Natural symmetries of the Yang–Mills equations

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A natural generalized symmetry of the Yang–Mills equations is defined as an infinitesimal transformation of the Yang–Mills field, built in a local, gauge invariant, and Poincaré invariant fashion from the Yang–Mills field strength and its derivatives to any order, which maps solutions of the field equations to other solutions. On the jet bundle of Yang–Mills connections a spinorial coordinate system is introduced that is adapted to the solution subspace defined by the Yang–Mills equations. In terms of this coordinate system the complete classification of natural symmetries is carried out in a straightforward manner. It is found that all natural symmetries of the Yang–Mills equations stem from the gauge transformations admitted by the equations. © 1995 American Institute of Physics.

I. INTRODUCTION

Yang–Mills theory, by which we mean any non-Abelian gauge theory, has provided a fruitful area of study for both physicists and mathematicians. Physicists have used Yang–Mills theory to describe the strong and electroweak interactions.1 Applications of the Yang–Mills equations in mathematics have been found in several areas; an important example is given by the recent discovery of an intimate relation between reductions of the Yang–Mills equations and a large class of integrable differential equations.2 Whether one is interested in physical or mathematical applications of the Yang–Mills equations, there are certain basic structural properties of these equations that one would like to understand. One of the most fundamental properties to be examined is the class of generalized symmetries admitted by the equations.3,4 Roughly speaking, by generalized symmetries we mean infinitesimal transformations of the fields that map solutions to solutions. The transformations are to be constructed in a local fashion from the fields and their derivatives to any finite order.5 Given a set of differential equations, the presence of symmetries is connected with the existence of conservation laws, the construction of solution generating techniques, and integrability properties of the equations.3,4,6,7

There are, of course, manifest symmetries that are built into the Yang–Mills equations, namely, the Poincaré and gauge symmetries. The Poincaré symmetry is responsible for ten conservation laws, while the gauge symmetry leads to trivial conservation laws. In recent years it has been found that many nonlinear differential equations admit “hidden symmetries.” For example, the sine-Gordon equation is a nonlinear wave equation with a built-in Poincaré symmetry group. Remarkably, this equation admits an infinite number of higher-order generalized symmetries and corresponding conservation laws,3 and this fact is intimately associated with the integrability of the sine-Gordon equation. In light of such examples, and given the strong connection between the Yang–Mills equations and integrable systems, it is tempting to speculate that the Yang–Mills equations will admit higher-order symmetries and conservation laws. On the other hand, Vinogradov has argued that, for nonlinear equations in more than two independent variables and with nondegenerate symbol, one generally cannot expect to find generalized symmetries.8 The Yang–Mills equations do have a degenerate symbol, so the argument of Ref. 8 cannot be directly applied here. By studying the Yang–Mills symmetries we can thus generate evidence for/against extensions of the Vinogradov argument to more general systems of differential equations.

In this article we begin a classification of all generalized symmetries admitted by the Yang–Mills equations on a flat four-dimensional space–time. Given the manifest gauge and Poincaré covariance of the Yang–Mills equations, it is reasonable to search for symmetries that are con-
structured in a gauge and Poincaré covariant manner from the Yang–Mills field strength and its
gauge-covariant derivatives. We call such symmetries natural generalized symmetries. In order to
classify natural symmetries we borrow techniques from a recent classification of all symmetries
for the vacuum Einstein equations. The principal tool used in Ref. 9 was an adapted set of spinor
coordinates on the jet space of Einstein metrics. These coordinates derive, in part, from Penrose's
notion of an "exact set of fields." As noted by Penrose, an exact set of fields exists for the
Yang–Mills equations, and this leads, via a relatively quick and straightforward analysis which is
very similar to that of Ref. 9, to a complete classification of all natural symmetries of the Yang–
Mills equations. Thus the power of combining spinor and jet space techniques has a more general
scope than merely in gravitation theory.

In Sec. II we summarize the preliminary results needed for our analysis. The requirement that
symmetries be built locally is handled by employing the jet bundle description of Yang–Mills
theory, and it is on the jet bundle that the adapted spinor coordinates are defined. Various technical
results needed for our analysis are also presented. In Sec. III we analyze the linearized Yang–Mills
equations and classify the natural symmetries. We find that all natural symmetries of the Yang–
Mills equations stem from the gauge transformations admitted by the equations. In Sec. IV we
comment on the generalizations needed to effect a complete classification of all symmetries of the
Yang–Mills equations.

II. PRELIMINARIES

We choose space–time to be the manifold \( M = \mathbb{R}^4 \) equipped with a flat metric \( \eta_{ab} \) of signature
(− + + +). The unique torsion-free derivative operator compatible with \( \eta_{ab} \) will be denoted by \( \partial_a \).
To define the Yang–Mills field we consider a principal bundle \( \pi: P \to M \) over space–time with
the structure group given by any Lie group \( G \). Because every bundle over \( \mathbb{R}^4 \) is trivial, we can
globally represent a connection on \( \pi: P \to M \) by a one-form \( A_a \) on \( M \) taking values in the Lie
algebra \( g \) of \( G \). We call this one-form the Yang–Mills field. The curvature of the connection is
represented by a two-form \( F_{ab} \) on \( M \) taking values in \( g \), which will be called the Yang–Mills field
strength. The field strength is given in terms of the Yang–Mills field by

\[
F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b], \tag{2.1}
\]

where \([\cdot, \cdot]\) is the bracket of \( g \). If \( \tau_a \) is a basis for \( g \), we write

\[
A_a = A^a \tau_a \quad \text{and} \quad F_{ab} = F^a_{ab} \tau_a. \tag{2.2}
\]

We then have

\[
F^a_{ab} = \partial_a A^b - \partial_b A^a + \kappa^a_{\gamma \delta} A^\gamma A^\delta, \tag{2.3}
\]

where \( \kappa^a_{\gamma \delta} \) are the structure constants of the Lie algebra \( g \).

Given a representation \( \rho \) of the group \( G \) we have an associated vector bundle \( \pi: E \to M \). The
Yang–Mills field defines a derivative operator \( \nabla_a \) on sections \( s: M \to E \) via

\[
\nabla_a s = \partial_a s + A_a \cdot s, \tag{2.4}
\]

where we use the raised dot (\( \cdot \)) to indicate the action of the Lie algebra on sections that is defined
by \( \rho \). The Yang–Mills field strength measures the failure of this derivative operator to commute;
we have the identity

\[
\nabla_{[a} \nabla_{b]} s = \frac{1}{2} F_{ab} \cdot s. \tag{2.5}
\]
The Yang–Mills field strength can be viewed as a (two-form-valued) section of the vector bundle defined by the adjoint representation of $G$. We thus have that

$$\nabla_a F_{bc} = \partial_a F_{bc} + [A_a, F_{bc}]$$

and the field strength satisfies the Bianchi identities

$$\nabla_a F_{bc} = 0.$$  \hspace{1cm} (2.7)

The Yang–Mills field equations are given by

$$\nabla^a F_{ab} = \partial^a F_{ab} + [A^a, F_{ab}] = 0.$$  \hspace{1cm} (2.8)

