_2 = o(1), u = o(1) \). The
continuity equation (2.1-1) then becomes \( \frac{2\pi}{\gamma R} = \frac{d}{d\gamma} \left( \frac{2\pi}{\gamma R} \right) \). The last
term is of \( o(\varepsilon) \). Therefore, for the other two terms to remain
in the continuity equation it is required that \( v_2 = o(\varepsilon^3) \).
The momentum \( \gamma \) equation (2.1-2) provides a balance between the
centrifugal force \( \frac{2u_2^2}{R} \) and the radial pressure gradient. For
these two terms to be dominant in Eqn. (2.1-2) it is necessary that
\( u_2 \) be of \( o(\varepsilon) \) or higher. However, to make the momentum \( \gamma \)
equation of the same order as the continuity equation it is
Momentum $\hat{\theta}$:

$$v_0 \frac{\partial \theta}{\partial z} - \nu \left( \frac{\partial v_0}{\partial z} + \frac{\partial v}{\partial x} - \frac{v_0}{A^2} \right) = 0 \quad (2.3-3)$$

Momentum $\hat{\phi}$:

$$v_0 \frac{\partial \phi}{\partial z} - \nu \left( \frac{\partial v_0}{\partial z} + \frac{\partial v}{\partial x} \right) = 0 \quad (2.3-4)$$

By the nature of the approximations made, the vortex is being examined some distance downstream of its origin. Hence, it is sufficient to assume, according to Newman, that the vortex is suddenly generated at $z = 0$ as a free vortex of circulation $T_0$. Far downstream the vortex finally decays until all the perturbation velocities $v_0$, $v$, $u$ are again zero. The boundary conditions to be satisfied by Eqs. (2.3-1) through (2.3-4) are:

- $z = 0$: $v_0 = \frac{T_0}{2\pi z}$ (except at the singular point $\eta = 0$); also $u = 0$.

- $z > 0$: $v_0 \rightarrow 0$ for large $\eta$ (though small compared to $z$)
  
- $\eta \rightarrow 0$: $v_0 \rightarrow 0$, $v \rightarrow 0$, $u \rightarrow 0$ for all $\eta$.

The solution of Eqn. (2.3-3) which satisfies these boundary conditions is identical with that given by Lamb for the development with time of a two dimensional vortex which at $t = 0$ is irrotational and decays for $t > 0$, under the action of viscosity.

If $t$ is replaced by $z/v_0$ the solution of Eqn. (2.3-3) as previously given by Squire is:

$$v_0 (\eta, \gamma) = \frac{T_0}{2\pi z} (1 - e^{-\eta}) \quad (2.3-5)$$

where

$$\eta = \frac{v_0 z}{4\nu} \quad (2.3-6)$$

Eqn. (2.3-5) shows that when $\eta$ is small, $v_0 \approx \frac{T_0}{2\pi z} \frac{v_0}{\nu}$ which represents the flow in a forced vortex or vortex core while, when $\eta$ is large, $v_0 \approx \frac{T_0}{2\pi z}$ which is the flow in a free vortex.

The maximum value of $v_0$ occurs at $\eta = a$ where

$$a = \frac{v_0 z}{4\nu} = 1.25 \quad (2.3-7)$$

Hence, from Eqn. (2.3-7)

$$a(z) = 2.256 \sqrt{\frac{v_0 z}{\nu}} \quad (2.3-8)$$

while Eqn. (2.3-5) provides

$$v_{0, max} (z) = 0.0509 \frac{T_0}{\nu} \sqrt{\frac{v_0 z}{\nu}} \quad (2.3-9)$$

Eqn. (2.3-8) shows that the growth of the vortex core behaves as a laminar boundary layer in zero pressure gradient, i.e., it varies as $z^{1/2}$. The circumferential velocity distribution (2.3-5) can also be written

$$\frac{v_0}{v_{0, max}} = 1.399 \frac{z}{a} \left[ -e^{-1.25\frac{z}{a}} \right]^{1/2} \quad (2.3-10)$$

and this is shown plotted in Fig. 2.3.

The pressure variation across the flow results from Eqn. (2.3-2).
Using Eqns. (2.3-5) and (2.3-6) gives
\[ C_p(\theta, \phi) = \frac{\chi(\theta, \phi) - P_0}{\frac{1}{2} \rho V_w^2} = \frac{\int_0^{\infty} \left(1 - e^{-\gamma}\right)^2 d\gamma}{16\pi^2 \nu V_w^2} \]
(2.3-11)

The integral is defined as \( I(\gamma) \) (see Eqns. (2.3-1) and (2.3-12)) and evaluated as Eqns. (2.3-4). Hence
\[ C_p(\theta, \phi) = \frac{\chi(\theta, \phi) - P_0}{\frac{1}{2} \rho V_w^2} = -\left(\frac{\gamma}{\frac{1}{2} \rho V_w^2}\right)^2 \frac{T_w}{16\pi^2 \nu V_w^2} \]
(2.3-12)

With the aid of Eqn. (2.3-7) it is shown that
\[ \frac{T_w}{16\pi^2 \nu V_w^2} = 1.25 \left(\frac{\gamma}{2\pi \nu V_w}\right)^2 \]
(2.3-13)

hence the pressure coefficient can be written in the alternate form
\[ C_p(\theta, \phi) = \frac{\chi(\theta, \phi) - P_0}{\frac{1}{2} \rho V_w^2} = -1.25 \left(\frac{\gamma}{2\pi \nu V_w}\right)^2 \]
(2.3-14)

Since \( I(\gamma) = 2 \ln 2 \) (Eqn. (2.3-12)) the minimum pressure coefficient, which exists at the center of the vortex is
\[ C_{p_{\text{min}}} = \frac{\chi - P_0}{\frac{1}{2} \rho V_w^2} = -1.74 \left(\frac{\gamma}{2\pi \nu V_w}\right)^2 \]
(2.3-15)

This relation seems of the same magnitude as that given by McCormick (Eqn. (2.2-18)). However, there is a fundamental difference: McCormick considers the core radius a constant (Eqn. (2.2-16)) whereas in Newman's analysis \( a = a(\theta) \) (Eqn. (2.3-8)).

The static pressure rise from the center of the vortex to the point of \( \frac{\gamma}{\gamma_{\text{max}}} \) (where \( \gamma = \infty \)) follows from
\[ \frac{P_0 - P_{\text{in}}}{} = \frac{P_0 - P_{\text{in}}}{\frac{1}{2} \rho V_w^2} = \frac{P_0 - P_{\text{in}}}{\frac{1}{2} \rho V_w^2} = \gamma_{\text{max}} - \gamma_{\text{in}} \]
(2.3-16)

Therefore the static pressure rise from the center of the vortex to the point of maximum circumferential velocity is more than one half the overall static pressure rise from the center of the vortex to \( \gamma = \infty \).

The solution of Eqn. (2.3-4) is
\[ \alpha(\gamma, \phi) = \frac{A}{2\pi} e^{-\gamma} \]
(2.3-17)

where \( A \) is a constant to be determined. From the continuity equation (2.3-1) with the aid of Eqn. (2.3-17)
\[ \alpha(\gamma, \phi) = \frac{A}{2\pi} e^{-\gamma} \]
(2.3-18)

The constant \( A \) is determined from the momentum theorem which relates the net force acting on a fluid within a region to the net rate of momentum outflow through the region. Since \( T_w \) does not appear in Eqs. (2.3-17) and (2.3-18) it is reasoned that these equations apply in the limit when \( T_w \rightarrow 0 \). Thus it is the drag of the body producing the vortex, when \( T_w \rightarrow 0 \), \( D \rightarrow D_0 \), the profile drag, which for axisymmetric flow is evaluated from.
\[ D = \int (\rho \vec{v} \cdot \vec{u}) \vec{u} \, d\vec{r} - \rho \vec{v} \int \vec{u} \, d\vec{r} \]  

(2.3-19)

After assuming \( \vec{v} = \vec{v}_o \), using Eqn. (2.3-17), integration provides

\[ A = \frac{D}{4\pi \nu^2} \]  

(2.3-20)

In summary, the velocity components are:

\[ u(\eta, z) = -\frac{D^2 \eta}{4\pi \nu^2} \]  

(2.3-21)

\[ u_\eta(\eta, z) = -\frac{D \eta \rho \eta \rho z^2}{4\pi \nu^2} \]  

(2.3-22)

\[ u_\rho(\eta, z) = \frac{77}{2\pi \rho} (1 - e^{-\eta}) \]  

(2.3-23)

2.4. Batchelor's Analysis (Ref. 2.4)

Batchelor\(^2\) repeated Newman's analysis\(^2,3\) (section 2.3) under the same orders of magnitude and boundary conditions with the sole change that the axial pressure gradient term \( \frac{\partial p}{\partial \rho} \) was included in the momentum equation. The equations Batchelor considered are:

Continuity:

\[ \frac{2\nu^2}{2\rho} + \frac{\nu^2}{\rho} + \frac{2u}{\rho^2} = 0 \]  

(2.4-1)

Momentum \( \vec{\rho} \):

\[ \frac{u^2}{\rho} + \frac{\partial p}{\partial \rho} = 0 \]  

(2.4-2)

Momentum \( \vec{\theta} \):

\[ V_\rho \frac{\partial u}{\partial \rho} - \nu \left( \frac{\partial^2 u}{\partial \rho^2} + \frac{2u}{\rho} - \frac{u^2}{\rho^2} \right) = 0 \]  

(2.4-3)

Momentum \( \vec{\rho} \):

\[ V_\rho \frac{\partial u}{\partial \rho} - \nu \left( \frac{\partial^2 u}{\partial \rho^2} + \frac{2u}{\rho} \right) = 0 \]  

(2.4-4)

The solutions for Eqns. (2.4-2) and (2.4-3) are the same as for the Newman analysis. Thus Eqn. (2.4-3) provides

\[ \frac{v_\theta(\eta, \rho)}{V_\rho} = \frac{17}{2\pi \rho} (1 - e^{-\eta}) \]  

(2.4-5)

where \( \rho = \frac{V_\rho R}{4\nu^2} \)  

(2.4-6)

The maximum value of \( v_\theta \) occurs at \( \eta = a \) with

\[ \rho_a = \frac{V_\rho a^2}{4\nu^2} = 1.25 \]  

(2.4-7)

Hence, from Eqns. (2.4-7),

\[ a(2) = 2.236 \sqrt{\frac{V_\rho}{V_\rho}} \]  

(2.4-8)

while Eqn. (2.4-5) provides

\[ \frac{v_{\theta_{\text{max}}}(2)}{V_\rho} = 0.0509 \frac{17}{2\pi \rho} \]  

(2.4-9)

Eqn. (2.4-5) can also be written:

\[ \frac{V_\rho}{V_{\text{max}}} = \frac{1.889}{\pi a} \left[ 1 - e^{-1.25 \left( \frac{2}{a} \right)^2} \right] \]  

(2.4-10)
Eqn. (2.4-2) provides the pressure coefficient in the two alternate forms:

\[ C_p (\gamma, \varepsilon) = \frac{P(\gamma, \varepsilon) - P_0}{\frac{1}{2} \rho v^2} = - I(\gamma) \frac{T_m^2}{16 \pi^2 v^2} \]  

(Eq 2.4-11)

or,

\[ C_p (\gamma, \varepsilon) = \frac{P(\gamma, \varepsilon) - P_0}{\frac{1}{2} \rho v^2} = - 1.25 I(\gamma) (\frac{T_m}{2 \pi \kappa v})^2 \]  

(Eq 2.4-12)

Eqn. (2.4-6) is considered next for the determination of the perturbation axial velocity. Using Eqn. (2.4-11) for \( P(\gamma, \varepsilon) \), Eqn. (2.4-4) can be written:

\[ \frac{\partial}{\partial n} \left( \frac{\partial^{2} u}{\partial n^{2}} + \frac{\partial v}{\partial n} \right) = - \frac{V_m T_m^2}{16 \pi^2 v^2} \left[ I(\gamma) + \gamma \frac{dI(\gamma)}{d\gamma} \right] \]

This equation is transformed from \( n \) to \( \varepsilon \) by use of relation (2.4-13). Thus

\[ \frac{\partial u}{\partial n} = \frac{V_m \kappa}{2 \pi v} \frac{\partial u}{\partial \varepsilon} \]

\[ \frac{\partial^2 u}{\partial n^2} = \frac{V_m}{2 \pi v} \left( \frac{\partial^2 u}{\partial \varepsilon^2} + \frac{\partial u}{\partial \varepsilon} \right) \]

and Eqn. (2.4-11) becomes

\[ \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial \varepsilon^2} - \frac{\partial u}{\partial \varepsilon} = - \frac{T_m^2}{16 \pi^2 v^2} \left[ I(\gamma) + \gamma \frac{dI(\gamma)}{d\gamma} \right] \]

(Eq 2.4-14)

This is a second order, linear, nonhomogeneous partial differential equation for \( u(\gamma, \varepsilon) \). Its complementary solution results by setting its right hand side (or, what is the same thing, the right hand side of Eqn. (2.4-13)) equal to zero. The resulting differential equation is then identical to Eqn. (2.3-4) so that the complementary solution

\[ u_c(\gamma, \varepsilon) \]  

is given by Eqn. (2.3-17), i.e.,

\[ u_c(\gamma, \varepsilon) = - \frac{A}{\varepsilon} e^{-\gamma} \]

(Eq 2.4-15)

where \( A \) is a constant to be determined. The particular solution \( u_p(\gamma, \varepsilon) \) satisfies the complete equation (2.4-14). An indication of the form of the particular solution is obtained by integrating all terms in Eqn. (2.4-14) over a cross sectional plane. Thus,

\[ \int_0^\infty \frac{\partial u}{\partial n} 2 \pi \kappa \, d\gamma = \frac{1}{2} \int_0^\infty \left( \frac{\partial^2 u}{\partial \varepsilon^2} + \frac{\partial u}{\partial \varepsilon} \right) 2 \pi \kappa \, d\gamma \]

\[ = - \frac{V_m T_m^2}{12 \pi^2 v} \int_0^\varepsilon \frac{d}{d\gamma} \left[ I(\gamma) \right] 2 \pi \kappa \, d\gamma \]

The variable of integration of the second two integrals is transformed to \( \gamma \). Assuming \( \frac{\partial^2 u}{\partial \varepsilon^2} = 0 \) at \( \gamma = \infty \) and using Eqns. (2.8-14) and (2.8-16) the above relation provides

\[ \frac{d}{d\varepsilon} \int_0^\infty u \kappa \, d\gamma = - \frac{T_m}{16 \pi^2 v} \]

(Eq 2.4-16)

Integrating with respect to \( \varepsilon \) and adjusting the constant of integration gives

\[ \int_0^\infty u(\gamma, \varepsilon) \, d\gamma = - \frac{T_m}{12 \pi^2 v} \left( \frac{\partial u}{\partial \varepsilon} \right)_0^\infty + \frac{C}{2} \]

(Eq 2.4-17)

The integration of the left hand side is performed at \( \varepsilon \) constant, with respect to \( \gamma \) over the definite limits \( 0 \) to \( \infty \) and results in a function of \( \varepsilon \) only on the right hand side. Thus
the particular solution is chosen in the form

\[ \alpha_r (\gamma, z) = - \sigma \frac{\Lambda \sqrt{\frac{3}{2}}}{c} Q_1 (\gamma) + \frac{c}{\varepsilon} Q_2 (\gamma) \]  

(2.4-18)

where \( \sigma \) is a constant, namely,

\[ \sigma = \frac{\gamma Q^2}{32 \pi^2} \]  

(2.4-19)

The functions \( Q_1 (\gamma), Q_2 (\gamma) \) are determined by substituting the particular solution (2.4-18) into the momentum \( \dot{z} \) equation as given by Eqn. (2.4-14) to obtain:

\[ \frac{\sigma}{2} \left( \Lambda \sqrt{\frac{3}{2}} \right) \left[ - \gamma Q'' + (\gamma + 1) Q' - Q_1 \right] \]

\[ + \frac{c}{\varepsilon} \left[ \gamma Q_2'' + (\gamma + 1) Q_2' + Q_2 + \frac{c}{\varepsilon} Q_1 - \frac{\sigma}{\varepsilon} I - \frac{\c}{\varepsilon} \frac{\alpha J}{J} \right] = 0 \]

Setting each square bracket equal to zero gives two equations for the two unknown functions \( Q_1 (\gamma), Q_2 (\gamma) \). The second square bracket contains the ratio of constants \( \frac{c}{\varepsilon} \) where \( \sigma \) is defined by Eqn. (2.4-19). Since the constant \( \varepsilon \) may be combined with \( Q_2 (\gamma) \), to form a new function of \( \gamma \), that is \( c Q_2 (\gamma) = Q_2 (\gamma) \) (say), it is permissible to assume \( \frac{c}{\varepsilon} = 1 \). Then

\[ c = \sigma = \frac{\gamma Q^2}{32 \pi^2} \]  

(2.4-20)

and the two equations for \( Q_1, Q_2 \) are

\[ \gamma Q_1'' + (\gamma + 1) Q_1' + Q_1 = 0 \]  

(2.4-21)

\[ \gamma Q_2'' + (\gamma + 1) Q_2' + Q_2 = - Q_1 + \varepsilon \gamma I' \]  

(2.4-22)

Eqn. (2.4-21) has as its solution

\[ Q_1 (\gamma) = e^{-\gamma} \]  

(2.4-23)

so that Eqn. (2.4-22) with the aid of Eqn. (2.4-23) can be written

\[ \frac{\sigma}{\gamma} \left[ \gamma (Q_2' + Q_1) \right] = e^{-\gamma} + \frac{c}{\varepsilon} (\gamma I') \]

Integrating from \( \gamma = 0 \) to \( \gamma \) gives

\[ Q_2' + Q_2 = \frac{e^{-\gamma} - 1}{\gamma} + I \]

whose integrating factor is \( e^\gamma \). Integrating again from \( \gamma = 0 \) to \( \gamma \) and setting the constant of integration \( Q_2 (0) = 0 \) gives

\[ Q_2 (\gamma) = e^{-\gamma} \int_0^\gamma \left( \frac{1}{\gamma} + I e^\gamma \right) d\gamma \]  

(2.4-24)

Using Eqns. (2.8-10) for \( I (\gamma) \) and simplifying provides

\[ Q_2 (\gamma) = e^{-\gamma} \left[ \frac{1}{\gamma} + \frac{e^{-\gamma}}{\gamma} + 2 e^{-\gamma} \left( e^\gamma - e^\gamma (\gamma) \right) \right] d\gamma \]

Using Eqns. (2.8-10) and (2.3-11) for \( e^\gamma (\gamma) \) and \( e^\gamma (\gamma) \)
respectively, noting that \[ \int_0^\gamma e^{-\nu} d\gamma = e^{\nu}(\gamma)\bigg|_0^{\infty} - e^{\nu}(\gamma) = -\nu + \ln \gamma + e^{\nu}(\gamma) \]

with the aid of Eqn. (2.4-7), enables the last equation to be written

\[ Q_2(\gamma) = e^{-\nu} \left[ -\nu - \ln \gamma - e^{\nu}(\gamma) + (2\Delta_n^2 - \nu) \right] \]

\[ -2 M(\gamma) + 2 N(\gamma) \]

where

\[ M(\gamma) = \int_0^\gamma e^{-\nu} \left[ \int_0^\gamma e^{-\nu} d\gamma \right] d\gamma = e^{\nu} \left[ -\nu - \ln \gamma - e^{\nu}(\gamma) \right] - \ln \gamma + \ln \gamma + \ln \gamma + \ln \gamma \]

\[ N(\gamma) = \int_0^\gamma e^{-\nu} \left[ \int_0^\gamma e^{-\nu} d\gamma \right] d\gamma = e^{\nu} \left[ -\nu - \Delta_n^2 - \ln \gamma - e^{\nu}(\gamma) \right] + \nu^2 + \ln \gamma + \ln \gamma + \ln \gamma \]

Hence, finally,

\[ Q_2(\gamma) = e^{-\nu} \left[ -\nu - \ln \gamma - e^{\nu}(\gamma) + (2\Delta_n^2 - \nu) \right] + 2 e^{\nu}(\gamma) - 2 e^{\nu}(\gamma) \]

\[ I(\gamma), \quad Q_1(\gamma), \quad Q_2(\gamma). \quad \text{Eqns. (2.4-14), (2.4-23), (2.4-27)} \]

respectively have been plotted in Fig. 2 of Batchelor's report and is repeated herein as Fig. 2.4. \( \nu \) is Euler's constant (\( \nu = 0.5772 \ldots \)) so \((2\Delta_n^2 - \nu) = 0.3091 \).

