Problem Set 9

Charles G. Torre
charles.torre@usu.edu

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PROBLEM SET 9

Problem 9.1

Compute the divergence and curl of the following vector fields:

(a) \( \vec{E}(\vec{r}) = \frac{\vec{r}}{r^3}, \quad r = \sqrt{x^2 + y^2 + z^2} > 0 \) (Coulomb electric field)

(b) \( \vec{B}(\vec{r}) = -\frac{y}{x^2+y^2}\hat{x} + \frac{x}{x^2+y^2}\hat{y}, \quad x^2 + y^2 > 0 \) (magnetic field outside a long wire)

(c) \( \vec{A}(\vec{r}) = -y\hat{x} + x\hat{y} \) (vector potential for a uniform magnetic field).

Problem 9.2

A common misconception, which perhaps stems from the notation \( \nabla \times \) for the curl, is that the curl of a vector field \( \vec{V} \) is everywhere orthogonal to \( \vec{V} \). Dispel this misconception by exhibiting a vector field \( \vec{V} \) whose curl is not orthogonal to \( \vec{V} \).

Problem 9.3

Show that

(a) \( \nabla \times (\nabla f) = 0 \), where \( f \) is a function;

(b) \( \nabla \cdot (\nabla \times \vec{V}) = 0 \), where \( \vec{V} \) is a vector field;

(c) \( \nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V} \).

Hints: Work in Cartesian coordinates. Since the results are coordinate independent, and the choice of coordinate axes is arbitrary, it is enough to show that the identities (a) and (c) hold for one component, e.g., the \( x \) component.

Problem 9.4

We derived the continuity equation for electric charge from the inhomogeneous Maxwell equations. Show that an analogous computation with the homogeneous Maxwell equations yields no new equations.

Problem 9.5

We used half the Maxwell equations to derive a wave equation for the magnetic field \( \vec{B} \). Using the other half of the equations, perform an analogous computation to derive a wave equation for \( \vec{E} \).
Problem 9.6

Suppose that you demand

\[ A \sin(\mathbf{k} \cdot \mathbf{r} + \alpha t) = B \sin(\mathbf{k}' \cdot \mathbf{r} + \beta t) \]

for all \( \mathbf{r} \) and \( t \). Show that \( A = B, \alpha = \beta, \) and \( \mathbf{k} = \mathbf{k}' \).

Problem 9.7

Consider an electromagnetic plane wave of the form

\[ \mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi), \]
\[ \mathbf{B}(\mathbf{r}, t) = B_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi). \]

Using (18.9), show that

\[ \mathbf{B}_0 = \frac{\mathbf{k}}{\mathbf{k}} \times \mathbf{E}_0. \]

Problem 9.8

Verify (19.4) – (19.7)

Problem 9.9

Using the Maxwell equations (17.2)–(17.5), and the electric charge continuity equation, (17.8), show that, when the Maxwell equations are satisfied,

\[ \frac{\partial}{\partial t} \left( \nabla \cdot \mathbf{E} - 4\pi \rho \right) = 0, \]

and

\[ \frac{\partial}{\partial t} \left( \nabla \cdot \mathbf{B} \right) = 0. \]

This implies that if the constraint equations are satisfied at any one time, then they are satisfied for all time by virtue of the evolution equations.

Problem 9.10

Show that the electric field in (20.14) traces out an ellipse.

Problem 9.11

Show that \( \mathbf{V} = x \hat{x} - z \hat{z} \) has vanishing divergence. Find a vector whose curl is \( \mathbf{V} \). Show that \( \mathbf{W} = xy^2 \cos z \hat{x} + x^2 y \cos z \hat{y} - \frac{1}{2} x^2 y^2 \sin z \hat{z} \) has vanishing curl. Find a function whose gradient is \( \mathbf{W} \).
Problem 9.12

Let $S$ be a closed surface (a surface with no boundary) and let $C$ be a closed curve (a curve with no endpoints). The fundamental theorem of calculus and Stokes theorem imply the famous results:

(a) $\oint_C \nabla f \cdot d\vec{l} = 0$,

(b) $\iint_S (\nabla \times \vec{V}) \cdot d\vec{S} = 0$.

Verify (a) in the case where $f(x, y, z) = x$ and $C$ is a circle (center at the origin) in the $x$-$y$ plane. Verify (b) in the case where $\vec{V} = y\hat{x} - x\hat{y}$ and $S$ is a sphere (center at the origin).

Problem 9.13

Let $S$ be a closed surface enclosing a charge $Q$. Let $A$ be a surface through which a current $I$ passes; $A$ has the (closed) boundary curve $L$. Using the divergence theorem and Stokes’ theorem, derive the integral form of Maxwell’s equations:

$$\iint_S \vec{E} \cdot d\vec{S} = 4\pi Q, \quad \text{(Gauss)},$$

$$\oint_L \vec{B} \cdot d\vec{l} = 0,$$

$$\oint_L \vec{B} \cdot d\vec{l} - \frac{1}{c} \frac{\partial}{\partial t} \int_A \vec{E} \cdot d\vec{A} = \frac{4\pi}{c} I, \quad \text{(Ampere – Maxwell)},$$

$$\oint_L \vec{E} \cdot d\vec{l} + \frac{1}{c} \frac{\partial}{\partial t} \int_A \vec{B} \cdot d\vec{A} = 0, \quad \text{(Faraday)}.$$}

Problem 9.14

Let $C$ be a circle of unit radius in the $x$-$y$ plane, enclosing the unit disk $D$, and let $\vec{V} = y\hat{x}$. Compute (i) the line integral of $\vec{V}$ around $C$ and (ii) the flux of $\nabla \times \vec{V}$ through $D$.

Hint: According to that guy Stokes, you should find

$$\oint_C \vec{V} \cdot d\vec{l} = \iint_D (\nabla \times \vec{V}) \cdot d\vec{S}.$$}

Problem 9.15

Suppose we redefined the Poynting vector $\vec{S} = c(\vec{E} \times \vec{B})$ via

$$\vec{T} := c(\vec{E} \times \vec{B}) + \nabla \times \vec{W},$$

$$= c(\vec{E} \times \vec{B}) + \nabla \times \vec{W},$$
where \( \vec{W} \) is some given vector field. Show that the continuity equation for energy-momentum still holds for \( U \) and \( \vec{T} \). Which vector field, \( \vec{S} \) or \( \vec{T} \), is the “real” energy-momentum current density?

**Problem 9.16**

Using elementary vector algebra manipulations, show that (18.21) and (18.19) are equivalent.