In terms of the basis $\tau^a$, we have

$$\nabla^a F_{ab} = \partial^a F_{ab} + \kappa^a_{\beta} A^{\beta a} F_{ab} = 0.$$  \hspace{1cm} (2.9)

Let $\pi: \mathcal{Q} \rightarrow M$ be the bundle of $g$-valued one-forms on $M$. A section $A: M \rightarrow \mathcal{Q}$ of this bundle is a Yang–Mills field $A_a(x)$. Let $J^k(\mathcal{Q})$ be the bundle of $k$th-order jets of sections of $\mathcal{Q}$. A point $\sigma$ in $J^k(\mathcal{Q})$ is defined by a space–time point $x$, the Yang–Mills field at $x$, and all its derivatives to order $k$ at $x$. A section $A: M \rightarrow \mathcal{Q}$ lifts to give a section $j^k(A): M \rightarrow J^k(\mathcal{Q})$, which is called the $k$-jet of $A$. If we write

$$A_{a,b_1 \cdots b_k}(j^k(A)(x)) = \partial_{b_1} \partial_{b_2} \cdots \partial_{b_k} A_a(x)$$

then a point $\sigma \in J^k(\mathcal{Q})$ is given by

$$\sigma = (x, A_a, A_a, b_1, \cdots, A_a, b_1 \cdots b_k).$$

The total derivative $D_{\sigma}f$ of a function

$$f = f(x, A_a, A_a, b_1, \cdots, A_a, b_1 \cdots b_k)$$

on $J^k(\mathcal{Q})$ is defined by

$$D_{\sigma}f = \frac{\partial f}{\partial x^c} + \frac{\partial f}{\partial A_a^c} A_a^c + \frac{\partial f}{\partial A_a^{b_1}} A_{a,b_1}^c + \cdots + \frac{\partial f}{\partial A_a, b_1 \cdots b_k} A_{a,b_1 \cdots b_k}^c.$$  \hspace{1cm} (2.13)

The main property of the total derivative is that it represents on the jet bundle the effect of the derivative operator $\partial_a$ on fields. More precisely, if $f: J^{k-1}(\mathcal{Q}) \rightarrow R$ is a smooth function and $A: M \rightarrow \mathcal{Q}$ is a Yang–Mills field with $k$-jet $j^k(A): M \rightarrow J^k(\mathcal{Q})$, then we have the identity

$$(D_{\sigma}f)^* j^k(A)(x) = \partial_a (f^* j^{k-1}(A)(x)).$$

The field equations (2.8) define a submanifold

$$\mathcal{R}^2 \hookrightarrow J^2(\mathcal{Q}),$$

which we call the equation manifold. The $k$th (total) derivative of the field equations defines the prolonged equation manifold

$$\mathcal{R}^{k+2} \hookrightarrow J^{k+2}(\mathcal{Q}).$$
A generalized symmetry for the field equations (2.8) is an infinitesimal map, depending locally on the independent variables, the dependent variables, and the derivatives of the dependent variables to some finite order, which carries solutions to nearby solutions. Geometrically, a generalized symmetry is a vector field on $J^\infty(Q)$ which is tangent to $\mathcal{R}^\infty$ and preserves the contact ideal associated to $J^\infty(Q)$. A generalized symmetry of order $k$ for the Yang–Mills equations can be represented as a map from $J^k(Q)$ into the bundle of $g$-valued one-forms on $M$. We denote this map by $C_a = C_a(a,b_1,...,b_k)$, and we write

$$C_a = C_a(x,A_a,A_{a,b_1},...,A_{a,b_k}).$$

We say a generalized symmetry is trivial if it vanishes on the prolonged equation manifold. Two generalized symmetries are deemed equivalent if they differ by a trivial symmetry. Any generalized symmetry of the form (2.15) is equivalent to a generalized symmetry obtained by restricting Eq. (2.15) to $\mathcal{R}^k$, that is, we can assume that $C_a$ is a map from $\mathcal{R}^k$ into the bundle of $g$-valued one-forms on $M$.

The following proposition is easily established from the theory of generalized symmetries.3

**Proposition 2.1:** The functions

$$C_a = C_a(x,A_a,A_{a,b_1},...,A_{a,b_k})$$

represent a $k$th-order generalized symmetry for the Yang–Mills field equations if and only if

$$\nabla^b \nabla_b C_a - \nabla^b \nabla_a C_b + [C_b, F_b] = 0 \quad \text{on} \quad \mathcal{R}^{k+2},$$

(2.16)

where

$$\nabla_b C_a = D_b C_a + [A_b, C_a].$$

(2.17)

Note that the defining equations (2.16) for a generalized symmetry are the linearized field equations.

Familiar examples of symmetries of the Yang–Mills equations stem from the gauge and conformal invariance of these equations. If $A_a(x)$ is a solution to Eq. (2.8), and $\phi: M \rightarrow M$, is a conformal isometry of the space–time $(M, \eta)$, then $\phi^* A_a(x)$ is also a solution to Eq. (2.8). Here we define $\phi^* A_a(x)$ to be the pullback of $A_a$ in which $A_a(x)$ is viewed as a collection of one-forms on $M$. The infinitesimal form of this conformal symmetry leads to the following proposition.

**Proposition 2.2:** Let $\xi^a(x)$ be a conformal Killing vector field for the space–time $(M, \eta)$, then

$$C_a = \xi^b C_a F_b$$

(2.18)

is a generalized symmetry of the Yang–Mills equations.

The infinitesimal transformation defined by Eq. (2.18) is the “gauge covariant Lie derivative” of $A_a$ along $\xi^b$.

Let $U: M \rightarrow P$ be a section of the principal bundle. If $A_a$ is a solution to the Yang–Mills equations (2.8) then

$$A_a^U = U^{-1} A_a U + U^{-1} \partial_a U$$

(2.19)

is also a solution to Eqs. (2.8). $A_a^U$ is called the gauge transformation of $A$. The infinitesimal form of the gauge transformations leads to the following proposition.

**Proposition 2.3:** Let $\Lambda(x) = \Lambda^a(x) \tau_a$ be a $g$-valued function on $M$, then

$$C_a = \nabla_a \Lambda = \partial_a \Lambda(x) + [A_a, \Lambda(x)]$$

(2.20)
is a generalized symmetry of the Yang–Mills equations.

The symmetries exhibited in these two propositions should really be viewed as evolutionary forms of point symmetries. The gauge symmetry of Proposition 2.3 can, however, be generalized to the case where \( \Lambda \) is constructed locally from the Yang–Mills field and its derivatives to any order, thus providing examples of bona fide generalized symmetries for the Yang–Mills equations.

Proposition 2.4: Let

\[ \Lambda = \Lambda(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \cdots b_{k-1}}) \]

be a \( g \)-valued function on \( J^{k-1}(\mathcal{Q}) \), then

\[ C_a = \nabla_a \Lambda = D_a \Lambda + [A_a, \Lambda] \]  

(2.21)

is a \( k \)-th order generalized symmetry of the Yang–Mills equations.

We will call these symmetries generalized gauge symmetries.

