Problem Set 5

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PROBLEM SET 5

Problem 5.1

Let \( \vec{A} \) be a given constant vector field (Cartesian components are constants) and let \( c \) a given constant scalar. Show that the equation \( \vec{A} \cdot \vec{r} = c \) is the equation of a plane. If \( \vec{A} = \hat{x} + \hat{y} + \hat{z} \), and \( c = 0 \), where is the plane?

Problem 5.2

Prove that the gradient of a function \( f(\vec{r}) \) is always orthogonal to the surfaces \( f(\vec{r}) = \) constant. (Hint: This one is easy; think about the directional derivative of \( f \) along any direction tangent to the surface.)

Problem 5.3

Consider the sphere defined by \( x^2 + y^2 + z^2 = 1 \). Compute the gradient of the function

\[
f(x, y, z) = x^2 + y^2 + z^2
\]

and check that it is everywhere orthogonal to the sphere. Consider a linear function

\[
f(\vec{r}) = \vec{a} \cdot \vec{r},
\]

where \( \vec{a} \) is a fixed, constant vector field. Compute the gradient of \( f \) and check that the resulting vector field is perpendicular to the plane \( f(\vec{r}) = 0 \).

Problem 5.4

Compute the divergence of the following vector fields:

(a) \( \vec{E}(\vec{r}) = \frac{\vec{r}}{r^3}, \quad r = \sqrt{x^2 + y^2 + z^2} > 0, \) (Coulomb electric field)

(b) \( \vec{B}(\vec{r}) = -\frac{y}{x^2 + y^2} \hat{x} + \frac{x}{x^2 + y^2} \hat{y}, \quad x^2 + y^2 > 0, \) (magnetic field outside a long straight wire)

(c) \( \vec{D}(\vec{r}) = \vec{r} \) (electric field inside a uniform ball of charge).

Problem 5.5

Derive (9.15) and (9.20).
Problem 5.6

Consider a spherically symmetric function \( f = f(r), r = \sqrt{x^2 + y^2 + z^2} \). Show that its Fourier transform takes the following form:

\[
h(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3 x e^{-i\vec{k} \cdot \vec{r}} f(\vec{r}) = \sqrt{\frac{2}{\pi \vec{k}}} \int_0^\infty dr r f(r) \sin(kr).
\]

(Hint: Use spherical polar coordinates, choosing your \( z \) axis along \( \vec{k} \).) Note that the transform is spherically symmetric also in \( \vec{k} \) space. Use this formula to compute the Fourier transform of a 3-dimensional Gaussian

\[
f = e^{-a^2(x^2+y^2+z^2)}.
\]

Problem 5.7

Derive (9.25) from (9.21). In particular, express \( c(\vec{k}) \) in terms of the initial data.

Problem 5.8

Let \( f \) and \( g \) be two functions. We can take the gradients of \( f \) and \( g \) to get vector fields, \( \nabla f \) and \( \nabla g \). We can multiply these vector fields by the functions \( f \) and \( g \) to get more vector fields, e.g., \( f \nabla g \). As with any vector field, we can make a function by taking a divergence, e.g., \( \nabla \cdot (f \nabla g) \). Using the definitions of gradient, divergence and Laplacian show that

\[
\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g.
\]

(10.3)

and

\[
f \nabla^2 g - g \nabla^2 f = \nabla \cdot (f \nabla g - g \nabla f).
\]