5-6-2016

The Equivalence Problem: EinsteinMaxwell Solutions

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May 6, 2016

Abstract

The "Equivalence Problem" is part of the Digital Einstein Project. The goal of this project is to create a digital and interactive library of all known solutions to the Einstein field equations in general relativity. The "Equivalence Problem" involves determining when two solutions are physically equivalent. This requires calculating physical and geometric features to characterize each solution independently of any coordinate system. One of the principal features used to characterize the solutions is the degree of symmetry or the isometry group of the space-time metric. We have focused on the solutions to the Einstein-Maxwell field equations and compared the isometry group of the space-time metric to the symmetry group of the electromagnetic fields for all known solutions. To further characterize these solutions, we have determined whether the electromagnetic fields are null. These characterizations have been added to the library of solutions of the Einstein field equations.

Einstein's theory of general relativity is the most widely accepted theory of space, time, and gravitation. It is the framework for our current understanding of the universe, the nature of condensed massive objects, gravitational radiation, and how galaxies distort light passing by them. Without this theory, GPS technology, which is a hundred-billion-dollar industry, would not be accurate. General relativistic models involve complicated geometry and tensor analysis. Calculations, if done by hand, are long and tedious, offering ample opportunities for error along the way. Modern computational systems, such as Maple and Mathematica, allow users to perform these operations in hours or even minutes; calculations that might otherwise take several weeks.

The Einstein field equations, which come from general relativity, describe the effect of spacetime curvature on gravitation and the solutions to these equations are the spacetime geometry, or metric, of a system. Many solutions to the Einstein equations have been found by making assumptions, such as symmetry. Researchers Charles Torre, my mentor, and Ian Anderson and their students in the Mathematical Physics group, are working on what is called the Digital Einstein Project.[1] The purpose of this project is to create an interactive computer-based encyclopedia of all of the thousands of known solutions to the Einstein field equations. The current goal of the Digital Einstein Project is to incorporate and verify all of the results contained in *Exact Solutions of Einstein's Field Equations*[2] into an easily accessible digital encyclopedia. This text represents all of the known gravitational fields (to date) for different sources, symmetries, and boundary conditions.

An issue that arises as a central part of the Digital Einstein Project is determining whether two gravitational fields are physically distinct. In mathematical terms, this means establishing when two metric tensors, or gravitational fields, are different only because of the choice of coordinate system. This is known as the "Equivalence Problem." As the digital encyclopedia is assembled, each entry includes details about its physical and geometric properties. These details characterize the solution. Within the Mathematical Physics group at USU, it has been shown that only a finite number of these properties are required to characterize each solution uniquely. All of these properties can be calculated using the Differential Geometry[3] (DG) package in Maple.[4] Once this characterization has been done, it can be determined if a solution is equivalent to another, thus creating at answer to the Equivalence Problem.

One of the principal features used to characterize spacetimes is its degree of symmetry. This is more precisely called the isometry group of the spacetime metric. The bulk of these symmetries are continuous symmetries. The isometry group of a metric can be characterized in terms of its infinitesimal generators, which are known as Killing vector fields. These Killing vector fields generate the symmetries of each gravitational field. As part of the Digital Einstein Project, the Killing vectors are built into the encyclopedia of exact solutions to the Einstein field equations and can be retrieved when needed for calculations.

A significant number (130) of the known solutions to the Einstein equations include electromagnetic fields as well as gravitational fields, known as Einstein-Maxwell solutions. A famous example is the Reissner-Nordstrom solution which gives the gravitational field surrounding a spherical, electrically charged, massive object. When there are electromagnetic sources of gravity, such as in these Einstein-Maxwell solutions, the degree of symmetry of the electromagnetic field can be less than the degree of symmetry of the gravitational field it creates. In fact, the symmetries of the electromagnetic field is always a subset of the symmetries of the gravitational field. This is an unusual physical situation. There is no analog of this situation in Newton's theory of gravity. This means that for the solutions of the Einstein equations that involve electromagnetic fields, the Einstein-Maxwell solutions, there is an important geometric property that must be determined: whether or not the electromagnetic field "inherits" the symmetry of the gravitational field. This was the goal of my research: to compute this property for all the known Einstein-Maxwell solutions to the Einstein field equations.