In this article we will classify natural generalized symmetries. These are generalized symmetries that have a simple behavior under Poincaré and gauge transformations of the Yang–Mills field. More precisely, the gauge transformations and isometries can be lifted (by prolongation) to act on \( J^k(\mathcal{Q}) \), and, in terms of these lifted actions, we have the following definition of a natural generalized symmetry.

Definition 2.5: Let \( \phi: M \to \mathcal{M} \) be an isometry of the space–time \( (M, \eta) \), and \( U: M \to P \) a section of the principal bundle. A natural generalized symmetry is a function

\[ C_a = C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \cdots b_k}) \]

satisfying Eq. (2.16), and such that for any \( \phi \)

\[ C_a(\phi^{-1}(x), \phi^*A_a, \phi^*A_{a,b_1}, \ldots, \phi^*A_{a,b_1 \cdots b_k}) = \phi^*C_a(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \cdots b_k}) \]  

(2.22)

and for any \( U \)

\[ C_a(x, A_a^U, A_{a,b_1}^U, \ldots, A_{a,b_1 \cdots b_k}^U) = U^{-1}C_a U(x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \cdots b_k}) \].  

(2.23)

We remark that, according to this definition, a generalized gauge symmetry can be a natural generalized symmetry, but the conformal symmetry of Proposition 2.2 is not a natural symmetry. We also note that we could have defined a natural symmetry using the full conformal group; by only using the Poincaré subgroup we put fewer restrictions on the allowed symmetries.

To elucidate the structure of a natural generalized symmetry we will construct a set of adapted coordinates for \( J^k(\mathcal{Q}) \). To this end, let us define

\[ A_{\alpha b_0 b_1 \cdots b_k} = \partial_{(b_1} \cdots \partial_{b_k} A_{\alpha b_0)}, \quad k = 0, 1, \ldots, \]

(2.24)

and

\[ Q_{\alpha b_0 b_1 \cdots b_k} = \nabla_{(b_k} \nabla_{b_{k-1}} \cdots \nabla_{b_2} F_{b_1 b_0}^\alpha), \quad k = 1, 2, \ldots \]

(2.25)

Both \( A_{\alpha b_0 b_1 \cdots b_k} \) and \( Q_{\alpha b_0 b_1 \cdots b_k} \) depend on the Yang–Mills field and its first \( k \) derivatives; we denote these variables by \( A^k \) and \( Q^k \). Each of these variables is algebraically irreducible in the sense that

\[ A_{\alpha b_0 b_1 \cdots b_k} - A_{\alpha (b_0 b_1 \cdots b_k)}, \]

(2.26)
\[ Q_{b_0, b_1, \ldots, b_k}^\alpha = Q_{b_0, (b_1, \ldots, b_k)}^\alpha \quad \text{and} \quad Q_{(b_0, b_1, \ldots, b_k)}^\alpha = 0. \quad (2.27) \]

We have the identity
\[
\partial_{b_1} \cdots \partial_{b_k} A_{b_0}^\alpha = A_{b_0}^\alpha b_1 \cdots b_k + \frac{k}{k+1} Q_{(b_0, b_1, \ldots, b_k)}^\alpha + L_{b_0 b_1, \ldots, b_k}^\alpha, \quad (2.28)
\]
where \( L_{b_0 b_1, \ldots, b_k}^\alpha \) depends on the Yang–Mills field and its derivatives to order \( k - 1 \). From this identity it is straightforward to show that coordinates for \( J^k(\mathcal{Q}) \) are given by
\[
(x, A_{b_0}^\alpha, A_{b_0 b_1}^\alpha, Q_{b_0, b_1}^\alpha, \ldots, A_{b_0 b_1, \ldots, b_k}^\alpha, Q_{b_0, b_1, \ldots, b_k}^\alpha). \quad (2.29)
\]

Here we have taken the convenient liberty of using the same symbols \( Q \) and \( A \) to denote the fields on space–time and functions on jet space. Every generalized symmetry can be expressed as a function of the variables (2.29)
\[
C_a^\alpha = C_a^\alpha(x, A_{b_0}^\alpha, A_{b_0 b_1}^\alpha, Q_{b_0, b_1}^\alpha, \ldots, A_{b_0 b_1, \ldots, b_k}^\alpha, Q_{b_0, b_1, \ldots, b_k}^\alpha). \quad (2.30)
\]

We can now characterize natural generalized symmetries as follows.

**Proposition 2.6:** Let \( C_a^\alpha \) be a natural generalized symmetry of the Yang–Mills equations of order \( k \). Then \( C_a^\alpha \) can be expressed as a function of the variables \( Q_{b_0, b_1, \ldots, b_k}^\alpha \) for \( l = 1, 2, \ldots, k \), that is,
\[
C_a^\alpha = C_a(x, A_{b_0}^\alpha, A_{b_0 b_1}^\alpha, Q_{b_0, b_1}^\alpha, \ldots, A_{b_0 b_1, \ldots, b_k}^\alpha, Q_{b_0, b_1, \ldots, b_k}^\alpha). \quad (2.31)
\]

**Proof:** We begin by analyzing the requirement (2.23). Let \( U(t) : K \times M \rightarrow P \) be a one-parameter family of gauge transformations such that \( U(0) \) is the identity transformation. The derivative
\[
\Lambda = \left. \frac{dU}{dt} \right|_{t=0}
\]
is a \( g \)-valued function on \( M \) defining an infinitesimal gauge transformation. Under an infinitesimal transformation \( \Lambda \) associated to \( U(t) \) we have that
\[
\frac{d}{dt} A_{b_1, \ldots, b_k}^\alpha|_{t=0} = \Lambda_{b_1, \ldots, b_k}^\alpha + \{ \star \}, \quad (2.33)
\]
where \( \Lambda_{b_1, \ldots, b_l}^\alpha \), for \( l = 0, 1, 2, \ldots, k \) (along with \( x \) defines the \( k \)-jet of \( \Lambda \), and \( \{ \star \} \) denotes terms involving \( A^l \) and \( \Lambda_{b_1, \ldots, b_l}^\alpha \), for \( l = 0, 1, 2, \ldots, k-1 \). We also have that
\[
\frac{d}{dt} Q_{a, b_1, \ldots, b_k}^\alpha|_{t=0} = \kappa_{\gamma \beta}^\alpha \Lambda_{b_1, \ldots, b_k}^\gamma Q_{a, b_1, \ldots, b_k}^\beta. \quad (2.34)
\]

We now demand that Eq. (2.23) holds for any \( U(t) \) and differentiate this equation with respect to \( t \) to find
\[
\frac{\partial C_a^\alpha}{\partial A_{b_0, b_1, \ldots, b_k}^\beta} \Lambda_{b_0, b_1, \ldots, b_k}^\beta + R_a^\alpha = \Lambda_{b_1, \ldots, b_k}^\gamma \kappa_{\beta \gamma}^\alpha C_a^\beta, \quad (2.35)
\]
where \( R_a^\alpha \) is independent of the variables \( \Lambda_{b_0, b_1, \ldots, b_k}^\beta \). Equation (2.35) must hold for all values of
\( \Lambda_{b_0, b_1, \ldots, b_k}^\beta \) and this implies that
A simple induction argument then establishes that

$$\frac{\partial C^\alpha_a}{\partial A^\beta_{b_0\cdots b_k}} = 0.$$  

(2.36)

for $l = 0, 1, \ldots, k$. Thus we have

$$C^\alpha_a = C^\alpha_a(x, Q^\alpha_{b_0, b_1}, Q^\alpha_{b_0, b_1 b_2}, \ldots, Q^\alpha_{b_0, b_1 \cdots b_k}).$$  