From the respective relations the following limiting values result:

\[ I(0) = 2 \Delta_n^2, \quad I(\infty) = 0, \quad Q_1(0) = 1, \quad Q_1(\infty) = 0, \quad Q_2(0) = 0, \quad Q_2(\infty) = 0. \]

The complete solution to Eqn. (2.4-14) is the sum of Eqns. (2.4-15) and (2.4-18). Thus

\[ u(\gamma, \rho) = -\frac{A}{2} e^{-\nu} \left[ \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \right] \frac{\Delta_n^2}{\nu} \frac{\nu}{2} + \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \frac{\nu}{2} \]

(2.4-28)

This term is usually negative. Since, by definition, \( u_\infty = V_\infty + u \)

it follows that \( u_\infty < V_\infty \), i.e. there will usually be a deficit of longitudinal velocity in the trailing vortex generated by the wing.

The radial velocity component results from the continuity equation (2.4-1), which after integration with respect to \( \rho \) can be written

\[ u_\infty (\gamma, \rho) = -\frac{2 \Delta_n^2}{\nu} \int_0^\gamma \frac{2E}{2} d\gamma \]

(2.4-29)

From Eqn. (2.4-28)

\[ \frac{2E}{2} = \frac{A}{2} \frac{d}{d\gamma} \left( e^{\nu} \right) + \frac{\Delta_n^2}{2} \left( \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \right) \frac{e^{\nu}}{2} - \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \frac{e^{\nu}}{2} \]

Therefore, integration provides

\[ u_\infty (\gamma, \rho) = -\frac{2 \Delta_n^2}{\nu} \int_0^\gamma \frac{2E}{2} \left[ e^{\nu} \left( \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \right) \frac{e^{\nu}}{2} - \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \frac{e^{\nu}}{2} \right] \frac{2E}{2} \frac{\nu}{2} \frac{\Delta_n^2}{\nu} \frac{e^{\nu}}{2} \]

(2.4-30)
The first term in this equation is identical to that obtained by Newman (Eqn. (2.3-18)); the additional terms are those resulting from the inclusion of the axial pressure gradient. This relation satisfies the condition \( \nu_0 \to 0 \) as \( \eta \to 0 \) regardless of the value of \( \varepsilon \); it also satisfies the condition \( \nu_0 \to \infty \) as \( \varepsilon \to \infty \) for all values of \( \eta \). By noting Eqns. (2.4-5) and (2.4-28) \( \nu_0 \) can be expressed in terms of \( \nu_0 \) and \( \nu_0 \) as follows:

\[
\nu_0 ( \eta, \varepsilon, z ) = - \frac{\nu_0}{2 \pi V_0 z} - \frac{2 V_0 u}{V_0 z^2} \tag{2.4-31}
\]

Summary:

\[
u_0 ( \eta, \varepsilon ) = \frac{\nu_0}{2 \pi z} \left( 1 - e^{-\eta} \right) \tag{2.4-5}
\]

\[
u_0 ( \eta, \varepsilon ) = - \frac{A}{2} e^{-\eta} - \frac{\nu_0}{2 \pi z} (\gamma (\eta) \frac{A}{V_0} \frac{V_0^2}{z^2} + \nu_0^2 \gamma (\eta) \frac{2 \nu_0}{z^2 V_0} \tag{2.4-28}
\]

\[
u_0 ( \eta, \varepsilon, z ) = - \frac{\nu_0}{2 \pi V_0 z} - \frac{2 V_0 u}{V_0 z^2} \tag{2.4-31}
\]

\[
\zeta ( \gamma, \varepsilon ) = \frac{P(\gamma, \varepsilon) - P_0}{V_0^2} = -1.25 \left( \nu_0 \frac{V_0^2}{2 \pi a V_0} \right)^2 \tag{2.4-12}
\]

These relations show that: as \( \eta \to \infty \) (regardless of the value of \( \varepsilon \)) or as \( \varepsilon \to \infty \) (regardless of the value of \( \eta \)), \( \nu_0 \), \( \nu_0 \), \( \nu_0 \), and the static pressure increase until \( P(\gamma, \varepsilon) \to P_0 \). The maximum value of \( \nu_0 \) occurs at \( \eta = a (\varepsilon) = 2.336 \left( \frac{V_0^2}{V_0^2} \right) ; \) the maximum value of \( \nu_0 \), from

\[
\frac{d u}{d \eta} = 0 \quad \text{occurs on the axis of symmetry} \quad \eta = 0, \quad \text{and is}
\]

\[
u_0 ( \gamma, \varepsilon ) = - \frac{A}{2} - 0.625 V_0 \left( \eta \frac{V_0}{2 \pi a V_0} \right)^2 \tag{2.4-32}
\]

The total pressure \( P = p + P \frac{V_0}{2} \) where \( V_0 = \nu_0^2 + \nu_0^2 + \nu_0^2 \). The change in the total pressure is

\[
\frac{d}{d \gamma} \frac{1}{2} \rho \frac{V_0^2}{2} = \zeta (\gamma, \varepsilon) + 2 \frac{\nu_0}{V_0} + \frac{\nu_0^2 + \nu_0^2 + \nu_0^2}{V_0^2} \tag{2.4-33}
\]

where \( \zeta (\gamma, \varepsilon) \) is given by Eqn. (2.4-12). From Eqn. (2.4-5) since \( \eta = 0 (\varepsilon) \) and \( \nu_0 = 0 (\varepsilon) \), \( \nu_0 = 0 (\varepsilon) \). Eqn. (2.4-12) then provides, since \( a = 0 (\varepsilon) \), \( \zeta = 0 (\varepsilon) \). Therefore, considering the orders of magnitude of the velocity components in Eqn. (2.4-33) it is found that, within the prescribed orders of the present analysis, the deficit of the total pressure in the vortex is associated solely with the axial motion:

\[
\frac{d}{d \gamma} \frac{1}{2} \rho \frac{V_0^2}{2} = \frac{u}{V_0} \tag{2.4-34}
\]

The maximum deficit of the total pressure occurs at the center of the vortex. If the complete expression (2.4-33) is used to calculate the maximum deficit of the total pressure then, since \( \nu_0 \) and \( \nu_0 \) are both zero at \( \eta = 0 \), it follows that
\[
\frac{I(\alpha^2) - I_0}{\frac{1}{2} \rho V_0^2} = C_\alpha + \frac{2 \alpha}{V_0} + \left(\frac{\alpha}{V_0}\right)^2 \tag{2.4-35}
\]

where \(C_\alpha = -1.74 \left(\frac{I_0}{2\pi \alpha V_0}\right)^2\) and \(\alpha\) results from Eqn. (2.4-32).

### 2.5. Hoffmann and Joubert Analysis (Ref. 2.5)

Hoffmann and Joubert\(^2\) analyzed the flow in a single trailing vortex assuming fully turbulent flow therein. They solved Eqns. (2.1-1) through (2.1-4) for incompressible, steady flow assuming axial symmetry \((\frac{\partial}{\partial \phi} = 0)\) and two dimensionality \((\frac{\partial}{\partial \theta} = 0)\). Under these assumptions the continuity equation (2.1-1) is satisfied identically and the momentum \(\hat{\alpha}\) equation (2.1-2), neglecting the Reynolds stresses \((-\rho \hat{u}_\alpha \hat{u}_\alpha')\) and \((-\rho \hat{u}_\theta \hat{u}_\theta')\), gives the balance between the centrifugal force and the radial pressure gradient, namely, \(-\rho \hat{u}_\alpha \hat{u}_\alpha' + \frac{2\rho}{\partial \phi} = 0\); under the above assumptions the momentum \(\hat{\alpha}\) equation (2.1-1) for fully turbulent flow (neglecting the molecular shear stress with respect to the turbulent shear stress) gives \(-\rho \hat{u}_\alpha \hat{u}_\alpha' = 0\) while the momentum \(\hat{\theta}\) equation (2.1-4) provides \(-\rho \hat{u}_\theta \hat{u}_\theta' = 0\). Hoffmann and Joubert consider only a solution to the momentum \(\hat{\theta}\) equation which, after a single elementary integration, becomes

\[-\rho^2 \hat{v}_\theta \hat{v}_\theta' = C_1 \tag{2.5-1}\]

Designating the Reynolds turbulent stress \((-\rho \hat{u}_\theta \hat{u}_\theta')\) by \(\hat{\tau}_\theta\), Hoffmann and Joubert assume the Reynolds stress related to the axial component of the fluid vorticity \(\hat{\theta}\) by

\[-\rho \hat{u}_\alpha \hat{u}_\alpha' \equiv \hat{\tau}_\theta = \rho \hat{v}_\theta \hat{v}_\theta' \tag{2.5-2}\]

where \(\hat{v}_\theta\) is the turbulent kinematic viscosity. Under the present assumptions, \(\hat{\theta} = \frac{2}{\alpha} \frac{\partial}{\partial \phi}(\rho \hat{u}_\theta)\). Therefore Eqn. (2.5-1) becomes

\[-\rho \hat{v}_\theta \hat{v}_\theta' = \hat{\tau}_\theta = C_1 \tag{2.5-2}\]

Assuming \(\hat{v}_\theta\) constant, integration provides

\[\hat{v}_\theta (\alpha) = \frac{C_1}{\hat{v}_\theta} \frac{I_0}{\alpha^2} + C_2 \frac{1}{\alpha}\]

where \(C_2\) is another constant of integration. Assuming the velocity maximum occurs at \(\alpha = \alpha\) gives

\[\hat{v}_\theta (\alpha) = \frac{I_0 (\alpha/\alpha)}{\alpha^2} + \frac{1}{\alpha} \tag{2.5-3}\]

By definition, the circulation \(I = 2 \pi \alpha \hat{v}_\theta\). Denoting

\[I = 2 \pi \alpha \hat{v}_\theta \tag{2.5-3}\]

Eqn. (2.5-3) becomes

\[\frac{I (\alpha)}{I_0} = \frac{I_0 (\alpha/\alpha)}{\alpha^2} + \frac{1}{\alpha} \tag{2.5-4}\]
Thus a logarithmic variation for $\tilde{u}_\theta (\tau)$ and $T' (\tau)$ are obtained.

Hoffman and Joubert carried out a series of experiments to check Eqsns. (2.5-3) and (2.5-4). Fig. 2.3 herein is Fig. 6 in Hoffman and Joubert’s report and shows $u_\theta' / u_{\theta, max}$ versus $\tau / a$. The encircled points are test data while the solid curve is the theoretical result under the assumption of laminar flow as given by Newman and Batchelor and corresponds to Eqs. (2.3-10) herein.

Fig. 2.5 herein is Fig. 4 in Hoffman and Joubert. This figure provides test data for the dimensionless circulation $\Gamma' / \Gamma_{a}$ as a function of the dimensionless distance $\tau / a$. This figure is given in semilog coordinates and confirms the linearity of the theoretical result (2.5-4) over a limited range of $\tau / a$.

Hoffman and Joubert realized the limitations of Eqsns. (2.5-3) and (2.5-4) since they gave $u_\theta \to -\infty$ as $\tau \to 0$ and $T' \to \infty$ as $\tau \to \infty$. They state: “As both of these are impossible ... there is a change of mechanism of the flow at radii both less and greater than the logarithmic region” in order to satisfy the conditions $u_\theta = 0$ at $\tau = a$ and $T' = T_a'$ at $\tau = \infty$.

2.6 Proposed Extension

The analyses in sections 2.2 through 2.4 above are for laminar flow. However, it seems reasonable to postulate that the flow in the trailing vortices would actually be turbulent, especially at some distance downstream.

A turbulent analysis does exist (section 2.5) but it is highly restrictive: it assumes two dimensional fully turbulent flow with axial symmetry. Batchelor has shown in his laminar analysis (section 2.4) that the axial pressure gradient is important, and it seems reasonable to postulate that it would be important also in turbulent flow.

This proposed extension considers steady, incompressible, axisymmetric turbulent flow in cylindrical coordinates $\tau$, $\theta$, $\rho$ retaining variations with respect to $\theta$.

The equations to be solved are Eqsns. (2.1-1) through (2.1-4). Experience with turbulent boundary layers has shown that the turbulent normal stresses, namely $(-\tilde{\rho} \tilde{u}_{\theta}' \tilde{v}'_\rho')$, $(\tilde{\rho} \tilde{u}_{\theta}' \tilde{v}'_\rho')$, $(\tilde{\rho} \tilde{u}_{\theta}' \tilde{v}'_\rho')$ may be neglected. The Newman order of magnitude analysis (section 2.3) is applied to the time averaged terms. Thus for turbulent flow it is assumed that

$$\begin{align*}
\tau &= o (\epsilon) \\
\rho &= o (\epsilon^2) \\
\nu &= o (\epsilon^2) \\
\varepsilon &= o (\epsilon^2) \\
\theta &= o (\epsilon^2) \\
\rho' &= o (\epsilon^2) \\
\nu' &= o (\epsilon^2)
\end{align*}$$

(2.6-1)
In addition an order of magnitude is assigned to the turbulent shear stress terms which are all assumed of equal order. The Hoffmann and Joubert analysis (section 2.5) shows that the turbulent shear stress term \(-\bar{\rho} \bar{u}' \bar{v}'\) is dominant in their analysis. To retain this term in the present analysis it must be of the same order \(O(\varepsilon)\) as the other terms retained in Eqn. (2.1-3); thus \(\bar{\rho} \bar{u}' \bar{v}' = O(\varepsilon^2)\). This establishes the order of magnitude of all the turbulent shear stress terms:

\[-\bar{\rho} \bar{u}' \bar{v}' = O(\varepsilon^2)\]
\[-\bar{\rho} \bar{u}' \bar{v}' = O(\varepsilon^2)\]
\[-\bar{\rho} \bar{u}' \bar{v}' = O(\varepsilon^2)\] (2.6-2)

Applying relations (2.6-1) and (2.6-2) to Eqns. (2.1-1) through (2.1-4) gives the following equations to be solved:

Continuity:
\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} = 0
\] (2.6-3)

Momentum \(\bar{u}\):
\[
-\bar{\rho} \frac{\partial \bar{u}}{\partial t} = \frac{\partial \overline{p}}{\partial x} - \bar{u} \frac{\partial \bar{u}}{\partial x} - \bar{v} \frac{\partial \bar{u}}{\partial y} - \bar{w} \frac{\partial \bar{u}}{\partial z}
\] (2.6-4)

Momentum \(\bar{v}\):
\[
\frac{\partial \bar{v}}{\partial t} \bar{u} \frac{\partial \bar{v}}{\partial x} - \bar{v} \frac{\partial \bar{v}}{\partial y} - \bar{w} \frac{\partial \bar{v}}{\partial z} = 0
\]

Momentum \(\bar{w}\):
\[
\frac{\partial \bar{w}}{\partial t} \bar{u} \frac{\partial \bar{w}}{\partial x} - \bar{v} \frac{\partial \bar{w}}{\partial y} - \bar{w} \frac{\partial \bar{w}}{\partial z} = 0
\] (2.6-5)

The nature of the approximations restricts the applicability of Eqns. (2.6-3) through (2.6-6) to some distance downstream of its origin at \(Z = 0\). Hence, it is sufficient to assume, consistent with Newman (section 2.3) that the vortex is suddenly generated at \(Z = 0\) as a free vortex of circulation \(T_{\infty}\). Far downstream the vortex finally decays until all the time averaged velocities \(\bar{u}_x\), \(\bar{u}_y\), \(\bar{u}_z\) are again zero. The boundary conditions to be satisfied by these equations are those employed by Newman:

\[Z = 0 : \bar{u}_y = \frac{T_{\infty}}{\pi \bar{y}} (\kappa \neq 0), \bar{u} = 0\]
\[Z > 0 : \bar{u}_y \to 0 \quad \text{for } \kappa \text{ large but small compared to } Z\]
\[Z \to \infty : \bar{u}_x \to 0, \bar{u}_y \to 0, \bar{u} \to 0 \quad \text{for all } \kappa\]

To integrate the system of Eqns. (2.6-3) - (2.6-6) it is necessary to relate the turbulent shear stress terms to the properties of the main flow. A survey of the literature is now underway to find some reasonable relations.