The first step was to determine the degree of symmetry of the gravitational field and then compare it to the degree of symmetry of the electromagnetic field. This was done by calculating the Lie Derivative of each metric solution in the direction of the Killing vectors. Then, I would compare those results with the Lie Derivative of the electromagnetic field for each solution. Because I was working with a large number of solutions, I wrote code in Maple that enabled me to check and compare these Lie Derivatives quickly. An example of this code is shown below.

```
Data:=MetricSearch(["PrimaryDescription" = "EinsteinMaxwell"]):

for j from 1 to nops(Data) do print(j, op(Data[j]));
g,   F,   KV:=   op(Retrieve(op(Data[j]),   manifoldname=P,   output   =
["Metric", "ElectromagneticField", "KillingVectors"]));
LDg:=LieDerivative(KV,g);
LDF:=LieDerivative(KV,F);
print(LDg);
end do:
```

The Lie derivative of the electromagnetic field in the direction of the Killing vectors will generate the symmetries for the electromagnetic field. When the degree of symmetry of the electromagnetic field is the same as the degree of symmetry of the gravitational field, this means the electromagnetic field takes on or *inherits* the symmetry of the gravitational field. For solutions where this is true, the spacetime is classified as "Inheriting." Otherwise, the spacetime is "NonInheriting."

To further characterize each solution, I also determined whether or not each electromagnetic field was null. An electromagnetic field is null when the electric field and the magnetic field are perpendicular and of the same magnitude. For solutions where this is true, the spacetime is classified as “Null.” Otherwise, the spacetime is “NonNull.” This characterization was done in a few ways, in order to verify the results. The first way was to find out if the solution is already listed as null. I wrote code in Maple that allowed me to retrieve the descriptions of each solution. If the solution is listed as “Pure Radiation”, this implies that it is null. Next, I calculated whether the solution was null. This was done in two different ways. First, if the square of the Ricci Tensor of the solution is equal to zero, this implies that it is null. Second, I directly checked that the electric field and magnetic field were orthogonal and that they had the same magnitude using the electromagnetic field strength tensor and the permutation symbol. Because of the large number of solutions, I wrote code that enabled me to do these calculations quickly for all of the solutions. An example of this code is shown below.

```
for j from 1 to nops(NonInheriting) do print(NonInheriting[j]);
g, F:=op(Retrieve(op(NonInheriting[j]), manifoldname=P, output =
["Metric", "ElectromagneticField"]));
alpha:=TensorInnerProduct(g, F, F);
if alpha = 0 then
PossiblyNullNonInheriting:=[op(PossiblyNullNonInheriting),
NonInheriting[j]];
end if;
end do:

for j from 1 to nops(PossiblyNullNonInheriting) do
print(PossiblyNullNonInheriting[j]);
g, F:=op(Retrieve(op(PossiblyNullNonInheriting[j]), manifoldname=P,
output = ["Metric", "ElectromagneticField"]));
G:=InverseMetric(g);
RaisedF:=RaiseLowerIndices(G,F,[1,2]);
EPS:=evalDG(PS &t RaisedF &t RaisedF);
beta:=ContractIndices(EPS, [[1,5],[2,6],[3,7],[4,8]]);
print(beta);
if beta = 0 then NullNonInheriting1:=[op(NullNonInheriting1),
PossiblyNullNonInheriting[j]];
end if;
end do:
```

Of the 130 Einstein-Maxwell solutions that are currently listed in the digital encyclopedia, 76 are “Inheriting” and 47 are “NonInheriting”. The rest were situations for which the Killing vector fields do not exist, and so the investigation did not apply, or the solutions are still analytically not under control and so those solutions have yet to be determined. Of the 130 Einstein-Maxwell solutions that are currently listed in the digital encyclopedia, 37 are "Null" and 86 are "NonNull". Again, there were a few situations that analytically are still not under control.

This project gave many opportunities to add to and refine the digital encyclopedia. Each step and each calculation required testing and troubleshooting of both my code and the encyclopedia. Whether it was fixing typos or making the entries more uniform and therefore easier to search, this project contributed to getting the encyclopedia closer to becoming the "gold standard" reference for solutions to the Einstein field equations.

There also seemed to be an interesting correlation between "Inheriting" and "NonNull" as well as "NonInheriting" and "Null." Of the 76 "Inheriting" solutions, 72 are "NonNull." Of the 47 "NonInheriting" solutions, 32 are "Null." Looking at it the other direction, of the 37 "Null" solutions, 32 are "NonInheriting." Of the 86 "NonNull" solutions, 72 are "Inheriting." It seems that, generally, if a solution is "Inheriting" it tends to also be "NonNull" and if a solution is "NonInheriting" it tends to also be "Null" and vice versa.[5] The cause of this correlation is unknown and requires further investigation. The fact that there could be some relation was unknown before this research was conducted.

