Darboux Integrability - A Brief Historical Survey

Ian M. Anderson
Utah State University

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Darboux Integrability —

A Brief Historical Survey

Symmetry in Variational Problems and Differential Equations

Ian Anderson

Utah State University

May 22, 2011
History Past

The classical method of Monge. Given

\[ F(x, y, u, u_x, u_y) = 0 \quad \text{add} \quad G(x, y, u, u_x, u_y) = 0 \]

so that \( u_x = A(x, y, u) \), \( u_y = B(x, y, u) \) is consistent [Jacobi, Lie].
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The generalization of the Monge method to 2nd order PDE leads to the methods of Ampère and Darboux (and [Cartan, 1910])

\[ F(x, y, u, p, q, r, s, t) = 0 \quad \text{add} \quad G(x, y, u, p, q) = 0 \]
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\[ F(x, y, u, p, q, r, s, t) = 0 \quad \text{add} \quad G(3\text{rd ord}) = 0 \]

The compatible equation \( G = 0 \) is called an intermediate integral (order, complete, general ...)

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- DI systems always appear in geometric studies of PDE and in equivalence problems [eg. D. The].

- Many papers in the theoretical physics literature (σ-models) on integrable systems unwittingly arrive at DI systems.
Vessiot’s Fundamental Discovery

Prior to 1939, the method of Darboux was always viewed from the viewpoint of compatibility theory. The method had no group theoretic interpretation.

\[ \begin{align*}
  u_{xy} &= e^u, \\
  u_{xx} - \frac{1}{2}u_x^2 &= f(x), \\
  u_{yy} - \frac{1}{2}u_y^2 &= g(y), \\
  p_x &= f(x) + \frac{1}{2}p_y^2, \\
  p_x &= a(x) + b(x)p + c(x)p_y.
\end{align*} \]

This is a Ricatti equation. Riccatti equations are ODE of Lie type, associated to \( SL_2 \).

Vessiot’s great idea was to turn this around. For each equation of Lie type associated to a Lie group (dim \( \leq 3 \)), he produced a DI equation of the form

\[ u_{xy} = f(x, y, u, u_x, u_y). \]

He reproduced the classical classification of Goursat and even integrated one of the equations which the master was unable to solve.

But the groups arising in Vessiot’s approach are not symmetry groups in the usual sense.
Vessiot’s Fundamental Discovery

Prior to 1939, the method of Darboux was always viewed from the viewpoint of compatibility theory. The method had no group theoretic interpretation.

This changed with a fundamental observation of Vessiot.

\[ u_{xy} = e^u, \quad u_{xx} - \frac{1}{2}u_x^2 = f(x), \quad u_{yy} - \frac{1}{2}u_y^2 = g(y) \]

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• gives a general derivation of the Vessiot group.

• proves that the Vessiot group is an invariant of any DI system.

• uses the Vessiot group to construct a non-linear superposition formula for any DI system.

• gives a completely algorithmic integration procedure, much better that the classical one.
Example

Starting from $u_{xy} = e^u$, the theory tells us to
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- Consider two copies of jet space

$$J^3(\mathbb{R}, \mathbb{R})[x, X, X', X'', X''']$$ and $$J^3(\mathbb{R}, \mathbb{R})(y, Y, Y', Y'', Y''')$$
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- Look to the diagonal action of $SL(2)$ with infinitesimal generators

$$\partial_X + \partial_Y, \quad X\partial_X + Y\partial_Y, \quad \frac{1}{2}X^2\partial_X + \frac{1}{2}Y^2\partial_Y,$$
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- Look to the diagonal action of \( SL(2) \) with infinitesimal generators

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- Calculate the reduced differential system \( (J^3 \times J^3)/SL(2) \), that is, calculate joint differential invariants.

\[
u = \log \frac{X'}{Y'}(X - Y)^2
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- Calculate the reduced differential system $(J^3 \times J^3)/SL(2)$, that is, calculate joint differential invariants.

- In the context of this simple example, the lowest order joint differential invariant gives the general solution.

$$u = \log \frac{2X'Y'}{(X - Y)^2}$$
Every intermediate integral for any DI system is in fact a differential invariant for the Vessiot group action.
Intermediate Integrals and Differential Invariants

Every intermediate integral for any DI system is in fact a differential invariant for the Vessiot group action.

All the classical work of Goursat on studying intermediate integrals is in fact (essentially) covered by Lie’s work on differential invariants and invariant differential operators [Olver].
Classification

From this new viewpoint:

There are as many DI EDS as there are symmetry groups of differential equations!
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From this new viewpoint:

There are as many DI EDS as there are symmetry groups of differential equations!

BUT, only certain symmetry groups of very special DE will lead to DI EDS representing a desired type of equation.
Primitive and Imprimitive Actions

We have calculated all systems of DI equations arising from vector field systems in the plane [Lie, GLKO].
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- Vessiot groups with imprimitive actions give ”triangularized” DI systems – essentially known examples
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We have calculated all systems of DI equations arising from vector field systems in the plane [Lie, GLKO].

- Vessiot groups with imprimitive actions give "triangularized" DI systems – essentially known examples
- Vessiot groups with primitive actions give genuinely new examples.
Cauchy Problem

The Vessiot group dictates the solvability of the Cauchy problem.
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- Let $\mathcal{I}$ be a DI integrable system. If the Vessiot group is solvable then the Cauchy problem can be solved by quadratures.
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- Let $\mathcal{I}$ be a DI integrable system. If the Vessiot group is solvable then the Cauchy problem can be solved by quadratures.

\[ u_{xy} = e^u \]

\[ 3 \times u_{xx} u_{yy}^3 + 1 = 0 \]
Bäcklund Transformations

The subgroups of the Vessiot group can be used to construct Bäcklund transformations for any DI integrable system.
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- All previously constructed examples can easily be derived by symmetry reduction.
- Many new examples can easily be derived by symmetry reduction.

The equation
\[ u_{xy} = \left(1 - u^2 \right) \sin u \]
has Vessiot group \( SO(3) \). But \( SO(3) \) has no real 2 dimensional subalgebras and therefore it does not admit a 1-dimensional Bäcklund transformation to the wave equation.
Bäcklund Transformations

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History Future

• clean up the theory of generalized symmetries for DI systems.
• verify Sokolov’s classification using group theoretical methods.
• analyze completely the Toda lattice systems (parabolic geometries associated to simple Lie algebras).
• study multi-soliton solutions from our group theoretic non-linear superposition viewpoint.
• decide what to do about ‘parabolic’ DI systems ([Cartan, 1911])
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The Method of Laplace


Jacobi – Meyer


Lie Equations


The Method of Darboux - Classical Theory


The Method of Darboux - Via Group Theory


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**The Method of Darboux - Classification**


[34] , *Lá méthode de Darboux et les équations $s = f(x, y, z, p, q)$*, Mémorial de Sciences Mathématique 12 (1926).


The Method of Darboux - Transformation Theory


The Method of Darboux - Symmetries and Conservation Laws