2.7. Conclusions

It is inconclusive at this point whether the turbulent approach is more accurate than the laminar. The turbulent approach of Hoffmann and Joubert is highly simplified neglecting as it does downstream variations of the flow properties. The proposed extension of section 2.6 represents an ambitious attempt at a more realistic turbulent model. Relations between the turbulent shear stress terms and the main properties of the flow are now being sought.
References


Fig. 2.1. Paired trailing vortices in wake of aircraft. (Figure from p. 80, Ref. 2.1.)
Fig. 2.2a. McCormick’s model. (Ref. 2.2)

Fig. 2.2b. McCormick’s model. (Ref. 2.2)

Fig. 2.3. Comparison of laminar velocity profile with test data. --- laminar; ○ test data. (Figure is Fig. 6, Ref. 2.5)

Fig. 2.4. The functions $I(\eta)$, $Q_2(\eta)$, $Q_3(\eta)$. (Figure is Fig. 7, Ref. 2.4)
Appendix 2.A. Evaluation of Irrotational Kinetic Energy

Proof that:  \[ K_{n,e} = -\frac{1}{2} \int \vec{v} \cdot \vec{\nabla} \phi \, dV \]

Consider any number of finite solid bodies in a flow and let \( \Sigma \) be a closed surface enclosing these bodies. Then from Stokes' divergence theorem

\[ \int \nabla \cdot \vec{A} \, d(V) = \int \vec{A} \cdot d\Sigma \]

where \( d(V) \) is the volume included between the surfaces \( S_1, S_2, \ldots, \Sigma \). In two dimensional flow for a unit distance normal to the paper, \( d(V) = \text{area} \) included between the curves \( C_1, C_2, \ldots, C \) and \( d\Sigma = \vec{m} \, dl \) along \( C_1, C_2, \ldots, C \). Hence for two dimensional flow of unit thickness Eqn. (2.A-1) becomes

\[ \int \nabla \cdot \vec{A} \, d(\text{area}) = \oint \vec{A} \cdot \vec{m} \, dl \]

Let \( \vec{A} = \vec{\phi} \) where \( \vec{\phi} \) is the velocity potential and \( \vec{\phi} \) is the velocity vector. Then Eqn. (2.A-2) provides
\[ \int \nabla \cdot (\psi \mathbf{v}) \, d(area) = \oint \psi \, \mathbf{v} \cdot d\mathbf{s} \]

or,

\[ \int \left( \psi \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \psi \right) \, d(area) = \oint \psi \, \mathbf{v} \cdot d\mathbf{s} \]

From continuity \( \nabla \cdot \mathbf{v} = 0 \) and from the irrotationality condition \( \mathbf{v} = \nabla \phi \), therefore the last relation becomes

\[ \int \mathbf{v} \cdot d(area) = \oint \psi \, \mathbf{v} \cdot d\mathbf{s} \]

(2.A-3)

The differential of the stream function along any line in the flow field is given by

\[ d\psi = \mathbf{v} \cdot d\mathbf{s} \]

(2.A-4)

whereby Eqn. (2.A-3) becomes

\[ \int \mathbf{v} \cdot d(area) = \oint \psi \, d\mathbf{s} \]

(2.A-5)

Integration by parts provides

\[ \int \mathbf{v} \cdot d(area) = \oint \psi \, d\mathbf{s} - \oint \psi \, d\phi \]

(2.A-6)

The first term on the right hand side is zero because \( \phi \) and \( \psi \) are single valued. Thus

\[ \int \mathbf{v} \cdot d(area) = - \oint \psi \, d\phi \]

Therefore the desired result is obtained, namely,

\[ K_{\text{int}} = -\frac{\beta}{2} \oint \psi \, d\phi \]

(2.A-6)

Eqn. (2.A-6) is applied to the dashed curve in Fig. 2.2b. Start at point A. Then

\[ K_{\text{int}} = -\frac{\beta}{2} \left[ \oint \psi \, d\phi \right] \]

When the cuts are closed

\[ K_{\text{int}} = -\frac{\beta}{2} \left[ \oint \psi \, d\phi \right] \]

On the vortex surfaces \( \psi = \psi_a = \text{const.} \) and \( \oint \phi \, d\phi = 0 \) by definition. Hence
\[ K_{\text{inst}} = \left( \frac{\rho}{2} \left[ -u_a \hat{\tau}^r - u_b \hat{\tau}^r + \lambda n \int_{\Gamma} e^{-s} \hat{\omega} \, ds \right] \right) \]  

(2.3-7)

The velocity component \( u_b \) for the combined flow results from the complex potential which is

\[ \omega (z) = \frac{\hat{\gamma}_b}{2\pi} \ln \frac{z - b'}{z - b'} \]  

(2.3-8)

Then, the derivative of \( \omega (z) \) provides the velocity components \( u_a, u_b \) as follows:

\[ \frac{d\omega}{dz} = (u_a - iu_b) e^{-i\theta} \]  

(2.3-9)

Differentiating Eqn. (2.3-8) and expressing it in the form of Eqn. (2.3-9) gives

\[ u_a = -\frac{b'}{\pi} \cdot \frac{\left( r^a + b'^3 \right) \cos \theta}{\left( r^a \cos \theta - b'^3 \right)^2 + \left( r^a \sin \theta \right)^2} \]  

(2.3-10)

\[ u_b = \frac{b'}{\pi} \cdot \frac{\left( r^b - b'^3 \right) \cos \theta}{\left( r^b \cos \theta - b'^3 \right)^2 + \left( r^b \sin \theta \right)^2} \]  

(2.3-11)

Then \( r \to \infty \), Eqn. (2.3-5) shows that \( \psi \to 0 \); also when \( r \to \infty \), Eqn. (2.3-11) shows that \( ru_b \to \frac{i}{r} \to 0 \).

Hence it is concluded that the last integral in Eqn. (2.3-7) is zero with the result that

\[ K_{\text{inst}} = \left( \frac{\rho}{2} \hat{\tau}^r \right) \]  

(2.3-12)

Appendix 3.B. Evaluation of the Integral \( I (\gamma) \)

Define:

\[ I (\gamma) = \int \frac{(1 - e^{-\gamma})^2}{\gamma} \, d\gamma \]  

(3.5-1)

\[ I (\gamma) = \int \frac{d\gamma}{\gamma} - 2 \int \frac{e^{-\gamma}}{\gamma} \, d\gamma + \int \frac{e^{-2\gamma}}{\gamma} \, d\gamma \]  

(3.5-2)

Let

\[ I_1 = \int \frac{d\gamma}{\gamma} = \ln \gamma \]  

\[ I_2 = \int \frac{e^{-\gamma}}{\gamma} \, d\gamma = e^{-\gamma} - \frac{e^{-\gamma}}{\gamma} \]  

\[ I_3 = \int \frac{e^{-2\gamma}}{\gamma} \, d\gamma = -e^{-2\gamma} \]  

By definition

\[ e\gamma (\gamma) = \int \frac{e^{-\gamma}}{\gamma} \, d\gamma \]  

(3.5-3)

\[ I_0 (\gamma) = \frac{1}{\gamma} \]  

\[ I_1 (\gamma) = \frac{e^{-\gamma}}{\gamma} - e\gamma (\gamma) \]
\[ I_2(\gamma) = \frac{e^{-\gamma}}{\gamma} - 2 \, e^i(2\gamma) \]

\[ I(\gamma) = \frac{1}{\gamma} - 2 \, \frac{e^{-\gamma}}{\gamma} + 2 \, e^i(\gamma) + \frac{e^{-2\gamma}}{\gamma} - 2 \, e^i(2\gamma) \]

\[ I(\gamma) = \frac{\left(1 - e^{-\gamma}\right)^2}{\gamma} + 2 \, e^i(\gamma) - 2 \, e^i(2\gamma) \] \hspace{1cm} (2.8-6)

From Eqn. (35.3) of Spiegel:

\[ e^i(\gamma) = -\gamma - \ln \gamma + \sum_{m=1}^{\infty} \frac{(\gamma \ln \gamma)^m}{m \, m!} \] \hspace{1cm} (for \( \gamma \to 0 \)) \hspace{1cm} (2.8-5)

From Eqn. (35.9) of Spiegel:

\[ e^i(\gamma) = \frac{e^{-\gamma}}{\gamma} \cdot \sum_{m=0}^{\infty} \frac{(\gamma \ln \gamma)^m}{m \, m!} \] \hspace{1cm} (for \( \gamma \to \infty \)) \hspace{1cm} (2.8-6)

From Eqn. (2.8-5)

\[ e^i(\gamma) \bigg|_{\gamma \to 0} = -\gamma - \ln \gamma \bigg|_{\gamma \to 0} \] \hspace{1cm} (2.8-7)

\[ e^i(2\gamma) \bigg|_{\gamma \to 0} = -\gamma - \ln 2 - \ln \gamma \bigg|_{\gamma \to 0} \] \hspace{1cm} (2.8-7)

\[ e^i(3\gamma) \bigg|_{\gamma \to 0} = 0 \] \hspace{1cm} (2.8-8)

From Eqn. (2.8-3) or (2.8-6)

\[ e^i(3\gamma) \bigg|_{\gamma \to 0} = 0 \] \hspace{1cm} (2.8-8)

From the definition (2.8-4)

\[ e^i(\gamma) = \int_{0}^{\gamma} \frac{e^{-\gamma'}}{\gamma} \, d\gamma' = \int_{0}^{\gamma} \frac{e^{-\gamma'}}{\gamma} \, d\gamma' - \int_{0}^{\gamma} \frac{e^{-\gamma'}}{\gamma} \, d\gamma' \]

\[ = e^i(\gamma) \bigg|_{\gamma \to 0} - \int_{0}^{\gamma} \frac{e^{-\gamma'}}{\gamma} \, d\gamma' \]

\[ e^i(2\gamma) = e^i(\gamma) \bigg|_{\gamma \to 0} - \int_{0}^{\gamma} \frac{e^{-2\gamma'}}{\gamma} \, d\gamma' \] \hspace{1cm} (2.8-9)

\[ e^i(2\gamma) = -\gamma - \ln \gamma \bigg|_{\gamma \to 0} - \int_{0}^{\gamma} \frac{e^{-2\gamma'}}{\gamma} \, d\gamma' \] \hspace{1cm} (2.8-10)

\[ e^i(2\gamma) = -\gamma - \ln \gamma \bigg|_{\gamma \to 0} - \int_{0}^{\gamma} \frac{e^{-2\gamma'}}{\gamma} \, d\gamma' \] \hspace{1cm} (2.8-11)

\[ I(\gamma) \bigg|_{\gamma \to 0} = \int_{0}^{\gamma} \frac{(1 - e^{-\gamma})^2}{\gamma} + 2 \, e^i(\gamma) \bigg|_{\gamma \to 0} - 2 \, e^i(2\gamma) \bigg|_{\gamma \to 0} \]

\[ I(\gamma) \bigg|_{\gamma \to 0} = 2 \, \ln 2 \] \hspace{1cm} (2.8-12)
Using Eqs. (2.3-4) and (2.3-5)

\[ I = (1 - e^{\gamma})^2 - 2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

\[ + 2 \gamma^2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

\[ \Rightarrow I = (1 - e^{\gamma})^2 = 2 \gamma^2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

\[ o = 0 \]

\[ \Rightarrow I = 2 \gamma^2 \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

Using Eqs. (2.3-4) and (2.3-6)

\[ I = (1 - e^{\gamma})^2 + 2 (\gamma - e^{-\gamma}) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

\[ - 2 (\gamma - e^{-\gamma}) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

\[ \Rightarrow I = 2 (\gamma - e^{-\gamma}) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \gamma^m \]

\[ \Rightarrow I = 1 \]

3. VORTEX TYPE CONCENTRATORS

Introduction

Two types of wind energy concentrators are currently under study at West Virginia University, namely, an obstruction type and a vortex type. Some preliminary work related to the vortex type concentrators is summarized here. For these devices, the emphasis is not on improving the rotor performance but on creating a region of low pressure around the rotor. The local wind kinetic energy increases in this area of low pressure and the amount of wind power harnessed per square foot of rotor area can be increased five fold. The area of low pressure around the rotor can be created by a nonrotating wind energy concentrator adjacent to the rotor. The area of low pressure around the rotor should be exposed to the incoming wind with a minimum of viscous losses.

None of these nonrotating wind energy concentrators increase the stagnation pressure of the wind. Only the dynamic pressure is increased locally by lowering the static pressure around the rotor.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>radius of maximum rotational velocity in vortex</td>
</tr>
<tr>
<td>A</td>
<td>dimensionless total concentrator plus turbine area</td>
</tr>
<tr>
<td>AR</td>
<td>wing aspect ratio 2b/c</td>
</tr>
<tr>
<td>b</td>
<td>semi wing span</td>
</tr>
<tr>
<td>C</td>
<td>mean aerodynamic wing chord</td>
</tr>
<tr>
<td>CL</td>
<td>average wing lift coefficient</td>
</tr>
<tr>
<td>C_P</td>
<td>pressure coefficient based on free wind speed</td>
</tr>
<tr>
<td>C_Pa</td>
<td>average pressure coefficient around rotor</td>
</tr>
</tbody>
</table>
Discussion

Natural wind energy concentrators, such as tornados, are vortex type rotational flows. The curvature of the flow sets up a radial pressure gradient which results in a low pressure region in the center of the vortex. One can also think of the centrifugal force creating the vacuum in the core of the vortex.

Vertical vortices, like tornados, lack the high axial velocity required for high power extraction. Some power could be extracted if the tornado was kept stationary, and air was ducted through a turbine and exhausted in the low pressure core of the vortex. The inlet of the duct should align itself with the direction of the wind so as to get maximum total pressure at the inlet of the turbine.

Horizontal vortices like those trailing behind the wing tips of an aircraft have a similar low pressure region in the core, but in addition also have a high axial velocity close to that of the relative wind. Such vortices with high axial mass flow rates are much more suitable for power extraction.

The trailing vortex behind an aircraft wing starts with the cross flow set up from the high pressure region below the wing to the low pressure region above it. For straight wings this cross flow occurs around the wing tip but for swept wings the cross flow initiates around the leading edge. The strength of the trailing vorticity equals the maximum value of the bound vorticity. The magnitude of the obtainable low pressure inside the vortex is found to be proportional to the square of the lift coefficient generated by the wing. A high aspect ratio, high lift airfoil wing is therefore capable of generating the strongest trailing vortices. A rapid vortex roll up also enhances the vacuum inside the vortex. Wings with nearly uniform spanwise loading have all their trailing vorticity shed near the wing tip and therefore create the strongest vacuum. In order to obtain both maximum lift coefficient and rapid roll up one needs a straight wing with inverse...
taper and twist and a reasonably high aspect ratio. Delta wings have low aspect ratio and thus a limited lift coefficient. They also have a highly nonuniform spanwise loading and thus slow roll up and limited vacuum in the trailing vorticity.

The wind can also generate a horizontal vortex similar to that of an aircraft wing when a high lift finite span airfoil is placed either horizontally or vertically at the optimum angle of attack to the wind direction. Trailing vortices of this kind are under extensive study; see, for example, Ciffone and Orloff \(^{3.1}\) and Chigier, \(^{3.2}\) see Fig. 3.1. The general goal of these works is to minimize the vortex strength, the present work is applied toward maximizing it.

The rate of roll up of the vortex sheet behind a wing can be computed and expressed as a function of the wing loading; see, e.g., Wilson and Loth. \(^{3.3}\) In the case of an elliptically loaded wing as much as 40% of the vortex sheet is rolled up into a single vortex after one chord length downstream of the trailing edge. For a wing loaded nearly uniformly along its span, almost all of the vorticity will be rolled up at that distance.

The trailing vortex system induces a downwash velocity on the bound vortex in the wing, which results in an induced drag, designated by \(D_i\). Owing to the high stability of the trailing vortex system, the vortex rotational velocities can still be measured hundreds of wing spans downstream. Some excellent far field data for such a vortex system has been presented by Ciffone and Orloff. \(^{3.1}\)

Based on this data, some numerical calculations have been carried out which are presented in Appendix 3.A. The results permit some useful conclusions.

Even in the far field the axial momentum deficit in the vortex is small and accounts for less than 10% of the induced drag. More than 90% of it is in the form of a pressure drag created by the vacuum in the vortex. Centrifugal effects maintain this vacuum up to great distances from the vortex core. The core is in near solid body rotation and is defined by the maximum tangential velocity, \(v_{\text{max}}\), at a radius \(\alpha\). Significant viscous effects extend to about 4.5 times the core radius from the center, and the circulation (Fig. 3.2) at radius \(\alpha\) is only about half that in the irrotational part of the vortex. Only 16% of the induced drag is due to the vacuum level in the viscous region, while 74% of it is due to the vacuum in the near inviscid irrotational part of the vortex. It is interesting to note that inside the irrotational part of the vortex, the pressure drag due to the vacuum equals the local rotational kinetic energy, which can be derived from the Bernoulli equation. However, inside the viscous region of the vortex the rotational kinetic energy is only 63% of the pressure drag due to the vacuum in that region. Therefore, only 84% of the induced drag manifests itself in the form of rotational energy of the wing tip vortex. Similar values can be obtained by using the analytical models of Newman \(^{3.4}\) or Batchelor. \(^{3.5}\)

The wind turbine required to harness the rotational vortex kinetic energy must satisfy the following design criteria. The amount of rotational kinetic energy entering the turbine depends on its diameter \(d\), and has been computed as a fraction \(f\) of \(D_i\) as shown in Fig. 3.3:

\[
f = \frac{\frac{\pi}{2} \rho u^2 2\pi r \, dr}{D_i} = \frac{\text{Vortex rotational KE}}{\text{Semi-span induced drag}}.
\]
In order to indicate the degree of concentration of wind energy obtained, a power concentration ratio $R$ should be defined. It is defined here as the reduction in rotor area permitted due to the presence of the concentrator, assuming no change in power output.

To get an idea of the obtainable order of magnitude of $R$, assume the flow to be inviscid and $\frac{V_{\text{avg}}}{V_{\text{avg}}}$ as the average tangential velocity at the rotor inlet. Approximate the axial velocity by $V_a$.

Then the average pressure coefficient is

$$C_p = \frac{1}{2} \left( \frac{V_{\text{avg}}}{V_{\text{avg}}}^2 + \frac{V_a}{V_{\text{avg}}}^2 \right).$$

At the rotor inlet the wind power density available for harnessing is given by the kinetic energy times the mass flow rate per unit area; this is

$$\rho V_a \left( \frac{V_{\text{avg}}}{Z} \right).$$

Note that for the vortex type concentrator all the rotational kinetic energy ($\frac{V_{\text{avg}}}{2}^2$) can be harnessed without reducing the mass flow rate through the rotor. Without the wind energy concentrator the power density is

$$\rho V_a (\frac{V_{\text{avg}}}{Z}).$$

Thus the power concentration ratio, defined as the ratio of power densities, is

$$R = \frac{\frac{V_{\text{avg}}}{Z} + \frac{V_a}{Z}}{\frac{V_{\text{avg}}}{Z}} \approx 1 - C_p.$$

Viscous effects limit the magnitude of $C_p$ in the vortex.

To compute the power concentration ratio $R$ as a function of the increase in area of the wind machine due to presence of the concentrator, it is convenient to define $Z$ as the ratio of the induced drag to the free wind kinetic energy in a stream tube of cross-sectional area equal to that of the turbine inlet:

$$Z = \frac{D_l}{\frac{1}{2} \rho V_{\text{avg}}^2 \frac{d}{2}} = \frac{1}{2} \rho V_{\text{avg}}^2 \frac{d}{2} \frac{C_l b c}{\rho V_{\text{avg}}^2 \frac{d}{2}} = \frac{C_l R}{\left( \frac{d}{Z} \right)^{\frac{1}{2}}}.$$  

Here $\varepsilon$ is the spanwise loading efficiency which should be as low as possible. Then the power concentration ratio $R$ can be computed as

$$R = \frac{\text{axial wind K.E.} + \text{rotational K.E.}}{\text{free wind K.E.}} = 1 + f \cdot Z.$$

The corresponding total area ratio of wind concentrator plus turbine to that of the turbine is given by

$$A = \frac{b c + \frac{1}{2} \frac{d}{2} \frac{d}{2}}{\frac{1}{2} \pi (\frac{d}{2})^2} = 1 + 2 R \frac{A}{\pi (\frac{d}{2})^2}.$$

Both $R$ and $A$ have been computed for a uniformly loaded high lift semi-span wing with the turbine placed at a distance of one mean aerodynamic chord downstream of the trailing edge and concentric with the vortex centerline. The vortex core radius $\frac{d}{2}$ at that location is found from experiments to be 0.03. Using this value one can proceed to compute $f$, $Z$, and $R$ as a function of $C_l^2/\varepsilon$ and turbine diameter $\frac{d}{2}$. The results are plotted in Fig. 3.4 for various values of $C_l^2/\varepsilon$. To obtain high values of $R$ one
requires high lift coefficients, which, in turn, calls for high aspect ratio. Unfortunately, as the aspect ratio increases, the corresponding area ratio \( A \) increases. Only an economic trade-off study can optimize the design. The maximum value of \( R \) is found when \( d = 3a \). The corresponding value of \( R \) is

\[
R_{\text{max}} = 1 + \frac{3}{2} \left( \frac{V_{\text{max}}}{V_{\text{c}}^2} \right)^2 \approx 1 + 0.2 + \frac{C_L}{c} \approx 1 + 0.5 \frac{C_L}{c}.
\]

A shroud around the turbine is essential in order to stabilize the vortex and to maintain the low pressure at the turbine outlet even after the rotational energy has been extracted. The presence of the turbine will have an adverse effect on the obtainable maximum lift coefficient of the wing. Using lifting line theory it is estimated that the presence of the turbine at a distance of one chord length downstream reduces \( \frac{C_L}{c} \) by 13%. Fig. 3.5 shows a model of a high lift wing vortex type concentrator. The airfoil chosen was a modified Liebeck airfoil as is described by Smith.\(^{3,6} \) The vertical arrangement of the high lift airfoil is structurally simpler. The wing has a height equal to two and one-half times its chord length and an aspect ratio of five. It pivots around a vertical axis to maintain an optimum angle of attack to the oncoming wind.