1. Funded by NSF grant # OCI-1148331.
2. "Exact Solutions of Einstein's Field Equations", H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Cambridge University Press (2003).
3. "New symbolic tools for differential geometry, gravitation, and field theory," Ian Anderson and Charles Torre, Journal of Mathematical Physics, **53**, 013511 (2012).
4. "Introduction to Maple," Andre Heck, Springer (2003).
5. See Table 1 in Appendix.

Appendix

Table 1 - Comparison of results

	Inheriting	NonInheriting
Null	5	32
NonNull	72	15

Table 2 - Summary of all the information I determined.

	Inheriting	NonInheriting	Null	NonNull	Notes
["HawkingEllis", 1, [5, 26, 1]]	x			x	
["Stephani", 1, [12, 7, 1]]	x		x		
["Stephani", 1, [12, 12, 1]]		x	x		
["Stephani", 1, [12, 12, 2]]		x	x		
["Stephani", 1, [12, 12, 3]]		x	x		
["Stephani", 1, [12, 12, 4]]		x	x		
["Stephani", 1, [12, 16, 1]]	x			x	
["Stephani", 1, [12, 18, 1]]	x			x	
["Stephani", 1, [12, 19, 1]]	x			x	
["Stephani", 1, [12, 21, 1]]		x		x	
["Stephani", 1, [12, 36, 1]]		x	x		
["Stephani", 1, [12, 37, 2]]		x	x		
["Stephani", 1, [12, 37, 4]]		x	x		
["Stephani", 1, [12, 37, 6]]		x	x		
["Stephani", 1, [12, 37, 7]]		x	x		

["Stephani", 1, [12, 37, 8]]		x	x		
["Stephani", 1, [12, 37, 9]]		x	x		
["Stephani", 1, [12, 38, 3]]	x		x		
["Stephani", 1, [13, 46, 1]]		x	x		
["Stephani", 1, [13, 48, 1]]	x			x	
["Stephani", 1, [13, 69, 1]]	x			x	
["Stephani", 1, [13, 71, 1]]	x			x	
["Stephani", 1, [13, 71, 2]]	x			x	
["Stephani", 1, [13, 71, 3]]	x			x	
["Stephani", 1, [13, 72, 1]]	x			x	
["Stephani", 1, [13, 73, 1]]	x			x	
["Stephani", 1, [13, 74, 1]]	x			x	
["Stephani", 1, [13, 74, 2]]	x			x	
["Stephani", 1, [13, 74, 3]]	x			x	
["Stephani", 1, [13, 74, 4]]	x			x	
["Stephani", 1, [13, 74, 5]]	x			x	
["Stephani", 1, [13, 74, 6]]	x			x	
["Stephani", 1, [13, 76, 1]]	x			x	
["Stephani", 1, [13, 77, 1]]	x			x	
["Stephani", 1, [13, 77, 2]]	x			x	
["Stephani", 1, [15, 12, 1]]	x			x	
["Stephani", 1, [15, 12, 2]]	x			x	
["Stephani", 1, [15, 12, 3]]	x			x	
["Stephani", 1, [15, 17, 1]]	x			x	
["Stephani", 1, [15, 17, 2]]	x			x	
["Stephani", 1, [15, 17, 3]]	x			x	
["Stephani", 1, [15, 18, 1]]		x	x		
["Stephani", 1, [15, 21, 1]]	x			x	
["Stephani", 1, [15, 21, 2]]	x			x	

["Stephani", 1, [15, 27, 2]]	x			x	
["Stephani", 1, [15, 27, 3]]	x			x	
["Stephani", 1, [15, 27, 4]]		x	x		
["Stephani", 1, [15, 27, 5]]		x	x		
["Stephani", 1, [15, 27, 6]]		x	x		
["Stephani", 1, [15, 28, 1]]	x			x	
["Stephani", 1, [15, 32, 1]]		x	x		
["Stephani", 1, [21, 4, 1]]	x			x	
["Stephani", 1, [21, 5, 1]]	x			x	
["Stephani", 1, [21, 6, 1]]	x			x	
["Stephani", 1, [21, 7, 1]]	x			x	
["Stephani", 1, [21, 10, 1]]	x			x	
["Stephani", 1, [21, 11, 1]]	x			x	
["Stephani", 1, [21, 16, 1]]	x			x	
["Stephani", 1, [21, 17, 1]]	x			x	
["Stephani", 1, [21, 20, 1]]	x			x	
["Stephani", 1, [21, 22, 1]]	x			x	
["Stephani", 1, [21, 24, 1]]	x			x	
["Stephani", 1, [21, 31, 1]]	x			x	
["Stephani", 1, [21, 35, 1]]	x			x	
["Stephani", 1, [22, 11, 1]]	x			x	
["Stephani", 1, [22, 12, 1]]	x			x	
["Stephani", 1, [22, 13, 1]]	x			x	
["Stephani", 1, [22, 14, 1]]	x			x	
["Stephani", 1, [22, 15, 1]]	x			x	
["Stephani", 1, [22, 16, 1]]	x			x	
["Stephani", 1, [22, 17, 1]]		x	x		
["Stephani", 1, [22, 64, 1]]	x			x	
["Stephani", 1, [22, 67, 1]]	x			x	