(2.38)

It remains to be shown that $C^\alpha_a$ is independent of $x$. Let $x^\mu$ be a global inertial coordinate chart on $\mathcal{M}$, and let $\xi^\mu$ be a translational Killing vector field. In the chart $x^\mu$ the components $\xi^\mu$ are any set of constants

$$\frac{\partial \xi^\mu}{\partial x^\nu} = 0.$$  

(2.39)

If we demand that Eq. (2.22) be satisfied for all translational isometries we have that

$$C^\alpha_a(x^\mu - \xi^\mu, Q^\alpha_{b_0, b_1}, Q^\alpha_{b_0, b_1 b_2}, \ldots, Q^\alpha_{b_0, b_1 \cdots b_k}) = C^\alpha_a(x^\mu, Q^\alpha_{b_0, b_1}, Q^\alpha_{b_0, b_1 b_2}, \ldots, Q^\alpha_{b_0, b_1 \cdots b_k}),$$  

(2.40)

for any constants $\xi^\alpha$, which implies that

$$C^\alpha_a = C^\alpha_a(Q^\alpha_{b_0, b_1}, Q^\alpha_{b_0, b_1 b_2}, \ldots, Q^\alpha_{b_0, b_1 \cdots b_k}).$$  

(2.41)

If $C^\alpha_a = C^\alpha_a(Q^\alpha_{b_0, b_1}, Q^\alpha_{b_0, b_1 b_2}, \ldots, Q^\alpha_{b_0, b_1 \cdots b_k})$ is a natural generalized symmetry of the Yang–Mills equations, then it must satisfy the linearized equations (2.16) at each point of $\mathcal{R}^{k+2}$. To classify solutions to the linearized equations we will construct an explicit parametrization of the prolonged equation manifolds.

In the following proposition $[Q^\alpha_{b_0, b_1 \cdots b_k}]_{\text{trace-free}}$ denotes the completely trace-free part of the tensor $Q^\alpha_{b_0, b_1 \cdots b_k}$ with respect to the metric $\eta_{ab}$.

**Proposition 2.7:** The variables

$$(x^\mu, A^\alpha_{b_0, b_1}, \ldots, A^\alpha_{b_0, b_1 \cdots b_k}, [Q^\alpha_{b_0, b_1}]_{\text{trace-free}}, \ldots, [Q^\alpha_{b_0, b_1 \cdots b_k}]_{\text{trace-free}})$$  

(2.42)

form a global coordinate system for $\mathcal{R}^k$.

**Proof:** The prolonged equation manifold $\mathcal{R}^k$ can be defined by $k$-jets which satisfy

$$\eta^{\mu \nu \cdots} \nabla_{(b_1} \cdots \nabla_{b_l)} \nabla_m F_{n a} = 0.$$  

(2.43)

for $l = 0, 1, 2, \ldots, k - 2$. We express these equations in terms of the variables $Q^j$ via the identity

$$\eta^{\mu \nu \cdots} \nabla_{(b_1} \cdots \nabla_{b_l)} \nabla_m F_{n a} = \left( \frac{l + 2}{l + 3} \right) \eta^{\mu \nu \cdots} \nabla_{Q_{a m b_1 \cdots b_l} - Q_{a b_1 \cdots b_l}} + L_{a b_1 \cdots b_l},$$  

(2.44)

where $L_{a b_1 \cdots b_l} = L_{a (b_1 \cdots b_l)}$ depends on $Q^j$ for $j = 1, 2, \ldots, l$. From Eq. (2.44) we have that
Let $S^p$ denote the vector space of tensors with the algebraic symmetries (2.27) of $Q^p$. Denote by $S_0^p$ the subspace of totally trace-free tensors. Let $T^p$ be the vector space of tensors with the symmetries of $L_{ab_1...b_p}$ and satisfying the trace condition (2.45). Define a linear map $\Psi^p : S^{p+2} \rightarrow T^p$ which takes $W_{a,mnb_1...b_p} \in S^{p+2}$ into $V_{ab_1...b_p} \in T^p$ by the rule

$$V_{ab_1...b_p} = \frac{(l+2)}{l+3} \eta^{mn} [W_{a,mnb_1...b_p} - W_{n,amb_1...b_p}].$$

It is straightforward to show that

$$\text{Ker } \Psi^p = S_0^{p+2} \quad (2.47)$$

and

$$\text{Im } \Psi^p = T^p. \quad (2.48)$$

By virtue of Eqs. (2.47) and (2.48), each point in $\mathcal{H}^k$, $k=2,3,...$, can be uniquely determined as follows. Let us begin with $\mathcal{H}^2$. Choose $x, A_a$ and $Q_{a,b_1}, A_{ab_1},$ and $A_{ab_1b_2}$ arbitrarily. In Eq. (2.44) with $l=0$ we have $L_a=0$, and so, from Eqs. (2.47) and (2.48), we solve Eq. (2.43) by setting all traces of $Q^2$ to zero. $\mathcal{H}^2$ is thus parametrized by

$$(x, A_a, A_{ab_1}, Q_{a,b_1}, A_{ab_1b_2}, [Q_{a,b_1b_2}]_{\text{trace-free}}). \quad (2.49)$$

Now we consider $\mathcal{H}^3$. We choose the coordinates (2.49) and $A^3$ arbitrarily. In the identity (2.44) for $l=1$ we have that $L_{ab_1}$ depends on $[Q^1]_{\text{trace-free}}$ only. By virtue of the surjectivity (2.48) of the map $\Psi^1$ we can solve Eq. (2.43). By virtue of Eq. (2.47) the solution will be uniquely parametrized by $[Q^3]_{\text{trace-free}}, A^3$, and the variables (2.49). By iterating this procedure, we can build every solution to Eq. (2.43), which is viewed as an equation on $J^k(M)$, and the solutions will be uniquely parametrized by the variables (2.42). ◼

In principle, the variables (2.42) can be used to analyze the linearized equation (2.16), but the resulting equations are still rather complicated. Considerable simplifications can be obtained by using a spinor representation of the variables $[Q^k]_{\text{trace-free}}$. Hence we now describe a spinorial coordinate system on $\mathcal{H}^k$. We remark that while all of the results presented to this point are essentially independent of the space-time dimension, our use of spinors will limit the validity of subsequent results to a four-dimensional space-time.