The original design which was considered had a ducted turbine inside the wing which exhausted inside the core of the trailing vortex. However, a theoretical analysis showed that the associated viscous duct losses did not justify such an arrangement. The subsequent design, shown in Fig. 3.5, has a turbine on the axis of the trailing vortex about one chord length downstream of the trailing edge. The turbine is shrouded to stabilize the vortex flow through the turbine and to maintain the low pressure around the turbine. This vortex type concentrator is capable of concentration ratios up to five. The airfoil and turbine design, however, is much more critical in this type of concentrator than in other types.

Most experimental vortex data available in the literature which apply to aircraft wings show a maximum rotational velocity in the vortex of about 60% of the free stream value. This rotational velocity increases with the wing lift coefficient. Therefore, regular aircraft wings with lift coefficients below 2.0 make poor wind vortex energy concentrators. For this purpose an improved high lift configuration is required as is used on some STOL aircraft. The associated high drag makes such wings unsuitable for most aircraft applications but the high drag does not interfere with its usefulness as a vortex energy concentrator. One promising high lift airfoil is the Liebeck type which has been extensively tested in the WVU wind-tunnel for the generation of a strong wing tip vortex. One-tenth scale models were used in the wind tunnel. Even at tunnel speeds as low as 1.34 m/s (3 mph) and low angles of attack, the local rotational velocity near the core and approximately one chord length downstream of the trailing edge has been found to exceed the free stream velocity by about 25%. At large lift coefficients rotational velocities up to two times the free stream value are possible. Fig. 3.6 shows top and side views of a vortex generated
by a flat plate at an angle of attack in the WVU smoke tunnel. Fig. 3.7 indicates what effect a downstream obstruction such as screens forming a high friction duct would have on the vortex characteristics entering the duct. Studies have shown that this is a typical low speed effect. At higher free stream velocities the effect becomes of little significance. The dots superposed on the helium soap bubble streaks on these pictures are used for accurate determination of axial and rotational velocities in the flow field. Fig. 3.8 shows the vortex in the free stream velocity as in the case of Fig. 3.7 but at a higher angle of attack. Notice the high rate of spin in the center of the vortex.

The wind tunnel test data are being used to design an appropriate wind turbine to extract most of the rotational vortex energy and some axial flow kinetic energy. The design rpm will be optimized to obtain maximum power output. A computer program has been developed to generate the required turbine blade profiles for any wing tip vortex type wind turbine. The input data constitute the turbine to vortex core diameter ratio and the ratio between the maximum angular velocity and the axial flow velocity. The magnitude of the vortex core and the maximum angular velocity depends on the wing lift coefficient and the turbine location downstream of the wing. Presently a 3.81 m (12.5 ft) semispan and 1.52 m (5 ft) chord high lift wing is under construction. The tip of the wing will be 5.5 m (18 ft) above ground level. A 1.22 m (4 ft) diameter turbine will be located 1.52 m (5 ft) downstream of the trailing edge. The concentrator-turbine assembly is designed to be mounted on a trailer with two 5.5 m (18 ft) long fold-out type stabilizing legs. This arrangement provides high mobility to different test sites. A model of the trailer wing assembly is shown on Fig. 3.9.

Conclusions

The theoretical studies of vortex wind energy concentrators undertaken so far have indicated that this concept holds considerable promise to allow significant energy concentrations. In order to more fully evaluate this concept extensive wind tunnel testing has been undertaken at WVU that is the basis for the design of a full-scale mobile wing-turbine assembly. While awaiting the completion of construction and test data, work is underway to develop a program that would theoretically compute the wing planform, twist and inverse taper required to achieve near uniform wing spanwise loading.

References


3.3. Wilson, J. D. and Loth, J. L., "Real Time Development of the Wake of a Finite Wing," TR-23, Department of Aerospace Engineering, West Virginia University, February, 1974.


Fig. 3.1. Top: Vertical, semispans, straight, high lift wing, with a single rolled-up trailing vortex. Bottom: Horizontal, full span, swept leading edge high lift wing, with two counter-rotating trailing vortices.

Fig. 3.2. Top: Typical tangential velocities in a rolled up trailing vortex. Bottom: Typical corresponding pressure level inside the vortex.
Fig. 3.3. Distribution of vortex rotational kinetic energy as a function of cross-sectional area and nondimensionalized by the induced drag.

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Fig. 3.4. "Vortex Type" wind energy concentrator performance, showing energy concentration ratio \( R \) and area ratio \( A \) as a function of rotor size.

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Fig. 3.5. "Vortex Type" wind energy concentrator model, using a high lift Liebeck type airfoil and shrouded rotor (power output estimate is preliminary).

Fig. 3.6. Two views of a vortex due to a flat plate at an angle of attack generated in the WVU smoke tunnel.
Fig. 3.5. "Vortex Type" wind energy concentrator model, using a high lift Liebeck type airfoil and shrouded rotor (power output estimate is preliminary).

Fig. 3.6. Two views of a vortex due to a flat plate at an angle of attack generated in the WVU smoke tunnel.
Fig. 3.7. Helium soap bubble pictures of a vortex due to a Liebeck type wing showing the effect of placing a high friction duct in the vortex at low Reynolds number.
Fig. 3.8. Vortex due to a Liebeck type wing at a higher angle of attack.
Fig. 3.9. A model of the trailer wing assembly.
Appendix 3.A.

Giffone and Orloff found that in the range of their investigation the circulation within the vortex core remained constant and the following relation is true:

$$\frac{V_{\text{max}}}{V_a} = \frac{a}{2b} = 0.0034$$

where $a$ is the core radius. They give the actual circulation as

$$\frac{V_a}{2b V_a} = 0.046$$

for $C_r = 0.7$ and their wing span of 61 cm. If the flow were irrotational the tangential velocity at the edge of the core, for the same circulation and core radius as the experiment, would have been

$$V_{\text{t}}^i = \frac{r_a}{2\pi a}$$

The superscript $i$ stands for irrotational and $e$ for experimental values. Hence $\frac{V_{\text{t}}^i}{V_a} = 2.15$, meaning that the irrotational tangential velocity at the core edge is 2.15 times larger than the actual tangential velocity.

Outside the core, the irrotational tangential velocity is given by

$$V_{\text{t}}^i = \frac{r_a}{2\pi a} \cdot \frac{a}{r}$$

and the experimental value can be approximated by

$$V_{\text{t}}^e \approx V_{\text{t}}^i \left( \frac{r}{r_a} \right)$$

At a certain distance $r_a$ from the origin they merge. This can be identified as the limit of viscous effects. Equating the two expressions gives

$$r_a = 4.62 a$$

Beyond this radius the flow can be considered irrotational. It can be easily shown that the tangential velocity at this limiting point is given by

$$V_{\text{t}}^e = V_{\text{t}}^i = V_{\text{t}}^e = 0.47 V_{\text{t}}^e$$

Since for $r > r_a$ the flow can be considered irrotational, the vortex wake pressure drag can be easily computed as

$$\int_A (p - p_0) dA = \int_0^{r_a} \frac{1}{2} \rho \left( V_{\text{t}}^e + V_{\text{t}}^i \right)^2 2\pi r dr$$

The upper limit of integration $\phi$ is the equivalent limit of integration when using a single vortex, while infinity is the proper limit for two counter-rotating vortices. The value of $\phi$ can be estimated using some simplified assumptions as follows:

It is assumed viscous effects are present only for $rC_a$. Then the induced drag, $D_i$, can be written as

$$D_i = \frac{\phi}{r} \int_0^{r_a} \left( p_{\text{t}} - p_0 \right) dA = \int_0^{r_a} \frac{\alpha}{\phi} \int_0^{\alpha} (p_{\text{t}} - p_0) dA$$

Since for $r > r_a$, the flow is irrotational one can write
The pressure relation for the core is obtained from the momentum equation

\[ \frac{dP}{dr} = \rho \frac{\alpha^2}{r} \]

Integrated, this gives, when \( U_{\text{max}} \frac{\alpha}{r} \) is substituted for \( U \),

\[ P_r - P_\alpha = \rho \frac{U_{\text{max}}}{2} \left( \frac{\alpha^2}{r^2} - 1 \right) \]

Then

\[ \int_{r=0}^{\alpha} (P_r - P_\alpha) \, dA = \int_{\phi=0}^{\alpha} \left[ P_r - P_\alpha + \rho \frac{U_{\text{max}}}{2} \left( 1 - \frac{\alpha^2}{r^2} \right) \right] \pi r \, dr \]

\[ = \frac{3}{4} \rho \frac{U_{\text{max}}}{2} \pi \alpha^2 \]

Hence,

\[ \frac{D_i}{2} = \rho \pi \alpha^2 U_{\text{max}}^2 \left( \frac{3}{4} + \ln \frac{\phi}{\alpha} \right) \]

But \( \frac{D_i}{2} \) is also given by

\[ \frac{D_i}{2} = C_{D_i} b \frac{1}{2} \rho V_a^2 = C_{D_i} \frac{\rho}{R} (b V_m)^2 \]

or

\[ \frac{D_i}{2} = C_{D_i} \frac{1}{R} \left( \rho \alpha^2 \pi U_{\text{max}}^2 \right) \frac{\pi}{(0.046)^2} \]

Equating the above two expressions for \( \frac{D_i}{2} \) one obtains

\[ \frac{3}{4} + \ln \frac{\phi}{\alpha} = 8.36 \]

or

\[ \frac{\phi}{\alpha} \approx 2000 \]

where the value of \( C_{D_i} \) is taken as 0.03 at \( C_r = 0.7 \) and \( b \) is the half span. The expression for \( \frac{D_i}{2} \) can further be written as

\[ \frac{D_i}{2} = C_{D_i} \frac{\rho}{R} \frac{\alpha^2 U_{\text{max}}^2}{(2 \times 0.0034)^2} = \left( \frac{\pi \alpha^2 \rho}{R} U_{\text{max}}^2 \right) \frac{C_{D_i}}{\pi R (2 \times 0.0034)^2} \]

or

\[ \frac{D_i}{2} = 38.74 \left( \frac{\pi \alpha^2 \rho}{R} U_{\text{max}}^2 \right) \cdot (3.14 - 1) \]

Now going back to the vortex pressure drag for \( r > r_L \) gives

\[ \int_{r_L}^{\phi} \frac{1}{2} \rho 2 \pi r \left( 2 \pi r \right)^2 \frac{\alpha^2}{r^2} U_{\text{max}}^2 \, dr \]

\[ = \pi \rho \frac{\alpha^2}{r^2} U_{\text{max}}^2 \left( 4.173 \ln \frac{\phi}{\alpha} \right) \]

where

\[ U_0 = U_{\theta_L} \frac{r}{R} = 0.47 U_{\text{max}} \frac{r}{R} = 2.171 U_{\text{max}} \frac{\alpha}{R} \]

With \( \frac{\phi}{\alpha} = 2000 \) this portion of the drag becomes 28.56 \( \left( \frac{\pi \alpha^2 \rho}{R} U_{\text{max}}^2 \right) \),

which is 73.7% of the total \( \frac{D_i}{2} \) when 38.74 in Eq. (3.1 - 1) is considered 100%. One can write, in general, for the induced drag

\[ \frac{D_i}{2} = C \left[ \pi \alpha^2 \rho U_{\text{max}}^2 \right] \]
where \[ C = C_0 + C_1 + C_2 + C_3 + C_4 \]

The individual contributions are defined as follows:
- \( C_0 \): due to irrotational vortex pressure drag for \( r < r_1 \)
- \( C_1 \): due to vortex wake pressure drag for \( r_1 < r < r_2 \)
- \( C_2 \): due to vortex core pressure drag for \( r_2 < r < r_3 \)
- \( C_3 \): due to vortex core axial momentum deficit in the region \( 0 < r < r_3 \)
- \( C_4 \): due to vortex axial momentum deficit in the region \( r < r_3 \)

Among these, \( C_0 \) has already been calculated and was found to be 73.7% of \( C \). It can be easily shown that \( C_1 \) is 5.07 or 13.1% of \( C \) and \( C_2 = 1.14 \), which is only 2.9% of \( C \). One should note that on the center line of the vortex

\[ \rho_u - p_0 = \frac{1}{2} \rho \gamma^2 \max (2.787) \]

Now

\[ C_3 + C_4 = C - [C_0 + C_1 + C_2] = 3.97 \]

or, \( C_3 + C_4 \) should account for 10.2% of \( C \). \( C_3 \) and \( C_4 \) can be obtained by graphical integration of Fig. 14 of Ciffone and Orloff\(^{4.1}\) with the value of \( \frac{\alpha}{2B} \) taken from Fig. 10 of the same paper as 0.025. An estimate of \( C_3 \) and \( C_4 \) indicate that \( C_3 + C_4 \) indeed account for approximately 10% of the value of \( C \). Fig. 3.2 shows some of the results obtained in this section in graphical form.

4. VERTICAL AXIS WIND MACHINE ANALYSIS USING STRIP THEORY

A parametric study of the performance of vertical axis wind machines with variable pitch blades was carried out using two dimensional quasi-steady strip theory. The study included the use of both conventional and circulation controlled airfoils\(^{4.1}\) for blades. For both types of blades, operation with the blades continuously adjusted to the optimum angle of attack and operation with two fixed settings were compared.

The basic principle of a circulation controlled airfoil, shown in Fig. 4.1, is the increase in circulation and lift by a jet of air blown out of a slot near the rear of a body with a rounded trailing edge. The jet flows around the trailing edge because of the Coanda effect. In the present application a symmetrical airfoil with slots on both the upper and lower surface is used. A valve directs the jet through the proper slot to increase the lift in the desired direction. A circulation controlled airfoil can give much higher lift coefficients than a conventional airfoil and the major objective of this phase of the project is to explore its potential use in wind machines. In particular it is hoped that circulation control will permit lower tip speeds and thereby reduce the structural problems. Fig. 4.2 shows a conceptual model of a large circulation controlled panemone.

Figs. 4.3a and 4.3b show the aerodynamic data reported by Kind\(^{4.2}\) for a typical configuration. The lift coefficient, \( C_L \), and drag coefficient, \( C_D \), are plotted against angle of attack for different blowing coefficients (defined below). However, it should be noted that the performance of a circulation controlled airfoil depends on
two basic parameters such as the blowing coefficient and slot size. Unfortunately, a more extensive set of experimental data by Englar was not available in time to use in these calculations.

The basic vector force diagram is shown in Fig. 4.4. At a position in the cycle, $\theta$, the torque per unit span produced by the blade is

$$T(\theta) = \frac{1}{2} \rho c R V_{rel}^2 \left[ C_L \cos \alpha_i + C_D \sin \alpha_i \right],$$

where $\alpha_i$ is the orientation of the radius to the blade, $\theta$ is the angle between the radius and the relative wind, $C_L$ and $C_D$ are the lift and drag coefficients of the blade, $\rho$ is the air density, $c$ is the blade chord, and $R$ is the radius of the wind turbine.

If the effect of the blades on the flow field is ignored, the relative velocity $V_{rel}$ is related to the wind velocity $V_W$ by

$$V_{rel} = \frac{V_W \cos \phi}{\cos \alpha_i},$$

Wilson and Lissaman suggest multiplying the wind velocity by a blockage factor $(1-a)$ in which

$$a = \frac{\tau/c}{2R} \frac{\omega R}{V_W},$$

where $\omega$ is the angular velocity, and $\tau$ is the number of blades.

The dimensionless grouping $\frac{\tau/c}{2R}$ is the solidity of the machine and $\frac{\omega R}{V_W}$ is the tip speed ratio. This blockage factor changes the relative velocity both directly through the $V_W$ term and indirectly through $\alpha_i$, which is determined by the relation

$$\alpha_i = \tan^{-1} \left( \frac{\sin \phi - \omega R/V_W}{\cos \phi} \right).$$

In the present calculation the blockage factor was taken as $a$, which agrees with the Wilson and Lissaman expression for small $\alpha$ but does not become negative for large $\alpha$.

Multiplying by the angular velocity and dividing by the wind kinetic energy per unit projected area, $\frac{1}{2} \rho V_W^2 J$, gives the expression

$$P_{r}(\theta) = \frac{c}{2R} \frac{\omega R}{V_W} \left( \frac{V_W}{c \cos \alpha_i} \right)^2 \left[ C_L \cos \alpha_i + C_D \sin \alpha_i \right],$$

for the dimensionless torque power. For a circulation controlled airfoil the blowing power must be subtracted from the torque power.

The consistent dimensionless expression for the blowing power is

$$P_{b}(\theta) = \frac{\tau/c}{2R} \frac{\omega R}{V_W} \left( \frac{V_W}{c \cos \alpha_i} \right)^3 \left[ C_L \cos \alpha_i + C_D \sin \alpha_i \right],$$

where $t$ is the size of the slot and $\gamma_t$ the blowing system efficiency. The overall power coefficient is given by

$$C_p = \frac{m}{N} \sum_{i=1}^{N} \left[ P_{r}(\theta_i) - P_{b}(\theta_i) \right],$$

where $N$ is the number of divisions used in the calculation and $\alpha$ the
number of blades. Within the assumptions of the present analysis $C_f$ depends on the solidity $\frac{C}{2R}$ rather than on the number of blades and the relative size of the blades independently, thus reducing the amount of calculation required somewhat. However, the calculation of the aerodynamic loading which requires adding the contribution for each blade must take the number of blades into account.

It should be noted that the combination $C_L \cos \alpha + C_D \sin \alpha$, in the expression for the power is the difference between two almost equal terms. While $C_D$ is smaller than $C_L$ over most of the cycle $\sin \alpha$, is close to 1 while $\cos \alpha$, is small. The resulting power is therefore quite sensitive to the assumptions made about the blockage factor correction, the aerodynamic coefficients, and the stall angle of the airfoil. Because of this the results presented below are not a reliable determination of the performance to be expected in practice, but it is believed that they present a correct qualitative assessment of the effect of the design parameters on the performance.

For the present calculations, the $C_L$ and $C_D$ values for the circulation controlled airfoil were taken from a report by Kind (Figs. 4.3a and 4.3b). An aspect ratio $(AR)$ correction $C_L = \frac{\alpha}{\pi AR}$ was added to the drag coefficient for both this and the conventional airfoil. The values for the conventional airfoils were based on the data reported for a NACA 0012 airfoil by Abbott and Von Doenhoff, but a higher value was used for the parasite drag. There is some evidence that such an increase is required in unsteady systems.

In the calculation in which the blade orientation was continuously adjusted for maximum torque, it was found that except close to $\theta = 90^o$ and $270^o$, where the sign (direction) of the required lift changed, the blades had to be set at the maximum allowed angle of attack. This is the reason for the dependence of the overall efficiency on the stall angle. In the early calculations of circulation controlled airfoils a check was made at each station whether the gain exceeded the required blowing power. It was found that there was a small gain if the blowing was omitted over small parts of the cycle, but it was felt that this did not justify the additional complexity and the results reported here are for blowing over the entire cycle.