["Stephani", 1, [22, 67, 2]]	x			x	
["Stephani", 1, [24, 21, 1]]	x			x	
["Stephani", 1, [24, 22, 1]]	x			x	
["Stephani", 1, [24, 35, 1]]	x		x		
["Stephani", 1, [24, 37, 1]]	x		x		
["Stephani", 1, [24, 37, 2]]		x	x		
["Stephani", 1, [24, 37, 3]]		x	x		
["Stephani", 1, [24, 37, 4]]		x	x		
["Stephani", 1, [24, 37, 5]]		x	x		
["Stephani", 1, [24, 37, 6]]		x	x		
["Stephani", 1, [24, 37, 7]]		x	x		
["Stephani", 1, [24, 37, 8]]		x	x		
["Stephani", 1, [24, 38, 2]]		x		x	
["Stephani", 1, [24, 46, 1]]	x		x		
["Stephani", 1, [24, 47, 1]]		x	x		
["Stephani", 1, [24, 51, 1]]		x	x		
["Stephani", 1, [26, 5, 1]]	-	-	-	-	Skipped
["Stephani", 1, [26, 6, 1]]	x			x	
["Stephani", 1, [28, 41, 1]]		x	x		
["Stephani", 1, [28, 43, 1]]		x	x		
["Stephani", 1, [28, 44, 1]]	x			x	
["Stephani", 1, [28, 44, 2]]	x			x	
["Stephani", 1, [28, 44, 3]]	x			x	
["Stephani", 1, [28, 44, 4]]	x			x	
["Stephani", 1, [28, 44, 5]]	x			x	
["Stephani", 1, [28, 44, 6]]	x			x	
["Stephani", 1, [28, 45, 1]]	x			x	
["Stephani", 1, [28, 45, 2]]	x			x	
["Stephani", 1, [28, 46, 1]]		x		x	

["Stephani", 1, [28, 46, 2]]		x		x	
["Stephani", 1, [28, 53, 1]]		x	x		
["Stephani", 1, [28, 53, 2]]		x	x		
["Stephani", 1, [28, 55, 1]]		x		x	
["Stephani", 1, [28, 55, 2]]		x		x	
["Stephani", 1, [28, 56.1, 1]]		x		x	
["Stephani", 1, [28, 56.2, 2]]		x		x	
["Stephani", 1, [28, 56.2, 3]]		x		x	
["Stephani", 1, [28, 56.3, 1]]		x		x	
["Stephani", 1, [28, 56.4, 1]]		x		x	
["Stephani", 1, [28, 56.5, 1]]		x		x	
["Stephani", 1, [28, 56.6, 1]]		x		x	
["Stephani", 1, [28, 58.2, 1]]	-	-	-	-	No Killing Vectors Exist
["Stephani", 1, [28, 58.3, 1]]	x			x	
["Stephani", 1, [28, 58.3, 2]]	x			x	
["Stephani", 1, [28, 58.4, 1]]	-	-	-	-	No Killing Vectors Exist
["Stephani", 1, [28, 60, 1]]		x		x	
["Stephani", 1, [28, 61, 1]]	x			x	
["Stephani", 1, [28, 64, 1]]		x		x	
["Stephani", 1, [28, 66, 1]]	x			x	
["Stephani", 1, [28, 67, 1]]	x			x	
["Stephani", 1, [28, 68, 1]]	x			x	
["Stephani", 1, [32, 59, 1]]	-	-	-	-	Skipped
["Stephani", 1, [32, 60, 1]]	-	-	-	-	Skipped
["Stephani", 1, [32, 61, 1]]	-	-	-	-	Skipped
["Stephani", 1, [35, 33, 1]]		x	x		
["Stephani", 1, [35, 35, 1]]	x			x	
["Stephani", 1, [37, 98, 1]]	x			x	