We begin with a brief summary of notation; for more details on spinors, see Ref. 11. The space-time metric and associated derivative operator have the spinor representation

$$\eta_{ab} \leftrightarrow \epsilon_{AB} \epsilon_{A'B'}$$

and

$$\partial_a \leftrightarrow \partial_{AA'}.$$ 

The $\epsilon$ spinors are skew symmetric and nondegenerate at each point of $M$. The Yang–Mills field and field strength have the spinor representation

$$A_b \leftrightarrow A_{BB'},$$

and
In Eq. (2.50) the $g$-valued spinor fields $\Phi$ and $\bar{\Phi}$ are symmetric

$$\Phi_{AB} = \Phi_{(AB)} \quad \text{and} \quad \bar{\Phi}_{A'B'} = \bar{\Phi}_{(A'B')}$$

and correspond to the self-dual and antiself-dual part of the field strength.

The Bianchi identities (2.7) take the spinor form

$$\nabla_A^B \Phi_{AB} = \nabla_A^{A'B'} \bar{\Phi}_{A'B'},$$

while the identities (2.5) become

$$\nabla_{X'(A} \nabla_{B')}^X U = \Phi_{AB} \cdot U$$

and

$$\nabla_{X'(A} \nabla_{B')}^X U = \bar{\Phi}_{A'B'} \cdot U.$$

Given the identities (2.52), the field equations (2.8) are equivalent to

$$\nabla_A^B \Phi_{AB} = 0 = \nabla_A^{A'B'} \bar{\Phi}_{A'B'}.$$

We now present a spinor representation of $[Q^k]_{\text{trace-free}}$.

**Proposition 2.8:** Let the $g$-valued tensor $Q^k$ be defined as in Eq. (2.25), and let $[Q^k]_{\text{trace-free}}$ have the spinor representation

$$[Q_{b_0, b_1, \ldots, b_k}]_{\text{trace-free}} = Q^{b_0, b_1, \ldots, b_k}_{b_0, b_1, \ldots, b_k},$$

then $[Q^k]_{\text{trace-free}}$ admits the unique spinor decomposition

$$Q^{b_0, b_1, \ldots, b_k}_{b_0, b_1, \ldots, b_k} = e^{b_0'(b_0) \Phi_{b_0 b_1 \ldots b_k}} + e^{b_0'(b_1) \bar{\Phi}_{b_0 b_1 \ldots b_k}},$$

where $\Phi_{b_1, \ldots, b_k}$ and $\bar{\Phi}_{b_1, \ldots, b_k}$ are the totally symmetric spinors

$$\Phi_{b_1, \ldots, b_k} = \nabla_{(b_1} \cdots \nabla_{b_k)}^{(b_1} \cdots \nabla_{b_k)(b_1+1)} \Phi_{b_1 \cdots b_k+1},$$

$$\bar{\Phi}_{b_1, \ldots, b_k} = \nabla_{(b_0} \cdots \nabla_{b_k)}^{(b_0} \cdots \nabla_{b_k)(b_k)} \bar{\Phi}_{b_0 \cdots b_k+1}.$$ 

**Proof:** From the first symmetry given in Eq. (2.27) and the trace-free requirement on the indices $b_1, \ldots, b_k$, it is readily shown that

$$Q^{b_0, b_1, \ldots, b_k}_{b_0, b_1, \ldots, b_k} = Q^{b_0, b_1, \ldots, b_k}_{b_0, b_1, \ldots, b_k} = Q^{b_0, b_1, \ldots, b_k}_{b_0, b_1, \ldots, b_k}$$

The requirement

$$\eta^{b_0 b_1} Q_{b_0, b_1, \ldots, b_k} = 0$$

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and the cyclic symmetry in Eq. (2.27) leads to the algebraic form (2.56). In Eq. (2.56) the spinors \( \Phi_{B_0 \cdots B_k}^{B_1' \cdots B_k'} \) and \( \Phi_{B_0' \cdots B_k'}^{B_1 \cdots B_k} \) are uniquely defined by

\[
\Phi_{B_0 \cdots B_k}^{B_1' \cdots B_k'} = \frac{k}{k+1} \epsilon_{B_0'B_1'} \Omega_{B_0'B_1'B_2 \cdots B_k}^{B_1' \cdots B_k'} \tag{2.61}
\]

and

\[
\Phi_{B_0' \cdots B_k'}^{B_1 \cdots B_k} = \frac{k}{k+1} \epsilon_{B_0B_1} \Omega_{B_0'B_1'B_2 \cdots B_k}^{B_1' \cdots B_k'} \tag{2.62}
\]

We use the decomposition (2.50) in the spinor representation of Eq. (2.25), then, using Eqs. (2.56), (2.61), and (2.62), we can solve for \( \Phi_{B_0' \cdots B_k'}^{B_1 \cdots B_k} \) and \( \Phi_{B_0 \cdots B_k}^{B_1' \cdots B_k'} \) to find Eqs. (2.57) and (2.58).

From Propositions 2.7 and 2.8 we can now define a spinorial coordinate system on \( \mathcal{M}^k \).

**Proposition 2.9:** The variables

\[
(\lambda, A_{b_0} A_{b_0 b_1}, \ldots, A_{b_0 \cdots b_k}, \Phi_{B_0 B_1}, \Phi_{B_0 B_1' B_2 \cdots B_k}, \Phi_{B_0 B_1' B_2 \cdots B_k}, \Phi_{B_0 B_1' B_2 \cdots B_k}) \tag{2.63}
\]

define a global coordinate chart on \( \mathcal{M}^k \).

We remark that to pass between the coordinates (2.42) and (2.63) we use any fixed soldering form \( \sigma^{AA'}_a \) such that

\[
\eta_{ab} = \sigma^{AA'}_a \sigma_{bAA'} \tag{2.64}
\]

The spinor variables \( \Phi_{B_1 \cdots B_k}^{B_1' \cdots B_k'} \) and \( \Phi_{B_1 \cdots B_k}^{B_1' \cdots B_k'} \) will play a fundamental role in our symmetry analysis. Their role as coordinates for \( \mathcal{M}^k \) stems from the fact that \( \Phi_A B \) and \( \Phi^{A'B'} \) form what Penrose calls an "exact set of fields" for the Yang–Mills equations. Henceforth we will call the fields (2.57) and (2.58) the Penrose fields and denote them by \( \Phi^k \) and \( \Phi^k \).

By virtue of the identities (2.52), (2.53), (2.54), and Eqs. (2.55), the Penrose fields satisfy the following structure equations on the prolonged equation manifolds. See Ref. 11 for details.

**Proposition 2.10:** The spinorial covariant derivative of \( \Phi_{B_1 \cdots B_k+1}^{B_1' \cdots B_k' -1} \) when evaluated on \( \mathcal{M}^k \), is given by

\[
\nabla^A \Phi_{B_1 \cdots B_k+1}^{B_1' \cdots B_k' -1} = \Phi_{AB_1 \cdots B_k+1}^{A'B_1' \cdots B_k' -1} \tag{2.65}
\]

where \( \{ \ast \} \) denotes a spinor (and \( g \))-valued function of the Penrose fields \( \Phi^1, \Phi^1, \ldots, \Phi^k, \Phi^k \). An analogous result holds for the complex conjugate Penrose fields \( \Phi^k \). Proposition 2.10 is central to our generalized symmetry analysis.

From Propositions 2.6–2.9 we now have the following restriction on the domain of natural generalized symmetries.