In the second operating mode the airfoil is flipped to a positive angle in the first and fourth quadrants and to the corresponding negative angle in the second and third quadrants in order to increase the desired lift component. When there is no flip the system corresponds to the Darrieus rotor 4.8, 4.9, 4.10. In this mode of operation a lower limit is set on the operating tip speed ratio $\frac{\omega R}{V_{in}}$ by the appearance of stall.

Fig. 4.5 is a set of universal curves (for different solidities) showing the relation between stall angle-flip angle, and tip speed ratio. The relation is independent of the aerodynamic properties of the airfoil but does depend on the blockage factor. To illustrate its use consider a machine with four blades, each having a chord of 1/16 the machine radius; the machine is to operate with a fixed blade setting and the stall angle is $12^o$. What is the minimum tip speed ratio? Since $\frac{\alpha}{2R} = \frac{\delta}{\theta}$, the middle curve is used. Entering at $\delta = 12^o$ gives a tip speed ratio of 3.3. Increasing the stall angle to $15^o$ decreases the minimum tip speed ratio to 1.9. On the other hand if the
blade is flipped 40° the curve is entered at \( \delta = 12° -4° = 8° \) and the minimum tip speed ratio increases to 4.4. In flip mode operation the highest efficiency is obtained at or close to the minimum tip speed ratio. Furthermore, a low tip speed ratio is desirable because the centrifugal force on the blade is given by

\[
F_c = W_e \left( \frac{\omega R}{V_{\infty}} \right)^2 \frac{V_{\infty}^2}{2gR},
\]

where \( W_e \) is the weight of the blade. The centrifugal loading is much larger than the aerodynamic forces on the blade and will therefore be the principal consideration in the structural design.

Figs. 4.6a, 4.6b, and 4.6c show the operation of the machine with conventional blades in both modes for three solidities. The aspect ratio was taken as 20 in these calculations and the stall angle as 11°. A comparison of the three curves shows that the tip speed ratio for optimum efficiency increases as the solidity decreases. The small variation in peak efficiency with solidity is probably not significant; much larger changes are obtained when slightly different values are used for stall angles, parasite drag coefficients, and blockage factor. The difference in efficiency between operating with the optimum angle of attack and with two position (flip) orientation is appreciable. It should be noted that the efficiency for 0° operation appears reasonably consistent with those reported for Darrieus rotors when the effect of mechanical losses and the end effect of the Darrieus configuration are taken into consideration. Any thin airfoil would have about the same lift coefficient but there would be small differences in the drag coefficient and stall angles that could affect the aerodynamic efficiency significantly.

Figs. 4.7a, 4.7b, and 4.7c show the corresponding results for the circulation controlled airfoil. In the calculation it is the value of the jet velocity which is fixed and the blowing coefficient, \( \Delta = \frac{2V_j}{V_{\infty}} \), varies because of the change in relative velocity as the blade rotates. The program was set up to interpolate the aerodynamic data of Fig. 4.3. The stall angle was taken as 15° because this was the limit of the given data. For negative angles of attack (2nd and 3rd quadrant) the drag is the same but the lift changes sign (direction). However, a circulation controlled airfoil does not exhibit a sharp stall; the lift simply becomes relatively constant for large angles of attack. Also, in some parts of the cycle for low tip speed ratios, the relative velocity becomes small and the blowing coefficient exceeded the range for which data was available. The conservative assumption was made of a small extrapolation to \( V_{\infty} = 1.12 \), but the power was computed on the basis of the larger blowing coefficient. A blowing system efficiency of .7 was used in the blowing power calculation.

The results presented are for the optimum value of \( V_j \) for each solidity. The maximum efficiency with the angle of attack continuously optimized appears somewhat lower than for the conventional blades, but it is believed that this difference is not significant in view of the limitations of the calculations. The important results are that the optimum efficiency occurs at a lower tip speed ratio, and that the difference between operation with continuous blade angle optimization and with a fixed blade orientation, but with the blowing direction changed at 90° and 270°, is much smaller than for the conventional

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blades. The lower tip speed ratio appears to be an important potential advantage of a wind machine with circulation controlled blades because of the resulting lower centrifugal stress.

It is planned to carry out further calculations for the machine with circulation controlled blades using the data published by Englar\textsuperscript{4,3} which covers a wider range of operating conditions than Kind's data.\textsuperscript{4,2} In particular the present calculations only covered one value of $\frac{c}{T}$, and this parameter is quite important as it affects the blowing power and therefore the efficiency appreciably. When the complementary theoretical and experimental studies now underway provide improved estimates of the blockage factor and aerodynamic coefficients, the calculation will be modified accordingly.

The numerical calculations to be performed as described in section 5 of this report, and measurements of the aerodynamic forces during operation of the proposed vertical axis prototype machine (see section 6), should enable this calculation procedure to be developed to the stage where it will give reliable predictions of the performance of machines with other blades and operating conditions.

Very little experimental work on vertical axis machines with adjustable blades has been reported. One early investigator \textsuperscript{4,11} claimed an efficiency of over 40\% for a small prototype, which is competitive with the best propeller type, and it is unlikely that his limited studies reached an optimum configuration.

It should be noted that vertical axis machines are not in principle subject to the Betz limit of $C_p \leq 0.593$ derived for horizontal axis machines by simple momentum consideration, since, for a vertical axis machine the actual capture area may exceed the projected area. This is shown in the calculations of Larsen\textsuperscript{4,12} who obtained efficiencies slightly above the Betz limit using a lifting line theory. While the $C_p$ values shown in Figs. 4.6 and 4.7 are below the Betz limit, small changes in the assumed aerodynamic coefficients, stall angle or blockage factor can increase $C_p$ substantially, bringing it above the Betz limit. It is therefore important to carry out further theoretical and experimental studies to determine the actual performance limits.

References


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Fig. 4.1. Circulation controlled airfoil cross section.

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Fig. 4.2. Circulation controlled panemone model.

Fig. 4.3. Kind's experimental data for a symmetrical circulation controlled airfoil (from Ref. 4.2).
Fig. 4.4. Panemone vector diagram.

Fig. 4.5. Universal stall limits for panemone with two blade position operation.
Fig. 4.6a. Performance with conventional airfoil blades: nc/2R = 1/4.

Fig. 4.6b. Performance with conventional airfoil blades: nc/2R = 1/8.
Fig. 4.6c. Panemone performance with conventional airfoil blades; nc/2R = 1/12.

Fig. 4.7a. Panemone performance with circulation controlled blades; nc/2R = 1/4.
Fig. 4.7b. Panemone performance with circulation controlled blades: nfc/R = 1/4.

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5. NUMERICAL SOLUTION FOR THE UNSTEADY LIFTING CHARACTERISTICS OF VARIABLE PITCH CROSS-FLOW WIND TURBINES

Abstract
A numerical procedure is developed for determining the unsteady lift and moment characteristics of cross-flow wind turbines. The method employs a force-free wake model, accounts for wake/blade interaction, reflects transient aerodynamics and accommodates time varying winds. Introduction of experimental drag data permits calculation of energy extraction capacity for such devices. Typical results are presented for a straight bladed Darrieus turbine and compared to those given by strip theory. Extension of the method to multi-bladed variable pitch turbines is a part of the continuing research. The developed model will aid the systematic investigation of vertical axis wind turbines for cost effective energy conversion.

Nomenclature
Roman and Greek Letters
\[ \begin{align*}
\text{c} & \quad \text{turbine blade chord; m (ft)} \\
D & \quad \text{drag coefficient} \\
C_L & \quad \text{lift coefficient} \\
C_m & \quad \text{moment coefficient} \\
C_p & \quad \text{power coefficient, } C_p = \frac{E}{\rho V^2 N [\text{Eqn. (S.A-27)}]} \\
D & \quad \text{drag, } D = \frac{1}{2} \rho V^2 c; \text{ N (lb_f)} \\
F_n & \quad \text{aerodynamic force normal to chord; N (lb_f)} \\
L & \quad \text{lift, } L = \frac{1}{2} \rho V^2 c; \text{ N (lb_f)} \\
M & \quad \text{moment, } M = \frac{1}{2} \rho V^2 c^2; \text{ N-m (lb_f-ft)} \\
p & \quad \text{fluid static pressure; N/m}^2 (\text{lb}_f/\text{ft}^2) \\
q & \quad \text{total fluid velocity relative to the inertial frame; m/s (ft/sec)} \\
r & \quad \text{number of blade revolutions} \\
R & \quad \text{turbine radius; m (ft)} \\
s & \quad \text{cartesian coordinates parallel/normal to blade chord} \\
\mathbf{s}_n & \quad \text{respectively, Fig. 5.A.4; m (ft)} \\
t & \quad \text{time measured in non-inertial frame; s} \\
u & \quad \text{fluid perturbation velocity relative to the inertial frame; m/s (ft/sec)} \\
U & \quad \text{total fluid velocity relative to the non-inertial frame; m/s (ft/sec)} \\
V_B & \quad \text{rectilinear blade velocity, } V_B = \omega R; \text{ m/s (ft/sec)} \\
V_R & \quad \text{relative inflow velocity, Fig. 5.A.1; m/s (ft/sec)} \\
V & \quad \text{freestream wind velocity; m/s (ft/sec)} \\
x, y, z & \quad \text{cartesian coordinates of non-inertial frame, Fig. 5.A.1; m (ft)} \\
\alpha & \quad \text{blade angle of attack with respect to relative inflow velocity, Fig. 5.A.1; }^\circ \\
\alpha_B & \quad \text{blade pitch angle measured from turbine radius vector, Fig. 5.A.1; }^\circ \\
\dot{\alpha} & \quad \text{time rate of change of blade angle of attack, } \frac{d\alpha}{dt} ; \gamma_t \\
\beta & \quad \text{angle between } \mathbf{s} \text{ and } x \text{ axes, Fig. 5.A.1; }^\circ \\
\gamma & \quad \text{strength of discrete bound vortex; m}^2/\text{s (ft}^2/\text{sec)}
\end{align*} \]
\[ y' \]

strength of vortex sheet at a point on the camber line;
m/s (ft/sec)

\[ \Gamma \]

net circulation (airfoil and wake); \( \frac{m^2}{s} \) (ft\(^2\)/sec)

\[ \Gamma_0 \]

bound (airfoil) circulation; \( \frac{m^2}{s} \) (ft\(^2\)/sec)

\[ \Gamma_0 \]

wake circulation; \( \frac{m^2}{s} \) (ft\(^2\)/sec)

\[ \epsilon \]

inverse tip speed ratio, \( \epsilon = \frac{\omega R}{\omega} \)

\[ \xi \]

magnitude of position vector in x-y frame, Fig. 5.A.2; m (ft)

\[ \theta \]

angle between freestream wind and radius vector, Fig. 5.A.1,
\[ \theta = \omega t \]

\[ \xi, \eta \]

cartesian coordinates of inertial frame normal/parallel to
the free stream wind respectively, Fig. 5.A.1; m (ft)

\[ \rho \]

fluid mass density assumed everywhere equal to that of
freestream in the far field, \( \rho = \rho_0 \); ks/m\(^3\) (lb/sec\(^2\)/ft\(^4\))

\[ \phi \]

velocity potential; \( \frac{m^2}{s} \) (ft\(^2\)/sec)

\[ \omega \]

turbine rotational speed; rad/s

\[ \Omega \]

angular velocity of (x,y) axes with respect to (\( \xi, \eta \))
axes, \( \Omega = \frac{d\phi}{dt} \); rad/s

Subscripts and Superscripts

b

discrete bound vortex on camber line

c
"control" point on camber line where flow tangency
boundary condition is satisfied

\( f \)

free (shed) vortices in the wake

\( L, U \)

below and above a point on the camber line respectively
(lower and upper)

\( x, y \)

x and y vector components respectively

\( \xi, \eta \)
\( \xi \) and \( \eta \) vector components respectively

(\( \cdot \))

vector quantity

(\( \hat{\cdot} \))

unit vector

Introduction

Cross-flow wind turbines consist of several high aspect ratio
blades, usually of airfoil shape and uniform cross section. These
blades, placed at regular intervals along a disc of radius \( R \), may
be terminated at their spanwise extremities by an end plate; their
pitch about a spanwise axis may be fixed or modulated. Wind
induced aerodynamic forces produce a torque which causes the blades
to rotate in a circular orbit about a central axis. The descriptive
term "cross-flow" is used to indicate that the wind is normal to
the axis of rotation. Although emphasis here is on vertical axis
cross-flow turbines, results of the analysis apply equally to
horizontal axis cross-flow machines.

Interest in innovative concepts relates primarily to their
potential for energy conversion at lower cost than conventional
systems. Vertical axis cross-flow machines may simultaneously
achieve increased wind energy conversion efficiency and reduced energy
costs. There is general agreement that axial-flow turbines without
augmenting devices can extract no more than 40-45% of the available
wind energy in the blade swept area. It is not known, however,
what analogous limit may apply to cross-flow turbines. A popular
hypothesis supported in part by preliminary results of this report,
inference that cross-flow turbines entrain air outside the projected frontal area of the machine thus increasing energy extraction.

Optimum power extraction requires alignment of conventional systems with the incoming wind. Normally accomplished by mechanical means, the requirement adds considerable structural weight, mechanical complexity and cost. Vertical axis machines do not suffer similarly, since the desired response to changing wind direction is a simple adjunct to the existing blade pitch mechanism. The same control system serves as the blade feathering device necessary to avoid structural damage in high winds.

Aerodynamic and structural considerations dictate that propeller type machines use slender, tapered, highly twisted blades. Despite advances in blade fabrication techniques their cost is quite high. The 38m blade of the ERDA-NASA 100 kW experimental wind turbine cost $160,000, 32% of total system cost. Vertical axis machines employ symmetrical airfoil sections, constant along blade span. The resulting low structural weight and ease of fabrication should reduce costs.

Finally, there is the question of electrical power generation. System efficiency improves when the rotational axis of generating equipment directionally coincides with the turbine rotational axis. For this reason the electrical generating equipment for conventional turbines commonly resides in a pod atop the tower. The machinery is quite heavy and to a large extent drives the structural design of the tower. Further complexities arise from the aerodynamic interaction of the blades, the hub and the generator pod. Vertical axis turbines avoid these problems since generator and control systems may be housed on the ground and linked directly to the rotating vertical shaft.

The above considerations have prompted a systematic investigation of vertical axis wind turbines. Empirical methods cannot be used at present due to the lack of reliable experimental data. Experimental prototypes are being constructed and tested but the need for cost effective parametric design studies establishes the requirement for computational models for performance prediction.

Previous Investigations

Analytical performance methods must adequately consider the unsteady nature of the flow, the distorted wake shed from blade trailing edges, the interaction of this wake with turbine blades and the mutual blade interaction. E. C. James attempted to solve the requisite linearized differential equation for single bladed machines, but because of its complexity, approximate solutions were attempted only for the limiting cases of low and high tip speed ratios. Unfortunately, the useful information lies somewhere in between. Investigators generally agree that single bladed results may be extrapolated to multi-bladed systems provided that the mutual blade interference is small and can be neglected. Exactly where this assumption fails in practice is not known.

Brulle and Larsen, using vortex theory originally developed for cyclogiros, made simplifying assumptions regarding
the vorticity shed from the blades due to unsteadiness. They assumed, based on momentum considerations, that the wake was shed with some average velocity which remained constant, and that this wake was shed at only two instantaneous "blade flip" points during a complete orbit. Brulle reported that this assumed wake structure results in computer program idiosyncrasies which limit giromill investigation to \( \epsilon < 1 \).

Drees conducted a theoretical and experimental (wind tunnel) study of straight bladed Darrieus turbines having cyclic pitch. It was assumed that energy is extracted uniformly across a plane perpendicular to the wind direction at the center of rotation. This implies constant induced velocity which is contrary to the known occurrence of marked velocity gradients in that region.

Fanucci and Squire used strip theory to investigate variable pitch vertical axis machines employing both conventional and circulation control blades. This method uses a "blockage factor", suggested by Wilson and Lissaman and derived from momentum considerations, to account for the effects of local induced velocities. Power coefficient is extremely sensitive to this parameter and there is little assurance of its accuracy. Optimum tip speed ratio also varies considerably.

The above references assume a wake constrained in some fashion or other; actually the wake is force-free and individual shed vortices convect at their own local velocities. Because the wake is shed continuously as the blade revolves, it is uncertain what may result when the trajectories of wake elements shed at a previous time, or from another blade, intersect or interfere with the local flow of a given blade. Ignoring the resulting interaction may lead to serious errors in calculating the aerodynamic performance of the machine. Although the stream tube methods applied to the aerodynamics of cross-flow machines have served a useful purpose, the assumptions used in these analyses cannot account for most of the effects previously mentioned. The computational model described in the Appendix attempts to remove the assumptions of previous work by using a completely numerical solution of the nonsteady lifting surface problem. The effects of finite blade span are assumed negligible since configurations of interest generally have blade aspect ratios of twenty or greater.

**Results**

**Computational Procedure:** Computations begin at some initial time when no wake is present. As the calculations proceed in small time steps (may be thought of as \( \Delta \theta \) steps), vorticity is shed from blade trailing edges and the wake is developed. This transient phase corresponds to the starting motion of the turbine. When a periodic solution results for the case of periodic or constant wind velocity, the calculation is terminated. Several computational variables affect convergence to this repetitive solution. \( \epsilon, \Delta \theta \) and \( r \) appear to be the most important of these although the exact relationship required for convergence has not yet been determined.

**Lift and Moment Characteristics:** The three or four vortex
model (described in Appendix) closely approximates the correct $C_L$ values, but $C_M$ typically differs by 40% from the proper values. With eight blade segments the situation improves.

The number of revolutions necessary to reach $C_L$ convergence depends on the value of $E$ and the size of the $\Delta \theta$ steps used in the numerical procedure. For $\Delta \theta = 8^\circ$ a satisfactory solution results at $r = 5$ and requires approximately 30 minutes of CPU time. For $\Delta \theta = 4^\circ$, $r = 4$ gives satisfactory results but takes approximately 120 minutes of CPU time. Minimum computer time to achieve the periodic solution is obviously desired. For purposes of calculating turbine performance then, the smaller $\Delta \theta$ are not justified. However, the smaller $\Delta \theta$ solution reflects the finer details of the wake structure as well as the transient aerodynamics and may be useful for flutter analysis or other structural design data.

Fig. 5.1 shows the variation of calculated lift from the steady state value given by classical theory. The difference is obvious, but not startling, because the Darrieus turbine studied here does not exhibit highly unsteady aerodynamics, since the blades are fixed. Fig. 5.2 compares lift coefficient transients with the periodic solution. After $80^\circ$ of the first revolution the starting transients decay and for all subsequent orbits $C_L$ is well behaved for $0^\circ < \theta < 120^\circ$. In this illustrative one bladed case, the dynamics of the blade/wake interaction appear in $r = 2$ and are generally confined to the region $120^\circ < \theta < 240^\circ$. Transients diminish in amplitude and frequency until the repetitive solution occurs, usually after four revolutions.

Lift transients appear to exhibit rather long time constants suggesting the possibility of unsteady loadings with higher frequencies than the one or two per revolution type. This also leads to speculation concerning the time and spatial variations of the wind. If the wind phase velocity and wave length are of the same order as the turbine diameter and natural frequency, resonance phenomena may occur. These can have important consequences regarding wind turbine vibrations.

Wake Details: Another test for periodicity is the convergence of vortex trajectories, shed at a particular location on the blade orbit, to a predictable path. Generally wake convergence requires several revolutions more than $C_L$ convergence, but the last few revolutions produce a negligible effect on performance calculations.

Steps of $\Delta \theta = 4^\circ$ were used to generate the details of the transient wake shown in Fig. 5.3. The first few starting vortices meander downstream into a relatively quiescent flow field and traverse the orbital path at $\Theta = 200^\circ$ when the blade is at $r = 2$, $\Theta = 80^\circ$ ($t_4$ in Fig. 5.3). However, as the wake elements at $\Theta = 24^\circ$ traverses the orbital path it passes very close to the blade ($t_2$ and $t_3$). The interaction results in the erratic trajectory shown, inhibits downstream progression of the vortex and has an appreciable effect on blade $C_L$. Transients are exhibited similar to those shown in Fig. 5.2, but of greater amplitude and frequency. Note also that as the blade passes the vicinity of the $\Theta = 8^\circ$ vortex (at $t_4$) the oscillatory nature of the trajectory increases, indicating the
influence of the passing blade. Skepticism of this erratic behavior or of the ability to predict it is not entirely unexpected. The reader is therefore referred to Djodjodharijo and Widnall\textsuperscript{5,9}, where it is seen that the calculations of Giesing\textsuperscript{5,10} and flow visualization studies of Bratt\textsuperscript{5,11} agree remarkably for the unsteady potential flow of oscillating airfoils.

The fully developed wake is shown in Fig. 5.4 for $\varepsilon = 6$. Solid lines downstream of the turbine represent the locus of the extremities of the shed vortex trajectories. Upstream of the machine the lines represent the path the vortices would have taken had they actually been fluid particles. Definition of the "wake" is somewhat arbitrary, but attempting to illustrate the region of influence of the turbine, the authors have shown in Fig. 5.5 the locus of points at which the windward component of the local velocity is 98\%, 99\% and 100\% that of the free stream wind. This region extends far beyond the projected frontal area of the turbine and suggests that major wind tunnel blockage corrections may be necessary for turbines tested in closed tunnels. Since small errors in tunnel speed produce large errors in $C_p$ calculations, further investigation should be pursued.

Velocity Defect and Blockage Factor: The term "blockage factor" is used to describe the reduction in the windward component of the freestream velocity. This blockage, of course, alters turbine performance appreciably. Rather than estimating blockage factor the present method develops aerodynamic characteristics based on calculated local flow conditions. Fig. 5.6 illustrates cross-wind velocity traverses near the wind turbine for the blade position shown. It can be seen that velocity defect is not symmetrical about the windward axis.

Energy Conversion Efficiency: The efficiency of a wind energy conversion system is given by $C_p$ which is the ratio of power extracted by the turbine to the theoretical maximum available wind energy within the blade swept area. Fig. 5.7 shows $C_p$ vs $\varepsilon$ for two different levels of aerodynamic drag. Recall that the present method gives lift characteristics only and drag must be obtained from some other source. Maximum $C_p$ and optimum $\varepsilon$ agree well with strip theory. However, this may only be true for Darrieus turbines which are not extremely unsteady devices (no cyclic blade pitch changes). The Wilson-Lissaman factor gives disparate results from strip theory or the current vortex model numerical solution and its use cannot be recommended.

Concluding Remarks

Results for the straight bladed Darrieus turbine illustrate the developed capability to model the complex lifting characteristics of cross-flow turbines. Initial research has focused on development of the analysis and the required computer program. Extension of the method to multi-bladed machines, cyclic blade pitch changes and time variant winds is progressing. The concept can be modified to accomodate thick airfoils, variable camber or circulation control blades by changing the vortex model to one with bound vortices.
distributed on the blade surface rather than the camber line.

Since blade aspect ratio is large, the two-dimensional analysis should closely approximate the finite blade characteristics. The matter of drag prediction for these devices must be aggressively pursued; it is the remaining obstacle to complete analytical performance prediction. In this regard, the West Virginia University experimental wind turbine will be utilized in conjunction with lift predictions of the current method to deduce drag characteristics.

References


Fig. 5.1. Lift calculated from thin airfoil theory and vortex model numerical solution.

Fig. 5.2. Transients and periodic solution.
Fig. 5.3. Wakeblade interaction in the transient wake.

Fig. 5.4. Vortex trajectories in the developed wake.
Fig. 5.5. Lines of constant windward velocity around the turbine.

Fig. 5.6. Velocity defect in the vicinity of the turbine.
Thin-airfoil theory: The problem is approached in the spirit of classical potential theory (i.e., fluid viscosity is neglected). Lifting characteristics of the blades below the stall are negligibly influenced by viscosity and the resultant of the pressure forces is only slightly influenced by section thickness as long as thin airfoils with small camber are operating at small angles of attack. Furthermore, blade speeds are assumed small enough to treat the flow as incompressible. Under these circumstances the overall section lifting characteristics are well predicted by thin-airfoil theory and the airfoil is replaced with its mean camber line. Blades are assumed to be of infinite span since most machines have large blade aspect ratios. The problem is then resolved into one of finding a flow pattern having the instantaneous streamline coincident with the section mean camber line (the boundary condition at the body) and insuring that the flow will leave the trailing edge smoothly (Kutta condition).

Kinematic boundary condition: Following E. C. James a noninertial (x,y) coordinate system is selected, attached to the blade such that its origin traverses a circular orbit with respect to an inertial (x̂ , ŷ ) frame of reference. The x-axis is aligned with the relative inflow velocity  \( \mathbf{v}_r(t) \) and the blade is at angle of attack \( \alpha(t) \). Choice of this coordinate system facilitates the linearization of the boundary conditions. The equations below follow from the geometry (Fig. 5.A.1):
\[ V_c(t) = \omega [1 + \varepsilon^2 - 2\varepsilon \sin \Theta]^{1/2}, \]
and
\[ \alpha(t) = \alpha_n(t) + \omega t - \beta(t) - \pi/2. \]  

(5.A-1)

As a consequence of aligning the \( z \)-axis with \( \text{T}_R \), the \( (x, y) \) axes rotate with an angular velocity \( \Omega \) (with respect to the inertial frame) given by,
\[ \Omega(t) = \omega \frac{d\beta}{dt} = \frac{\omega [1 - \varepsilon \sin \Theta]}{[1 + \varepsilon^2 - 2\varepsilon \sin \Theta]}, \]

(5.A-2)

where \( \beta(t) = \arctan[(\sin \Theta - \varepsilon)/\cos \Theta] \).

The boundary condition on the cambered surface requires that the normal velocity of the blade relative to the \( (x, y) \) system be equal to the normal velocity of the fluid on the surface. In terms of the noninertial frame the equation is
\[ \frac{d \cdot f(x, y, t)}{dt} = \left( \frac{\alpha_x - \alpha_n}{x} - \frac{\alpha_y - \alpha_n}{y} \right) \cdot \cdot f(x, y, t) = 0. \]

(5.A-3)

In Eqn. (5.A-3), \( f(x, y, t) = 0 \) describes the mean camber line of the airfoil. Also,
\[ \alpha_x = -(V_w \sin \beta - u_x) \frac{x}{x} - (V_w \cos \beta - u_y) \frac{y}{y}, \]
\[ \alpha_y = \omega x \frac{x}{x} - \omega \cos(\Theta - \beta) \frac{x}{x} - \omega \sin(\Theta - \beta) \frac{y}{y}, \]
\[ \alpha_n = \alpha_n \frac{z}{z}, \]
\[ \alpha = \alpha_n \frac{z}{z}, \]
and
\[ \alpha_n \frac{z}{z}. \]

Of primary interest are uncambered airfoils for which \( y - x \tan \alpha = 0 \) and the exact boundary condition follows from Eqn. (5.A-3) and is
\[ u_y = x \sec^2 \alpha \frac{x}{x} + \tan \alpha (V_R + u_x + \alpha_n y) = 0. \]

(5.A-4)

The relationship between the velocities in the inertial and noninertial frames is,
\[ \vec{U} = V_x \vec{x} + V_y \vec{y} = \frac{x}{x} - \frac{y}{y} \vec{R} - \vec{\alpha}_n \vec{z}. \]

(5.A-5)

Inserting appropriate relationships in Eqn. (5.A-5) yields
\[ U_x = V_x + u_x + \alpha_n y \text{ and } U_y = u_y - \alpha_n x. \]

(5.A-6)


Assuming small angles of attack and neglecting second order terms yields the linearized boundary condition shown in Eqn. (5.A-8):
\[ U_y = x \sec^2 \alpha \frac{x}{x} + \tan \alpha (V_R + u_x + \alpha_n y); \]

(5.A-7)

\[ U_y = x \frac{x}{x} + V_R \alpha_n \frac{z}{z}. \]

(5.A-8)

Vortex Model: In potential theory, the loading on a lifting surface may be represented by a distribution of singularities. In this model, a distribution of bound vorticity was employed, and for numerical convenience it was assumed that this distribution could be represented by discrete bound vortices attached to the cambered surface (Fig. 5.A.2). The normal velocity ** induced at \( \overrightarrow{f} \) is given by

**For small angles of attack \( U_y \) is very nearly equal to the velocity normal to cambered surface.
If the boundary condition is enforced at the "control point" of each of the $n_s$ chord segments, no flow results through the surface. It has been assumed for the moment that no shed vortices are present as would be the case in the steady state situation. The mathematical representation of this situation gives equations which can be solved simultaneously for the unknown strengths of the bound vortices. For the $j$th control point,

$$
\sum_{i=1}^{n_s} \left[ \gamma_{i} / \left( \beta_{b_i} - \beta_{c_i} \right) \right] = -2 \pi \left( \psi_{b_i} \phi_{j} \right).
$$

(5.A-10)

James 5.11 has shown that the choice of the 1/4 point of the chord segment for the bound vortex and the 3/4 point for the control point is optimum for the solution of the vorticity distribution in steady two dimensional problems. Furthermore, the Kutta condition is automatically satisfied by this vortex model. Rudham 5.12 discusses the application of this computational model to the problem of unsteady airfoils.

Unsteady Flow Considerations: When changes in $\phi$ and/or $\psi$ occur, the flow is unsteady. The physical principle of primary importance in unsteady problems is the conservation of circulation. This implies that the time rate of change of the bound and wake circulation is zero, and that the net circulation around the airfoil and wake must remain constant:

$$
\frac{d\Gamma}{dt} = d\Gamma_b/dt + d\Gamma_w/dt = 0.
$$

(5.A-11)

$$
\Gamma_b = \frac{c/2}{-c/2} \int \gamma'(x) dx = \sum_{i=1}^{n_s} \gamma_i
$$

defines the bound circulation. Thus if the bound circulation changes due to airfoil motion or flow fluctuations, there must be an equal but opposite change in the wake. This is possible when $\Gamma_b$ changes, by shedding from the trailing edge and into the wake, vortices of appropriate strength and sense. The strengths of these shed vortices then remain fixed. Considering the blade motion in small $\Delta t$ time steps, it can be shown that for the $m$th time interval (in the one bladed case) the equations for determining the bound vorticity are of the form:

$$
\sum_{i=1}^{n_s} \frac{\gamma_{i}^u}{\beta_{b_i} - \beta_{c_i}} - \sin\phi_{i} \left[ \sum_{i=1}^{n_s} \gamma_{i}^u \Gamma_{i}^{m-1} \right] = 2 \pi \left( \psi_{b_i} \phi_{j} + \phi_{s} \right).
$$

(5.A-12)

$$
\sum_{k=1}^{m-1} \sin\phi_{k} \left[ \frac{1}{\beta_{b_k} - \beta_{c_k}} \right] = 2 \pi \left( \psi_{b_i} \phi_{j} + \phi_{s} \right).
$$

Here the linearized boundary condition is used including the term reflecting the time rate of change of angle of attack. Fig. 5.A.3 illustrates the situation including definition of the angle $\phi$.

The equation is written for the $j$th control point where $j(max) = n_s$.
and superscripts designate time steps. The first term in Eqn. (5.A-12) is the contribution of the bound vorticity to the normal velocity at the jth control point. The second term denotes the contribution of the shed vortex leaving the trailing edge in the current nth time step (the quantity in brackets is its strength written in terms of the unknown bound vorticity and the known strength of the previously shed vortex). The third term accounts for the induction due to the remaining shed vorticity with the understanding that \( f' = 0 \) at \( t = 0 \). For multi-bladed systems, Eqn. (5.A-12) is modified to account for all the bound and shed vorticity present in the flow, and at each time step, will lead to a matrix formulation of the problem of order \( n x n \) when all the control points are considered.

Computational Procedure: The numerical solution assumes that continuous blade motion and wake formation can be approximated by considering small discrete time steps. At time \( t = 0 \) there is no circulation on the airfoil or in the wake. At time \( t_1 \), corresponding to \( \Theta = 0^\circ \), \( \Theta \) or \( V_w \) changes and a starting vortex is shed whose strength is equal to the strength of the bound vorticity at that time:

\[
\Gamma_f^1 = -\Gamma_b^1 = -\frac{2\pi}{1}\Sigma_{i=1}^1 \Gamma_i.
\]

The strengths of the bound vortices are found from Eqn. (5.A-10) where flow conditions are calculated from Eqn. (5.A-1) and the geometry is known. The shed vortex immediately leaves the trailing edge and the blade advances from \( t_1 \) to \( t_2 \). In the intervening \( \Delta t \) the vortex convects with its own local velocity to some new position in space. Its displacement and velocity have in no way been constrained; it is "force-free".

With the strength and location of all vortices known, the velocity at any point in space can be determined using the Biot-Savart law. This is done for the vortex leaving the trailing edge and its displacement during the \( \Delta t \) time increment is added to its previous position vector to find its location at \( t_2 \). In the process it is necessary to transform velocity components and spatial coordinates from the \( (x,y) \) to the \( (\xi,\eta) \) system. The velocity transformation is given by

\[
\begin{align*}
(u_x)_{\xi} &= -(u_\xi \sin \Theta + u_\eta \cos \Theta), \\
(u_\eta)_{\xi} &= -(u_\xi \cos \Theta + u_\eta \sin \Theta).
\end{align*}
\]

The coordinate transformation is given by

\[
\begin{align*}
x &= -\xi \sin \Theta + \eta \cos \Theta + R \cos(\Theta - \Phi), \\
y &= -\xi \cos \Theta + \eta \sin \Theta - R \cos(\Theta - \Phi),
\end{align*}
\]

The absolute velocity of a wake element is then

\[
\begin{align*}
\overrightarrow{V}_f &= (V_\xi \xi + V_\eta \eta) = \overrightarrow{V}_f^\xi + (u_\xi)_{\xi} \overrightarrow{V}_f^\xi + (u_\eta)_{\xi} \overrightarrow{V}_f^\eta,
\end{align*}
\]

Since velocity is a function of position and the vortex is in motion, \( V_f^\xi \) changes constantly. An averaging technique is therefore used for \( V_f^\xi \).
and its position after the mth time increment is

\[
(y^m)'_x = (y^{m-1})'_x + (q^m)'_x \Delta t, \quad (5.A-16)
\]

\[
(y^m)'_y = (y^{m-1})'_y + (q^m)'_y \Delta t.
\]

At t2 the bound vortex strengths are found from Eqn. (5.A-12) and the relationship for the shed vortex is

\[
-f^2 = -f_1^2 - f_2^1 - \left( \sum_{j=1}^{m_s} \gamma^2 - \gamma_1^1 \right).
\]

This follows from the conservation of circulation. Computations for subsequent time steps proceed in a similar fashion. The complex arithmetic facility of Fortran IV is used for computer programming.

Storing vortex spatial coordinates and velocity components as complex variables simplifies the programming considerably.

It can be seen from the above developments, that at any time \( t, n \) the positions and strengths of all bound and shed vortices are known. Lift and moment coefficients, wake shape, vortex trajectories and velocity profiles can then be found.

Forces and moments: The generalized aerodynamic forces on the blade are a local lift normal to the inflow velocity, a local drag parallel to the inflow velocity, and a moment about the mid-chord. Their derivation, in part, follows.

Bernoulli's equation for unsteady, incompressible, inviscid flow written in terms of the \((\xi, \eta)\) frame is

\[
\frac{\partial \phi}{\partial t} + p/\rho + q^2/2 = F(t), \quad (5.17)
\]

where \( F(t) \) is a constant along a streamline. Neglecting small quantities, it can be shown that in terms of the \((x,y)\) systems and written in \((s,n)\) coordinates, Eqn. (5.17) becomes

\[
F(t) = p/\rho + q^2/2 + \partial \phi / \partial t - (V^s_B \phi / \partial n) + (V^s_B \phi / \partial n)
\]

\[
- \left[ (V^s_B \phi / \partial n) \right] \phi / \partial s.
\]

where \((V^s_B)\) and \((V^n_B)\) are components of the blade rectilinear velocity in the \( s \) and \( n \) directions.

For irrotational flow \( F(t) \) is constant everywhere in the field.

Considering the flow conditions below and above a control point (lower and upper surfaces of the blade) and defining that point as \( \Omega \equiv (s,n) \), it is possible to write

\[
\left[ \phi \right]_{\Omega} = \left[ (q^2 - q^2)/2 \right]_{\Omega} + \partial / \partial t \left[ \phi \right]_{\Omega} + (V^s_B \phi / \partial n) - (V^n_B \phi / \partial s)
\]

\[
= \left[ (V^s_B + \Delta s) \phi / \partial n \right]_{\Omega} - \left[ (V^n_B) \phi / \partial s \right]_{\Omega} + \phi / \partial \Omega - \phi / \partial \Omega,
\]

where \[ \left[ \right]_{\Omega} \] indicates that the quantity in brackets is evaluated at \( \Omega \). Potential theory provides the mechanism for evaluating the right hand side of Eqn. (5.A-19) in terms of the vortex strength. The relationships \( \partial / \partial s [\phi - \phi_L]_{\Omega} = -\gamma'(s), \partial / \partial n [\phi - \phi_L] = 0 \),

\[
\left[ \phi - \phi_L \right]_{\Omega} = \int_{-c/2}^{c/2} \gamma'(s) ds, \quad \text{and} \quad \left[ q_n^2 + q_s^2 \right]_{\Omega} = -2\left[ q \right]_{\Omega} \gamma'(s)
\]

give

\[
\left[ (p - p_L)/\rho \right]_{\Omega} = \left[ (V^s_B - q) \gamma'(s) - \partial / \partial t \int_{-c/2}^{c/2} \gamma'(s) ds \right].
\]
where \( y'(s) \) is the strength of the vortex sheet at any point on the blade chord and is a continuous rather than a discrete vortex distribution. The usual convention of \( y'(s) \) positive for clockwise circulation is reversed here so that positive circulation (counterclockwise) induces forces which produce torque in the direction of blade rotation. The following expressions for the normal force on the blade per unit span, the lift and moment can be derived from Eqn. (5.0-20) (Fig. 5.A.4 illustrates):

\[
F_n = \int_{-c/2}^{c/2} \left[ p_L - p_U \right] v_y'(s) ds + \int_{-c/2}^{c/2} \left[ v_y - q \right] v_y'(s) ds,
\]

\[
M = \int_{-c/2}^{c/2} \left[ p_L - p_U \right] v_y'(s) ds + \int_{-c/2}^{c/2} \left[ v_y - q \right] v_y'(s) ds,
\]

\[
\gamma = \int_{-c/2}^{c/2} \gamma'(s) ds,\]

expresses the average vortex strength over the segment. \([v_y - q]_{\gamma'}\) has been assumed constant over the segment.