**Proposition 2.11:** The spinor components

\[
C_a \rightarrow C_{AA'}
\]

of a natural generalized symmetry of order \( k \) are functions of the Penrose fields to order \( k \)

\[
C_{AA'} = C_{AA'}(\Phi_{B_0 B_1}, \Phi_{B_0 B_1'} B_2', \ldots, \Phi_{B_0 \cdots B_k}^{B_1' \cdots B_k'}, \Phi_{B_0' \cdots B_k'}^{B_1 \cdots B_k}, \Phi_{B_0' \cdots B_k'}^{B_1 \cdots B_k}, \Phi_{B_0' \cdots B_k'}^{B_1 \cdots B_k}) \tag{2.66}
\]
Let us note that the requirements (2.22) and (2.23) must still be satisfied by the generalized
symmetry (2.66). In particular, the Lorentz invariance requirement implies that the spinor form
of the generalized symmetry must be SL(2,C) covariant. More precisely, if $L^A$ is an element
of SL(2,C), then

$$C_{BB'}(L \cdot \Phi, \check{L} \cdot \check{\Phi}) = L^A \check{L}^A_{BB'}, C_{AA'}(\Phi, \check{\Phi}),$$

where $L \cdot \Phi$ and $\check{L} \cdot \check{\Phi}$ denote the action of SL(2,C) on the Penrose fields, e.g.,

$$[L \cdot \Phi]_{AB} = L^C_A L^D_B \Phi_{CD} \quad \text{and} \quad [\check{L} \cdot \check{\Phi}]_{A'B'} = \check{L}^C_{A'} \check{L}^D_{B'} \check{\Phi}_{C'D'}.$$  (2.68)

To take advantage of Proposition 2.11, we will use the following spinor form of the linearized
equations (2.16).

**Proposition 2.12:** The spinor (and g)-valued functions on $\mathbb{R}^k$

$$C_{AA'} = C_{AA'}(\Phi_{B_0 B_1', \ldots, B_k'}, \check{\Phi}_{B_0 B_1' \ldots B_k'})$$

define a kth-order generalized symmetry of the Yang–Mills equations if and only if

$$\nabla_{BB'} \nabla_{BB'} C_{AA'} - \nabla_{AA'} \nabla_{BB'} C_{BB'} + [\Phi^B_A, C_{BA'}] + [\check{\Phi}_{A'B'}, C_{AB'}] = 0 \quad \text{on} \quad \mathbb{R}^{k+2}. \quad (2.69)$$

Let us point out that in Eq. (2.69) the covariant derivatives are defined using total derivatives
as in Eq. (2.17). In this regard it is worth noting that the gauge invariance requirement (2.23)
implies that

$$\nabla_{BB'} \nabla_{BB'} C_{AA'} = \frac{\partial C_{AA'}}{\partial \Phi^\alpha_{00 C_1}} \nabla_{BB'} \Phi^\alpha_{00 C_1} + \frac{\partial C_{AA'}}{\partial \Phi^\alpha_{00 C_1'}} \nabla_{BB'} \Phi^\alpha_{00 C_1'} + \cdots + \frac{\partial C_{AA'}}{\partial \Phi^\alpha_{00 C_k}} \nabla_{BB'} \Phi^\alpha_{00 C_k} + \frac{\partial C_{AA'}}{\partial \Phi^\alpha_{00 C_k'}} \nabla_{BB'} \Phi^\alpha_{00 C_k'} + \cdots \quad (2.70)$$

Our analysis of the linearized equation (2.69) will involve its differentiation with respect to
the Penrose fields. Thus we need an efficient way to deal with symmetric spinors of arbitrary rank.
This will be done by viewing spinors as multilinear maps on complex two-dimensional vector
spaces. If $T^{\ldots A_q}_{A_1 \ldots A_p}$ is a spinor of type $(p, q)$ we write

$$T(\alpha_1, \alpha_2, \ldots, \alpha_p, \check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_q) = T^{\ldots A_q}_{A_1 \ldots A_p} \alpha_1^A A_1^A \ldots \alpha_p^A A_p^A \check{\alpha}_1^A A_1^A \ldots \check{\alpha}_q^A A_q^A.$$  (2.71)

If the spinor $S_{ABC}$ is symmetric in its first two indices, we write

$$S(\alpha, \beta, \gamma) = S_{ABC} \alpha^A \beta^B \gamma^C = S(\beta, \alpha, \gamma),$$  (2.72)

where we have dropped the comma between symmetric arguments of $S$. Note that in this case $S$
is completely determined by the values of

$$S(\alpha^2, \beta) = S(\alpha, \alpha, \beta)$$  (2.73)

for all $\alpha$ and $\beta$. Here we have introduced an exponential notation for repeated symmetric
arguments. More generally, if $V_{A_1 \ldots A_p B}$ is symmetric in its first $k$ indices, we will write
We extend our multilinear map notation to g-valued spinors as follows. If \( T = T^\alpha \tau_\alpha \) takes values in the Lie algebra \( g \), we will write
\[
T(v) = T^\alpha v_\alpha,
\]
where \( v_\alpha \) are the components of an element \( v \) of the dual vector space \( g^* \) to \( g \). If \( S \) takes values in \( g^* \) we will write
\[
S(w) = S_\alpha w^\alpha,
\]
where \( w^\alpha \) are the components of \( w \in g \).

The antisymmetric pairing of spinors defined by the \( e \) spinors is denoted by
\[
\langle \alpha, \beta \rangle = \epsilon_{AB} \alpha^A \beta^B = \alpha_A \beta^A \quad \text{and} \quad \langle \tilde{\alpha}, \tilde{\beta} \rangle = \tilde{\epsilon}^{AB} \tilde{\alpha}^A \tilde{\beta}^B = \tilde{\alpha}_A \tilde{\beta}^A. \tag{2.77}
\]

Next we develop a notation for derivatives of functions on \( J^k(\mathcal{C}) \) (or \( \mathcal{R}^k \)) with respect to the Penrose fields. If
\[
T^C = T_{C_1 \cdots C_p}^{C_1' \cdots C_p'} = T_{C_1' \cdots C_q}^{C_1' \cdots C_q'} (\Phi^1, \tilde{\Phi}^1, \ldots, \Phi^k, \tilde{\Phi}^k) \tag{2.78}
\]
is a natural spinor of type \((p,q)\) and order \( k \), then the partial derivative of \( T^C \) with respect to \( \Phi^l \) is a natural spinor of type \((p+1+1, q+1-1)\). We shall write
\[
[s^l T]_{C_1' \cdots C_q'}^{C_1' \cdots C_p'} (\psi^1, \ldots, \psi^{l+1}, \tilde{\psi}_1, \ldots, \tilde{\psi}_{l-1}, v) = \sum_{A_1, \ldots, A_{l-1}} \partial_{\Phi^{A_1}} \ldots \partial_{\Phi^{A_{l-1}}} \psi_1^{A_1} \ldots \psi_{l-1}^{A_{l-1}}. \tag{2.79}
\]

Further, let \( \phi^1, \ldots, \phi^q \) and \( \tilde{\phi}_1, \ldots, \tilde{\phi}_q \) be arbitrary spinors of type \((1, 0)\) and \((0, 1)\), respectively; we shall write
\[
[s^l T]^C_{C_1' \cdots C_q'} (\psi^1, \ldots, \psi^{l+1}, \tilde{\psi}_1, \ldots, \tilde{\psi}_q, v) = [s^l T^C_{C_1' \cdots C_q'}] (\psi^1, \ldots, \psi^{l+1}, \tilde{\psi}_1, \ldots, \tilde{\psi}_q) = \phi^{A_1} \ldots \phi^{A_{l+1}} \tilde{\phi}_1^{A_1} \ldots \tilde{\phi}_q^{A_{l+1}}. \tag{2.80}
\]
A semicolon will always be used to separate arguments corresponding to derivatives with respect to the coordinates \( \Phi^k \). Partial derivatives, \( \partial^l_\Phi \), with respect to \( \Phi^A_{\ldots A_{l-1}} \) will be similarly denoted.

We shall repeatedly need certain commutation relations between the partial derivative operators \( \partial_\Phi \) and \( \partial_{\tilde{\Phi}} \) and the covariant derivative operator \( \nabla_C^\tau \).

Lemma 2.13: Let
\[
T_{\cdots} : = T_{\cdots} (\Phi^1, \tilde{\Phi}^1, \ldots, \Phi^m, \tilde{\Phi}^m)
\]
be a natural spinor of order \( m \). Then, on \( \mathcal{R}^{m+1} \)
\[
[s^m_\Phi \nabla_C^\tau T_{\cdots}] (\psi^m+2, \tilde{\psi}^m, v) = \psi^C \tilde{\psi}_C [s^m_\Phi T_{\cdots}] (\psi^{m+1}, \tilde{\psi}^{m-1}, v) \tag{2.81}
\]
and
and similarly

\[ [\partial^m_{\phi} \nabla^C_{\phi}, T_{\phi}] (\psi^{m+1}, \bar{\psi}^{m-1}, v) = [\nabla^C_{\phi}, \partial^m_{\phi} T_{\phi}] (\psi^{m+1}, \bar{\psi}^{m-1}, v) + \psi^C \bar{\psi}^C [\partial^m_{\phi} T_{\phi}] (\psi^{m+1}, \bar{\psi}^{m-1}, v) \]  

(2.83)

and

\[ [\partial^m_{\phi} \nabla^C_{\phi}, T_{\phi}] (\psi^{m-1}, \bar{\psi}^{m+1}, v) = [\nabla^C_{\phi}, \partial^m_{\phi} T_{\phi}] (\psi^{m-1}, \bar{\psi}^{m+1}, v) + \psi^C \bar{\psi}^C [\partial^m_{\phi} T_{\phi}] (\psi^{m-1}, \bar{\psi}^{m+1}, v) \]  

(2.84)

Proof: These formulas follow directly from Eq. (2.70) and the structure equations (2.65).

We conclude this section by presenting a couple of elementary results from spinor algebra which we shall use in our symmetry analysis. See Refs. 13 and/or 11 for proofs.

Lemma 2.14: Let \( P(\psi, \alpha) \) be a rank \((k+1)\) spinor that is symmetric in its first \( k \) arguments. Then there are unique, totally symmetric spinors \( P^* \) and \( Q \), of rank \( k+1 \) and \( k-1 \), respectively, such that

\[ P(\psi, \alpha) = P^*(\psi) + (\psi, \alpha) Q(\psi) \]  

(2.85)

If \( P \) is a natural spinor of the Penrose fields \( \Phi^1, \tilde{\Phi}^1, \ldots, \Phi^{k+1}, \tilde{\Phi}^{k+1} \), then so are \( P^* \) and \( Q \).

Lemma 2.15: Let \( P(\psi, \alpha) \) be a rank \((k+1)\) spinor that is symmetric in its first \( k \) arguments. If \( P(\psi, \alpha) \) satisfies

\[ P(\psi^k, \alpha) = (\psi, \alpha) Q(\psi^{k-1}) \]  

(2.86)

then there is a totally symmetric spinor \( Q = Q(\psi^{k-1}) \) such that

\[ P(\psi^k, \alpha) = (\psi, \alpha) Q(\psi^{k-1}) \]  

(2.87)

If \( P \) is a natural spinor, then so is \( Q \).

III. SYMMETRY ANALYSIS

We suppose that

\[ C_{AA'} = C_{A}^{AA'} (\Phi^1, \tilde{\Phi}^1, \ldots, \Phi^k, \tilde{\Phi}^k) \]  

(3.1)

is a natural generalized symmetry of order \( k \). Keeping with our multilinear map notation we write

\[ C(\psi, \bar{\psi}, v) = C_{AA'}^{\alpha} \psi^A \bar{\psi}^{A'} v_{\alpha} \]  

(3.2)

On \( \mathbb{R}^{k+2} \) the linearized equation (2.69) is a gauge and SL(2,\( \mathbb{C} \)) invariant identity in the Penrose fields \( \Phi^l \) and \( \tilde{\Phi}^l \) for \( l = 1, \ldots, k + 2 \). Our analysis consists of differentiating this identity with respect to the Penrose fields \( \Phi^l \) and \( \tilde{\Phi}^l \) for \( l = k, k+1, k+2 \); we present the results in the following series of propositions. All equations in this section hold on \( \mathbb{R}^{k+2} \), i.e., modulo the field equations.

Proposition 3.1: Let \( C_{AA'} = C_{A}^{AA'} (\Phi^1, \tilde{\Phi}^1, \ldots, \Phi^k, \tilde{\Phi}^k) \) be a \( k \)th-order natural generalized symmetry of the Yang–Mills equations. Then there exist natural spinors \( G(\psi^l, \bar{\psi}^{l-2}, v, w), H(\psi^{k-2}, \bar{\psi}^{k+2}, v, w), A(\psi^k, \bar{\psi}^k, v, w), B(\psi^{k+2}, \bar{\psi}^{k-2}, v, w), D(\psi^{k-2}, \bar{\psi}^{k+2}, v, w), E(\psi^k, \bar{\psi}^k, v, w) \) such that
\[ [\partial^k \Phi C](\psi^{k+1}, \bar{\psi}^{k-1}, \nu; \alpha, \bar{\alpha}, w) = (\alpha, \bar{\psi}) \langle \bar{\alpha}, \bar{\psi} \rangle G(\psi^k, \bar{\psi}^k, \nu, w) + (\alpha, \psi) A(\psi^k, \bar{\psi}^{k-1}, \nu, w) \]

\[ + (\bar{\alpha}, \bar{\psi}) B(\psi^{k+1} \alpha, \nu, w) \]

(3.3)

and

\[ [\partial^k \Phi C](\psi^{k-1}, \bar{\psi}^{k+1}, \nu; \alpha, \bar{\alpha}, w) = (\alpha, \psi) \langle \bar{\alpha}, \bar{\psi} \rangle H(\psi^k, \bar{\psi}^k, \nu, w) + (\alpha, \psi) D(\psi^k, \bar{\psi}^{k+1}, \nu, w) \]

\[ + (\bar{\alpha}, \bar{\psi}) E(\psi^{k-1} \alpha, \nu, w). \]

(3.4)

With the symmetries as indicated, the spinors \( A, B, D, E, G, \) and \( H \) are uniquely determined by \( \alpha \) and \( C \). When \( k=1 \), Eqs. (3.3) and (3.4) hold with \( B=0, D=0, G=0, \) and \( H=0 \).