Wind Turbine Power: The torque developed by the wind turbine is equal to the moment of the blade aerodynamic forces about the \((\vec{z}, \vec{r})\) origin. Positive torque is in the direction of rotation, clockwise as seen from above the turbine:

\[
T = R \left[ L \sin(\theta - \phi) - D \cos(\theta - \phi) \right].
\]

Since potential flow analysis neglects viscous effects, drag data is supplied from experimental sources or is calculated by some other method.

A power input to the turbine results from applying a moment to the blade's pitching axis which maintains the blade at zero incidence to \(V_n\). Furthermore, energy must be added to overcome the resulting aerodynamic reaction. The total power input, \(P(t) = [A(t) + \Delta A(t)] \omega(t)\), is subtracted from the useful rate of working, \(W(t) = \omega(t) Q(t)\), to obtain the net rate of wind energy extraction, \(E(t) = W(t) - P(t)\).
Fig. 5.A.1. Geometry of inertial and noninertial coordinate systems.

Fig. 5.A.2. Vortex distribution on the cambered surface.

Fig. 5.A.3. Position vectors of shed vortex and control point on the camber line.

Fig. 5.A.4. The resultant of $F_n$ and the nose suction force.
6. VAT TEST MODEL

The results of the theoretical calculations, performed as described in sections 4 and 5 of this report, indicate several important characteristics of a vertical axis wind turbine. First, the results were very sensitive to changes in values of airfoil drag, and therefore, airfoil roughness and Reynolds number. In addition, the blockage factor used in the strip theory of section 4 had an important influence on the performance of the device. Consideration of these results suggested that a small-scale wind tunnel test of such a turbine would not allow proper simulation of airfoil drag at operational Reynolds number, could probably not simulate actual airfoil roughness, and would result in an uncertain correction to the test results due to the wind tunnel blockage factor.

Thus, it was decided that a larger scale model should be developed to be tested in the ambient natural wind. This concept would minimize the above-mentioned problems, and has some additional advantages. Correlation with theoretical results will be much more meaningful, thus allowing more dependence on the theory in future design parameter studies. The test model will allow development of practical system components, such as those used in the control system, and for safety devices. The resulting test model will be a flexible, multi-purpose test bed allowing common instrumentation for various design and configuration changes. Finally, the test model will hasten progress toward a practical full-scale wind energy conversion system, and will allow more accurate evaluation of costs and actual power output obtained over a period of time.

However, a natural-wind operated model has the recognized disadvantages of somewhat higher cost than small models, and also that its operation must be subjected to a natural wind where the velocity, direction, and steadiness are not controllable. It is felt that averaging data should resolve these problems to an acceptable degree, and that, on balance, such a test model is much preferable to a small scale wind tunnel model.

Sections 6.1 through 6.5 of this report describe the progress made to date in the design and construction of the WVU vertical axis wind turbine test model.
6.1 DESIGN CONSIDERATIONS

Introduction

This report section discusses the configuration selected for the turbine test model, and the planned schedule of use with blades of both conventional and circulation controlled airfoils. Aerodynamic and centrifugal blade loads are evaluated, and a cost estimate and anticipated power output is given.

Configuration

To assure the earliest possible operation of the test model, the configuration shown in Fig. 6.1.1 was selected. A two bladed machine is being used, with provisions to allow for easy future addition of blades, and changes of parameters such as blade length and machine radius. The main shaft is supported by two combination radial and axial thrust bearings housed in a steel cylinder. This cylinder is supported by eight steel angles which rest on steel pads. The pads are supported above the roof of a small building, as shown in Fig. 6.1.1. The top of the main shaft is attached to a bearing which is connected to three guy wires equally spaced around the bearing and attached to three support poles located about 10m (33 ft) from the building. The steel angles form the primary support system, while the guy wires are a safety feature which should be valuable in case of rotor vibration. Selection of the building as a site allows the instrumentation (section 6.2) to be located inside the building; a main rotor shaft extension passes down through the roof of the building and to the instrumentation below (Fig. 6.1.2). Measurements of the wind uniformity at the rotor location above the building will be made as part of an intensive site wind measurement program. Thus, any nonuniformities in the wind will be taken into account during the evaluation of the rotor performance.

The initial rotor consists of two of the conventional airfoil blades described in section 6.4 of this report. Blades are 3.25m (10'8") in length, and the rotor diameter is 3.05m (10 ft). The rotor bottom is located about 2.44m (8 ft) above the building roof. The blades are supported by two struts from the main shaft, which turns with the blades. A mechanism will be provided to "flip" the blades to the two operating attitudes required for improved operation (see section 4); the blades may also be operated with a constant attitude, i.e., a Darrieus rotor arrangement. After operating experience and data is obtained for this configuration, blades utilizing circulation control airfoils will be installed for testing.

The power output of such a device will, of course, vary with the exact configuration and wind conditions. The expected power is on the order of 0.5 kW with a wind of 6m/s (19.7 ft/s). Electrical power output is from an alternator located inside the building, with power fed into a resistance load bank.

Blade and Shaft Loads

Information as presented in sections 6.3 and 6.4 of this report is necessary for a complete analysis of the blade and shaft loads. For the initial rotor configuration previously described, the
inertial blade loads due to centrifugal forces will be several times
the blade aerodynamic loads. From nondimensional force data presented
in Fig. 6.3.8, a value for blade aerodynamic force of 22.6 kg/m
(15.2 lb/ft) of span is obtained for a wind velocity of 6 m/s (19.7
ft/s) and a tip speed to wind speed ratio of 6. For the same
operating condition and the conventional airfoil of section 6.4,
the centrifugal blade force (machine diameter of 3.05 m (10 ft)) will
be 167 kg/m (112 lb/ft) of span. Thus the centrifugal force is
about 7.4 times as great as the aerodynamic force, and will be the
predominant design factor.

From Fig. 6.3.9 it can be calculated that the maximum shaft
loads for the same operating condition will be on the order of
28.70 kg/m (19.3 lb/ft) of shaft length. In practice, there will
also be some centrifugal shaft force due to rotor unbalance, but
this can be minimized by proper design and initial rotor balancing.

The result of having much larger centrifugal forces than
aerodynamic forces, as shown in these calculations, emphasizes the
importance of possible machine operation at lower rotational
speeds. As stated in section 4, the circulation controlled airfoil
design shows promise of efficient operation at these lower rotational
speeds; therefore, the entire structural problem would be greatly
simplified. This is especially important for large radius machines.

Costs

At this stage of development, no definite costs can be estab-
lished for various sized vertical axis wind machines.

However, the cost of the test model itself, using the two conven-
tional airfoils as blades, has been determined during its construction.
Table 6.1-1 summarizes the costs. The costs do not include the
research-type instrumentation unique to the test model, but do include
the alternator required to generate power and the control system
required to program the blade flip. Several items included in this
cost analysis were available without cost, such as the alternator
and conventional airfoil blades; purchase costs have been estimated
and included in the analysis.

Several items should be considered when reviewing the cost
analysis. The test model is a singular unit, and no economy could
be attained by using mass-production techniques. The test model
is a research machine, and so in many instances the machine is
overdesigned in the sense that parts were fabricated to provide
versatility and large safety factors so that various tests can be
performed in the future. Also, in order to expedite construction,
many items were purchased from vendors where minimum time delays
would occur, and, therefore, not necessarily at the lowest possible
cost. Some extra costs were incurred due to the site (building
roof and surrounding terrain) selected, which could be minimized
with other sites.

For the test model, the cost is estimated to increase to
about $6000 with the circulation controlled blades and supporting
system. This system is currently being designed. The power output
is expected to increase significantly with this modification. From
Fig. 1.2 and Table 1–I of this report, it is seen that the costs are on the order of those of the smaller machines shown, especially if improvements from design optimization and mass production techniques are to be considered. While any such cost figures must be considered extremely tentative at this time, it is interesting to note that this first estimate is relatively near the costs for other machines having similar output. Also, from the ERDA sponsored study of the McDonnell Giromill design (Fig. 1.2), it is estimated that large machines of this general type will be competitive with horizontal axis turbines in initial cost per kilowatt.

In future work on this project at West Virginia University, considerable emphasis will be put on improving the cost analysis, with design optimization and other cost-cutting techniques to be considered.
<table>
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<th>Material</th>
<th>Labor</th>
<th>Total</th>
<th>% of Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Support Stand</td>
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<td>$ 638</td>
<td>$1124</td>
<td>23%</td>
</tr>
<tr>
<td>Rotor and Blades</td>
<td>908</td>
<td>772</td>
<td>1680</td>
<td>33%</td>
</tr>
<tr>
<td>Guy Wire System</td>
<td>492</td>
<td>181</td>
<td>673</td>
<td>14%</td>
</tr>
<tr>
<td>Power Generating System and Control System</td>
<td>1073</td>
<td>280</td>
<td>1353</td>
<td>26%</td>
</tr>
<tr>
<td>Assembly and Painting</td>
<td>33</td>
<td>160</td>
<td>193</td>
<td>4%</td>
</tr>
<tr>
<td><strong>TOTALS</strong></td>
<td><strong>$2992</strong></td>
<td><strong>$1991</strong></td>
<td><strong>$4983</strong></td>
<td><strong>100%</strong></td>
</tr>
</tbody>
</table>

% of Total | 60 | 40 | 100% |
6.2 INSTRUMENTATION AND CONTROL

Introduction

This section outlines the instrumentation and methods to be used to measure the performance characteristics of the test model articulated blade vertical axis wind machine while operating in the ambient natural wind. The system is designed to give real time data acquisition and reduction, and also provides automatic control of the blade articulation (blade "flip") under changing wind conditions.

Wind machine performance is usually measured by the relationship between the power coefficient, \( C_p \), given by

\[
C_p = \frac{P}{\frac{1}{2} \rho V_r^3 A}
\]

(6.2-1)

where
- \( P \) is the wind machine output power,
- \( \rho \) is the air density,
- \( V_r \) is the free stream wind velocity, and
- \( A \) is the projected area of the wind machine,

and the tip speed ratio (TSR) given by

\[
TSR = \frac{\omega R}{V_r}
\]

(6.2-2)

where
- \( \omega \) is the machine angular velocity, and
- \( R \) is the radius of the wind machine.

The wind machine power can be calculated by

\[
P = \omega R T
\]

(6.2-3)

where \( T \) is the torque output of the machine.

In order to calculate \( C_p \) and \( TSR \) the following quantities must be measured:

1. Wind machine angular velocity.
2. Wind machine torque.
3. The wind velocity.
4. The air temperature.
5. The air pressure.

In the system described here, the angular velocity, torque, wind velocity, and air temperature are measured automatically, and the air (barometric) pressure is measured manually. A block diagram of the system is shown in Fig. 6.2.1. A data acquisition/reduction system is used which is shown in Fig. 6.2.2; a block diagram of this general purpose system is shown in Fig. 6.2.3. The interface to this data acquisition/reduction system is the wind machine control box, shown on Fig. 6.2.1 and pictured in Fig. 6.2.4. The automatically measured quantities are transmitted in the form of voltage signals through the control box and to the Hewlett Packard (HP) 3480 digital voltmeter. These signals are then transmitted to the HP 9810A computer/calculator through a coupler device. The HP 9810A calculator is equipped with the proper program to receive, reduce, and output the wind machine data. Only the barometric pressure is entered manually by the operator. The typewriter is the normal data output device, and the operator may also select a graphical plot of the data. The typewriter output gives the following parameters:
1. Tip speed ratio (TSR)
2. Power coefficient (CP)
3. Power (P)
4. Angular velocity (ω)
5. Torque (T)
6. Free stream velocity, location 1 (Vw,1)
7. Free stream velocity, location 2 (Vw,2)
8. Wind direction at location 1
9. Wind direction at location 2
10. Air density (ρ)

If the plotter has been selected, the graph shows CP vs. TSR, and each data point measured is shown as a "+".

Angular Velocity

The angular velocity (ω) of the wind machine is measured by a tachometer-generator which produces 7 volts d.c. output per 1000 rpm. The generator is coupled to an extension of the wind machine shaft which is on the instrumentation package (Fig. 6.2.5) located in the building below the machine. Part of this package turns with the angular velocity of the wind machine, and a rubber "O" ring drive belt turns the tachometer-generator. The generated signal passes through the control box and is filtered by an R-C filter with a time constant of about 1.5 seconds before being sent to the HP 3480 DVM. The output voltage will be related to the angular velocity (ω) in radians/second by

\[ \omega = \frac{1000 \text{ rpm}}{7 \text{ hr}} \times \frac{2 \pi \text{ rad/s}}{60 \text{ min/min}} \times V_{\text{w}} \times G \]

where \( V_{\text{w}} \) is the generator output voltage and \( G \) is the drive pulley ratio.

Torque

It is anticipated that the wind machine will be loaded with an automotive type alternator. The torque (T) produced by the loaded machine will be measured by a strain gage load cell (Fig. 6.2.6). The load cell is made of type 2024-T3 aluminum with an inside diameter of 4.45 cm (1.75 in) and an outside diameter of 5.08 cm (2.00 in). Four 350 ohm strain gages are located on the cell in the form of a full bridge as shown in Fig. 6.2.7. The gages mounted in this configuration are sensitive to torque and are insensitive to temperature, axial load, and bending.

The strain gage bridge is excited by a Brush model 13-4312-00 D.C. bridge amplifier, which also amplifies the signal from the bridge. The amplifier output is connected to the HP 3480 DVM through the control box.

The load cell has been calibrated for the range of zero to 271.2 "\( \cdot \)m (200 ft/lb) and was found to be linear (Fig. 6.2.8).

In use, the torque load cell will be located above the instrumentation package and above the alternator belt drive takeoff where the machine is loaded. In this position the cell rotates with the main shaft, and the leads from the cell must pass through slip rings which will be located inside of the power slip ring assembly.
(Fig. 6.2.5). Slip rings for this use will be a ten ring assembly suitable for strain gage level signals.

Wind Velocity
The wind velocity \( V_w \) will be measured at two locations near the machine with cup type anemometers. The anemometers are part of wind direction/wind velocity indicators available commercially. The anemometer generator produces an a.c. signal related to the wind velocity. This signal is converted to d.c. in the control box by full wave bridge rectifier circuits. After passing through R-C filters with time constants of about 1.5 seconds, the signals are sent to the DVM.

Wind Direction
The indicator units described in the previous section have vane type sensors coupled to potentiometers. The potentiometer in the indicator will be connected as a voltage divider and supplied with 10 volts from a zero regulated source. The wind direction in degrees will be related to the output voltage \( V_o \) from the voltage divider by

\[
D = \frac{V_o}{V_i} \cos \alpha
\]

(6.2-4)

where \( V_i \) is the supply voltage to the divider, \( \alpha \) is the angle in degrees through which the potentiometer moves, and \( D \) is the wind direction in degrees. To make the output independent of fluctuations in the supply voltage, both \( V_o \) and \( V_i \) are measured by the DVM and the ratio \( V_o / V_i \) is computed.

Air Temperature
The air temperature \( T \) and pressure \( P \) must be measured in order to calculate the air density \( \rho \) from the equation of state:

\[
\rho = \frac{P}{RT}
\]

(6.2-5)

where \( R \) is the gas constant for air. The air pressure is measured manually with a barometer, and the value manually entered into the calculator.

The air temperature is detected with a thermistor probe which is connected in the gate circuit of an N-channel field effect transistor used as an amplifier. The temperature signal and the amplifier supply voltage are both recorded by the DVM.

The circuit was calibrated (Fig. 6.2.9) and the results represented by two straight lines for temperatures from \( 50^\circ \)C to \( 46^\circ \)C. A second curve of temperature output voltage vs. supply voltage was constructed so that corrections could be made for changes in supply voltage.

Blade Articulation Position Control
The position at which the blades of a vertical axis wind machine articulate with respect to the direction of the wind has a significant effect on the power output. Since the direction of the wind changes as a function of time it is necessary to make provisions
for control of the articulation position.

The direction of the wind is detected by a vane type wind direction indicator connected to a transmitting sellyn (Fig. 6.2.4). The direction information (i.e., wind direction) is then transmitted to a receiving sellyn located under the table of the bottom of the instrumentation package (Fig. 6.2.5) and which is attached to the table (i.e., fixed with respect to the ground). Brushes are attached to the shaft of the receiving sellyn which run on a commutator located in the section of the instrumentation package that moves with the wind direction. When the wind direction changes the brushes move in such a way as to generate a signal to the control box which causes one of the relays to pick and drive the wind direction drive motor (Fig. 6.2.5). The motor's motion is such that the wind aline section of the instrumentation package moves to align the brushes and commutator and turn off the motor. This action keeps the wind aline section of the instrumentation package in the direction of the wind.

Since the wind direction changes rapidly in time and may oscillate about a fixed heading it is necessary to include damping to prevent rapid seeking by the alignment system. The damping is provided by R-C time constants in the control circuit. Also, the control system may be disabled and operated manually to select any articulation position without regard to the wind direction.

The power and articulation information are sent up the wind machine shaft by a set of 5 power slip rings and one commutator ring (bottommost ring in Fig. 6.2.5). These rings rotate with the wind machine shaft and the brushes which power them move with the wind direction.
Fig. 6.2.2. Data acquisition/reduction system.
Fig. 6.2.3. Data acquisition/reduction system (DARS).
Fig. 6.2.4. Control box for panemone test model.
Fig. 6.2.5. Test model instrumentation package.
Fig. 6.2.6. Torque load cell and wind direction indicator.
a. Configuration sensitive to torque and insensitive to temperature, axial load, and bending.

b. Bridge connectors

Fig. 6.2.7. Torque load cell strain gauge configurations.
Fig. 6.2.8. Output vs. torque for load cell calibration.

Fig. 6.2.9. Output vs. temperature for thermistor and circuit calibration.
6.3 WIND MACHINE BLADE AND SHAFT LOAD CALCULATIONS

Introduction

A study was conducted to determine what aerodynamic forces would be present on the blades and the shaft of the proposed vertical axis wind machine. A computer program was written to do this and sample printouts and results are presented here.

Two load-calculating programs exist: FMP 1 and FMP 2. FMP 1 calculates the loads on a 2 position blade flip machine. FMP 2 calculates loads on a machine with constantly varying optimum blade attitude. These programs are both modifications of earlier programs which calculated only the machine efficiency. This report only contains a sample print outs for FMP 2 since the FMP 1 output is similar.

Nomenclature

Note: See Fig. 4.3 of section 4 of this report.

English Letters

\( C_d \) Drag coefficient
\( C_l \) Lift coefficient
\( c \) Chord of blade
\( D \) Drag
\( F_x \) Force in x direction
\( F_y \) Force in y direction
\( L \) Lift
\( R/c \) Ratio of radius of wind machine to chord of blade
\( V_c \) Free stream velocity

Greek Letters

\( \theta \) Angle of blade rotation with respect to free stream velocity
\( \rho \) Air density
\( \alpha \) Angle between \( V \)rel and chord line
\( \alpha_c \) Angle between \( V \)rel and radial line
\( \alpha_o \) Blade angle of incidence

Blade Forces

The aerodynamic forces on a blade of the wind machine were resolved into components perpendicular and parallel to the blade chord line to allow future calculation of the bending loads that will exist on the principal bending axis of the blade. Fig. 6.3.1 shows that the force along the blade chord is

\[
F_x = -L \sin \alpha + D \cos \alpha \quad ,
\]

and the force normal to the blade chord is

\[
F_y = L \cos \alpha + D \sin \alpha \quad .
\]

These equations can then be written in terms of lift and drag coefficients:

\[
F_x = L \frac{1}{2} \rho V_{rel}^2 C_l \left[ -C_l \sin \alpha + C_d \cos \alpha \right] \quad ,
\]

\[
F_y = L \frac{1}{2} \rho V_{rel}^2 C_l \left[ \cos \alpha + C_d \sin \alpha \right] \quad .
\]
For computer calculations, both sides of these equations were divided by \( \frac{1}{2} \sqrt{\frac{w}{\rho}} c \). This gave the nondimensionalized forms which were used to find the aerodynamic blade loads per unit length of blade span.

**Shaft Forces**

To determine the aerodynamic load on the wind machine shaft, the blade forces were first resolved into components parallel and perpendicular to the radial lines of the machine (Fig. 6.3.2). Since the perpendicular force component produces only a torque about the shaft, just the radial force need be considered in this analysis.

Taking outward forces on the shaft as positive, the radial force on the shaft can be written from Fig. 6.3.2 as

\[
F_{\text{Radial}} = \frac{1}{2} \sqrt{\frac{w}{\rho}} c \left[ C_L \sin \alpha - C_D \cos \alpha \right].
\]  

(6.3-5)

Resolving this force into with-wind and cross-wind components yields for the cross-wind force

\[
F_2 = -F_{\text{Radial}} \sin \theta.
\]  

(6.3-6)

and for the with-wind force

\[
F_1 = -F_{\text{Radial}} \cos \theta.
\]  

(6.3-7)

These equations were nondimensionalized by dividing by \( \frac{1}{2} \sqrt{\frac{w}{\rho}} c \) for calculations on the computer. The actual force on the shaft depends on the sum of the radial forces from all blades, so these must be added and the resultant force on the shaft found.

Fig. 6.3.3 is a sample computer printout of the forces on the blades and shaft for a two bladed machine. The top set of results list nondimensional values for the forces on the blades. The calculations were done for every 5° of blade rotation from 0° to 360°. The values of interest were \( \frac{w}{V_0} / V_0 \) (which is \( \frac{w}{V_0} \)), \( \frac{C_L}{C_D} \), \( \frac{C_L}{C_D} \) (forces parallel to the wind), \( \frac{C_L}{C_D} \) (forces along the blade chord), and \( \frac{C_L}{C_D} \) (forces normal to the blade chord). The lower results are nondimensional forces on the shaft. Calculations were done for blade rotation angles from 0° to 180° by increments of 5°. The printout lists values for the two force components, \( \frac{w}{V_0} \) and \( \frac{C_L}{C_D} \) (\( \frac{w}{V_0} \) and \( \frac{C_L}{C_D} \) respectively on the computer printout sheet).

Several graphs were prepared from the printouts for forces on the blade and forces on the shaft vs. the machine blade rotation angle. All are for cases of no bleeding and optimum \( \alpha_0 \). Figs. 6.3.4 and 6.3.5 are for values of \( \frac{w}{V_0} = 1 \) and \( \frac{R}{C} = 10 \); Figs. 6.3.6 and 6.3.7 are for \( \frac{w}{V_0} = 1 \) and \( \frac{R}{C} = 1 \); and Figs. 6.3.8 and 6.3.9 are for \( \frac{w}{V_0} = 1 \) and \( \frac{R}{C} = 10 \). Looking at Figs. 6.3.4 through 6.3.7 which have the same \( \frac{w}{V_0} \) values of 4 but different \( \frac{R}{C} \) values of 10 and 14 respectively, it can be seen that the curves are almost the same point for point. This
indicates the size of the wind machine has little effect on the nondimensional shaft and blade forces caused by the aerodynamic loads. Comparing Figs. 6.3.4, 6.3.5, 6.3.8, and 6.3.9, which have the \( R/C \) values of 10, but different \( \omega R/V_{\infty} \) values of 4 and 6 respectively, it can be seen that the blade and shaft forces are almost doubled at the higher rotational speed.

Calculated results for several different machine sizes and operating conditions have been completed, and are being used in the design of the test model (section 6.1).
Fig. 6.3.1. Blade force vectors.

Fig. 6.3.2. Blade force vectors causing shaft loads.
Fig. 6.3.4. Nondimensional blade forces vs. blade rotation angle: \( R/C = 10, \omega R/V_w = 4 \).

Fig. 6.3.5. Nondimensional shaft forces vs. blade rotation angle: \( R/C = 10, \omega R/V_w = 4 \).
For Optimum $< 0$
No Blowing

For Optimum $< 0$
No Blowing

Fig. 6.3.6. Nondimensional blade forces vs. blade rotation angle: $R/C = 14, u_{ref}/v_0 = 1$.

Fig. 6.3.7. Nondimensional shaft forces vs. blade rotation angle: $R/C = 14, u_{ref}/v_0 = 4$. 
For $\theta < \alpha$, No Blowing

**Fig. 6.3.9.** Nondimensional blade forces vs. blade rotation angle: $B/C = 10$, $\omega R/V_m = 6$.  

**Fig. 6.3.9.** Nondimensional shaft forces vs. blade rotation angle: $B/C = 10$, $\omega R/V_m = 6$.  

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6.4 BLADE STRUCTURAL TESTS

Introduction

The blades which were selected for initial operation of the panemone test machine (section 6.1) were originally manufactured for helicopter use. Since the original design requirements of the blades were not known, it was necessary to experimentally determine their allowable load levels before the panemone design could proceed. The blades are symmetrical airfoils as shown in Fig. 6.4.1, and are constructed of an unknown alloy of aluminum with a mass of 208 kg/m (1.4 lb/ft) of span. For the experiment, a blade was supported as shown in Fig. 6.4.2. In use on the wind machine, the blade will be loaded approximately uniformly with a combination of aerodynamic and centrifugal loads. The following experiment was performed to simulate these loads.

Procedure

First, the blade was examined to check for nonuniformities in the surface and internal structure of the blade. From the four blades available to be used for the test model one was selected that was the most representative of the four. Since little was known at that time of the support design required for the blades on the test model, a simple knife-edge support arrangement was used (Fig. 6.4.2), with supports 3.47 m (11.4 ft) apart. In addition to checking the center deflection of the blade, strain was determined from two opposing strain gages located mid-way between the blade supports on the top and bottom surfaces of the blade. This was to insure against possible premature or unexpected failure. The loading proceeded as follows:

Small bags filled with sand, each with approximate mass of 6.35 kg (14 lb), were placed along the length of the blade (Fig. 6.4.3). The mass of each bag was measured before placement on the blade; the bags were numbered 1 to 14. The mass and its location from the blade center, along with the deflection and strain gage readings, were recorded (Table 6.4-1). After the blade was loaded to about 89 kg (196 lb) it was allowed to stand for 30 minutes to check for possible metal creep (which was found to be negligible). Then the blade was unloaded by removing the sandbags in reverse loading order to check for possible hysteresis effect (none was found).

Results and Conclusions

The loading values used during the experiment were used to calculate the blade bending moments. The strain readings indicated that no apparent yielding took place at the highest stress location (the center point—mid-way between the supports) and thus the blade could have withstood higher loads. These results can be found tabulated in Table 6.4-11.

It is obvious from these results and deflections that struts or cable supports are going to be necessary for the blade to withstand the wind machine inertial loads with reasonable blade deflections.

The test model's maximum operational rotational speed will be determined taking the data obtained into account. The maximum allowable may well be a function of the design of the blade flip mechanism,
and not only dependent on the yield strength of the blade structure. The experimental data obtained will be used to calculate deflections and strains for proposed blade attachment configurations by calculating an effective blade modulus-inertia parameter for use in design calculations.

### TABLE 6.4-I

**EXPERIMENTAL LOADING DATA**

<table>
<thead>
<tr>
<th>No. of Mass Added</th>
<th>Cage #1 Strain</th>
<th>Cage #2 Strain</th>
<th>Deflection True (in.)</th>
<th>Position To Load (in.)</th>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9.8</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>14.12</td>
<td>150</td>
<td>9.5</td>
<td>+ 0.6</td>
</tr>
<tr>
<td>2</td>
<td>15.12</td>
<td>101</td>
<td>9.1</td>
<td>- 0.2</td>
</tr>
<tr>
<td>3</td>
<td>13.75</td>
<td>417</td>
<td>8.8</td>
<td>+ 16.5</td>
</tr>
<tr>
<td>4</td>
<td>14.0</td>
<td>530</td>
<td>8.5</td>
<td>- 17.0</td>
</tr>
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<td>621</td>
<td>8.2</td>
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<td>736</td>
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<td>- 28.0</td>
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<td>6.00</td>
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</tr>
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<td>0</td>
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<td>19</td>
<td>9.8</td>
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### Table 6.4-II
ExperimenTal Loading Results

<table>
<thead>
<tr>
<th>No. of Rags</th>
<th>Incremental Mass Added (lbs.)</th>
<th>Experimental Deflection $\delta$ (in.)</th>
<th>Load Moment (in-lbs.)</th>
<th>Experimental Strain Average at $\delta$ (min.)</th>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>1265.0</td>
<td>420</td>
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<td>1.3</td>
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<td>1.9</td>
<td>2226.6</td>
<td>728</td>
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<td>7</td>
<td>14.79</td>
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<td>2660.6</td>
<td>853</td>
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<td>15.0</td>
<td>2.30</td>
<td>2782.5</td>
<td>877</td>
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<td>1477</td>
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</table>

Maximum Thickness = 0.027 m (1.06 in).
15.5% Thick Symmetrical Airfoil
Maximum Thickness at 30% Chord

Fig. 6.4.1. Test model airfoil cross-section.
Fig. 6.4.2. Test model blade at start of bending test.
Fig. 6.4.3. Test model blade during bending test with loads.
6.5. ROTOR DESIGN ANALYSIS

Introduction

This section contains the structural design analysis for the test model rotor, as depicted in Fig. 6.1.1. A conservative structural design approach was selected in order that the rotor configuration could be varied in future tests with minimum design changes. The test rotor analyzed here uses blades with the symmetrical airfoil section shown in Fig. 6.4.1. The rotor will be extensively instrumented with strain gauges and load cells in order to verify the calculated stresses on the blades, struts, and in the cables.

Load on the Blade

A uniform outward load on the blade is assumed, consisting of the aerodynamic loads and centrifugal force. Two struts are attached to the rotating shaft and each blade. The attachments of the struts to the blades are through the housings of self-aligning bearings. The uniform load on a blade is given by the following equation (see Fig. 6.5.1):

$$ W = \frac{n}{2} \left( \frac{\omega R}{V_m} \right)^2 \left( C_L \sin \alpha - C_D \cos \alpha \right) $$

(6.5-1)

where

- $W$ = the uniform load per unit span,
- $n$ = the mass of the blade per unit span,
- $L$ = the lift per unit span,
- $D$ = the drag per unit span,
- $R$ = the radius of the rotor,
- $\alpha$ = the angle between relative wind velocity and the centrifugal force, and
- $\omega$ = the rotational speed.

Substituting the values of the lift and the drag in Eqn. (6.5-1) gives

$$ W = n \omega^2 R \left[ \frac{1}{2} \frac{\rho \omega^2 V_{rel}^2}{C_L} \sin \alpha - \frac{1}{2} \frac{\rho \omega^2 V_{rel}^2}{C_D} \cos \alpha \right] $$

(6.5-2)

where

- $\rho$ = the density of air at sea level,
- $V_{rel}$ = the relative wind velocity,
- $C_L$ = the lift coefficient, and
- $C_D$ = the drag coefficient.

Dividing Eqn. (6.5-2) by $1/2 \rho V_m^2$ gives

$$ \frac{W}{1/2 \rho V_m^2} = \frac{2n}{V_m} \left( \frac{\omega R}{V_m} \right)^2 \left( C_L \sin \alpha - C_D \cos \alpha \right) $$

(6.5-3)

where $V_m$ = wind velocity. As the blades rotate, the value of this nondimensional uniform load fluctuates. The maximum value of this load is obtained from the computer printout of the values of the expression on the right hand side of Eqn. (6.5-3) (see section 6.3).

Each blade, as shown in Fig. 6.5.2, is subjected to the uniform loading $W$. The blade itself is supported between the two self-aligning bearings whose housings are attached to the outer ends of the struts. To apply a tensile load $S$ along the blade, its ends are
attached to an inclined cable as shown in Fig. 6.5.2. Appropriate tension is applied to these cables.

The self-aligning bearings at the ends of the blade provide supports which are equivalent to simple supports. This is confirmed by comparing the deflection obtained by applying a concentrated load at the middle of the blade supported by bearings with the theoretical deflection. At the mid-point of the blade, bracing is provided by attaching a cable between the blade and the shaft. With this arrangement and bearings we have a beam continuously supported on three simple supports. The following two cases of loading are considered:

(a) Continuous Beam with a Combined Loading of Uniform Lateral and Longitudinal Tensile Loads.

This continuous beam is statically redundant. Assuming that the moment at mid-support is redundant, the following equation is derived for the deflection at any point between one end and the mid-point of the blade:

\[
\frac{\psi / W}{6 EI} = \frac{1}{U} \left( \frac{\cosh U (1 - 2 \gamma)}{\cosh U} - 1 \right) + \frac{2}{U} \gamma (1 - \gamma) + \frac{\eta_0}{U^2} \left( \frac{\sinh 2U\gamma}{\sinh 2U} - \gamma \right),
\]

where

\( \gamma = x/L \),
\( U = \sqrt{\frac{E}{EI}} \),
\( W \) = the tensile load along the blade,
\( \psi \) = the moment of inertia of the blade about an axis normal to the strut.

F = Young's modulus, and

\[
\eta_0 = \frac{1}{U} - \frac{1}{U^2} \tanh U \left( \frac{1}{2U} \right) - \frac{1}{2U^2}.
\]

The redundant moment at the mid-point of the blade is given by

\[
\eta_0 = \frac{W^2}{4} - \gamma \psi^2,
\]

where \( \gamma \) is a non-dimensional coefficient. \( \eta_0/4 \). This itself is the maximum bending moment required to calculate the allowable stress in the blade. Eqs. (6.5-4) through (6.5-6) are used for calculating the deflections and the stresses when an axial load \( S \) is applied.

When the axial load is zero, the following case is considered:

(b) Blade Loaded with only Uniform Lateral Load.

For this case the deflection at any given point, \( \gamma = x/L \), is given by

\[
\gamma = \frac{W^2}{24EI} \left( \gamma^4 - \frac{3}{2} \gamma^3 + \frac{1}{2} \gamma \right).
\]

The maximum deflection occurs at \( \gamma = 0,42 \) for which the above equation is reduced to

\[
\gamma_{\text{max}} = \frac{0.13 W^4}{24EI}.
\]

The maximum bending moment in this case is equal to \( \psi_{\text{max}} = \frac{1}{3} \), and the value of \( \gamma \) is equal to 0.125.
Calculation of Allowable Wind Velocity, Load and Deflection of the Blade

The maximum stress in the blade is given by the following equation:

$$\sigma_{\text{max}} = \frac{M_0 \sin \alpha_0 e}{I_2} \quad \text{(6.5-9)}$$

where $e$ is the distance between the extreme fiber and the neutral axis of the blade. Noting that $M_0 = \eta W L^2$, for the uniform load $W$ the following equation is obtained:

$$W = \frac{\sigma_{\text{max}} I_2}{\gamma K^2 \sin \alpha_0 e} \quad \text{(6.5-10)}$$

For $e = .0127$ m (.5 in.), $I_2 = 562 \times 10^{-7}$ m$^4$ (.135 in.$^4$), and $K = 1.6002$ m (63.0 in.), the above equation is simplified to

$$W (N/m) = \frac{1.728 \times 10^{-6}}{\gamma \sin \alpha_0 e} \sigma_{\text{max}} \quad \text{(6.5-11)}$$

If $L_o$ is substituted for the maximum value of the right hand side of Eqn. (6.5-3), the uniform load $W$ (per unit span) is obtained from

$$W = 1/2 \eta C L_o V_o^2 \quad \text{(6.5-12)}$$

For $\omega R/V_o = 6$ and $C = .1728$ m (.567 ft.) and $\eta = 1.2256$ kg/m$^3$ (.002378 slug/ft$^3$), $V_o = 34.43 V_o^2$, or

$$V_o (m/s) = \sqrt{\frac{W}{34.43}} \quad \text{(6.5-13)}$$

The maximum deflection can be computed from Eqns. (6.5-4) and (6.5-7). The maximum deflection occurs at $\gamma = 0.42$. For $\omega R/V_o = 6$ and an allowable tensile stress of $17.238 \times 10^7$ N/m$^2$ (25 ksi) in the blade, the maximum deflection, wind velocity, and uniform load on the blade are computed. They are given as follows:

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<tr>
<th>$S$ (lb)</th>
<th>$\gamma$</th>
<th>$W$ (lb/ft)</th>
<th>$V_o$ (m/s)</th>
<th>$V_{\text{max}}$ (m)</th>
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<tr>
<td>0 (0)</td>
<td>0.125</td>
<td>2406 (13.74)</td>
<td>6.648 (21.81)</td>
<td>0.0147 (.58)</td>
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<tr>
<td>4468 (1000)</td>
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<td>2476 (14.41)</td>
<td>6.721 (22.05)</td>
<td>0.137 (.54)</td>
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<tr>
<td>8896 (2000)</td>
<td>0.1123</td>
<td>2497 (14.26)</td>
<td>6.777 (22.72)</td>
<td>0.130 (.51)</td>
</tr>
</tbody>
</table>

Conclusions

It may be concluded that the selected structural configuration (Fig. 6.5.2) will allow testing at tip speed ratios of 6 or less at wind velocities of about 6.7 m/s (15 mph/22 fps) with acceptable blade stress levels and deflections. Tension (S) applied to the inclined cables (from the blade tips to the shaft) results in lower blade deflections and higher allowable wind velocities, and therefore, higher rotor rotational speeds.
7. CONCLUSIONS

Based on the results of preliminary theoretical analyses of both the vortex concentrator and the vertical axis wind machines, it is concluded that both concepts show promise of being rather efficient wind energy conversion devices. A strip theory analysis of the circulation controlled vertical axis turbine and a turbine with conventional blades as airfoils shows that the circulation controlled device can reach efficiencies at least as high as those for the conventional bladed turbine. Efficiencies for both types are of the same order as those for “conventional” horizontal axis rotor-type wind turbines, and more exact theory and experimental results could possibly result in efficiencies higher than those of horizontal axis turbines. Furthermore, these efficiencies are reached at a lower rotational speed, thereby considerably reducing the blade centrifugal forces. The vortex concentrator device could result in about a five fold concentration in energy, allowing considerable reduction in rotor (turbine) diameter and allowing higher rotor operating speeds.

Additional work is required to develop the theories to a degree so that they can be used as reliable performance estimators and tools in optimization of designs. Experimental verifications of the theories are required. Due to uncertainties in data which would result from small-scale wind tunnel tests of the vertical axis device, a larger scale test turbine which can be tested in the natural wind is highly desirable, and has been constructed.
addition, the costs and economics of the two innovative designs must be further investigated to insure competitiveness with other types of wind energy conversion systems.

8. RECOMMENDATIONS

It is recommended that continuing effort be expanded to allow more thorough evaluations of both circulation controlled vertical axis turbines and vortex concentrator devices. As more accurate theory is developed, and as larger scale test data becomes available, greater emphasis should be placed on optimization of designs and more detailed estimates of machine capital and operational costs.

It should be realized that compared to "conventional" horizontal axis wind turbines, both the vertical axis turbine and the vortex concentrator are in their infancy, and a continuing research and development program will be necessary if the full potentials of these innovative wind energy conversion systems are to be realized.