**Proof:** This proposition follows from an analysis of the dependence of Eq. (2.69) on \( \Phi^{k+2} \) and \( \Phi^{k+2} \). To this end we use the commutation relations (2.81) to find

\[ [\partial^k \Phi C_2 \nabla_{\bar{M}M}, \nabla_{\bar{N}N}, C^\alpha_{\bar{A}A'}](\psi^{k+3}, \bar{\psi}^{k+1}, \nu) = \psi_M \psi_N \psi_{M'} \psi_{N'} [\partial^k \Phi C^\alpha_{\bar{A}A'}](\psi^{k+1}, \bar{\psi}^{k-1}, \nu). \]

(3.5)

We use this result to compute the derivative of Eq. (2.69) with respect to \( \Phi^{k+2} \), and this implies that

\[ [\partial^k \Phi C](\psi^{k+1}, \bar{\psi}^{k-1}, \nu; \psi, \bar{\psi}, w) = 0. \]

(3.6)

Similarly, the derivative of Eq. (2.69) with respect to \( \Phi^{k+2} \) implies that

\[ [\partial^k \Phi C](\psi^{k-1}, \bar{\psi}^{k+1}, \nu; \psi, \bar{\psi}, w) = 0. \]

(3.7)

We use Lemmas 2.14 and 2.15 to decompose \([\partial^k \Phi C] \) and \([\partial^k \Phi C] \) into irreducible components. We then use Eqs. (3.6) and (3.7) and arrive at Eqs. (3.3) and (3.4). Uniqueness of the decompositions (3.3) and (3.4) is easily established.

**Proposition 3.2:** If \( C_{\bar{A}A'} = C_{\bar{A}A'} (\Phi^1, \Phi^1, \ldots, \Phi^k, \Phi^k) \) is a natural generalized symmetry of the Yang–Mills equations, then \( C_{\bar{A}A'} \) is linear in the top-order Penrose fields \( \Phi^k \) and \( \Phi^k \).

**Proof:** This result follows from the quadratic dependence of Eq. (2.69) on the Penrose fields \( \Phi^{k+1} \) and \( \Phi^{k+1} \). From Lemma 2.13 we deduce that

\[ [\partial^k \Phi \partial^k \Phi \nabla_{\bar{M}M}, \nabla_{\bar{N}N}, C^\alpha_{\bar{A}A'}](\chi^{k+2}, \bar{\chi}^k, \nu; \psi^{k+2}, \bar{\psi}^k, \nu) \]

\[ = (\psi_M \chi_N \psi_{M'}, \chi_{N'} + \psi_N \chi_M \psi_{N'}, \chi_{M'}) [\partial^k \Phi \partial^k \Phi C_{\bar{A}A'}](\chi^{k+1}, \bar{\chi}^{k-1}, \nu; \psi^{k+1}, \bar{\psi}^{k-1}, \nu). \]

(3.8)

We now differentiate Eq. (2.69) twice with respect to \( \Phi^{k+1} \) and use Eq. (3.8) to find

\[ 2(\psi, \chi) \langle \bar{\psi}, \bar{\chi} \rangle [\partial^k \Phi \partial^k \Phi C](\chi^{k+1}, \bar{\chi}^{k-1}, \nu; \psi^{k+1}, \bar{\psi}^{k-1}, \nu; \alpha, \bar{\alpha}, w) \]

\[ - (\psi, \alpha) \langle \bar{\psi}, \bar{\alpha} \rangle [\partial^k \Phi \partial^k \Phi C](\chi^{k+1}, \bar{\chi}^{k-1}, \nu; \psi^{k+1}, \bar{\psi}^{k-1}, \nu; \chi, \bar{\chi}, w) \]

\[ - (\chi, \alpha) \langle \bar{\chi}, \bar{\alpha} \rangle [\partial^k \Phi \partial^k \Phi C](\psi^{k+1}, \bar{\psi}^{k-1}, \nu; \chi^{k+1}, \bar{\chi}^{k-1}, \nu; \psi, \bar{\psi}, w) = 0. \]

(3.9)

The last two terms of this equation vanish by virtue of Eq. (3.6) and we then have

\[ [\partial^k \Phi \partial^k \Phi C](\chi^{k+1}, \bar{\chi}^{k-1}, \nu; \psi^{k+1}, \bar{\psi}^{k-1}, \nu; \alpha, \bar{\alpha}, w) = 0. \]

(3.10)

Similar computations, which involve applying \( \partial^k \Phi \partial^k \Phi \) and \( \partial^k \Phi \partial^k \Phi \) to the linearized equations, lead to
Proposition 3.3: The natural spinors $A, B, D, E$ in the decompositions (3.3) and (3.4) depend on the Penrose fields $\Phi^l$ and $\Phi^l$ for $l \leq k - 2$.

Proof: Using Proposition 3.2 it is straightforward to show from the commutation relations (2.81) and (2.82) that

$$
[\sigma^k_{\Phi} \sigma^{k+1}_{\Phi} \nabla_{MM'} \nabla_{NN'} C_{AA'}^\alpha,](x^{k+1}, x^{k-1}, u; \psi^{k+2}, \bar{\psi}^{k+1}, v)
$$

$$
= \psi_M \psi_{N'} \bar{\psi}_M \bar{\psi}_{N'} [\sigma^{k-1}_{\Phi} \sigma^k_{\Phi} C_{AA'}^\alpha](\psi^k, \bar{\psi}^{k-2}, v; \chi^{k+1}, \chi^{k-1}, u)
$$

$$
+ (\psi_M \chi_{N'} \bar{\psi}_M \bar{\chi}_{N'} + \chi_M \psi_{N'} \bar{\chi}_M \bar{\psi}_{N'})[\sigma^{k-1}_{\Phi} \sigma^k_{\Phi} C_{AA'}^\alpha](\chi^k, \chi^{k-2}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v). 
$$

We now differentiate the linearized equations (2.69) with respect to $\Phi^k$ and $\Phi^{k+1}$ and use Eqs. (3.13) and (3.6) to find

$$
2(\psi, \chi)(\bar{\psi}, \bar{\chi})[\sigma^{k-1}_{\Phi} \sigma^k_{\Phi} C](x^k, \bar{x}^{k-2}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v; \alpha, \bar{\alpha}, w)
$$

$$
- (\psi, \rho)(\bar{\psi}, \bar{\rho})[\sigma^{k-1}_{\Phi} \sigma^k_{\Phi} C](\psi^k, \bar{\psi}^{k-2}, v; \chi^{k+1}, \chi^{k-1}, u; \psi, \bar{\psi}, w)
$$

$$
- (\psi, \alpha)(\bar{\psi}, \bar{\alpha})[\sigma^{k-1}_{\Phi} \sigma^k_{\Phi} C](x^k, \bar{x}^{k-2}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v; \chi, \bar{\chi}, w) = 0.
$$

We set $\alpha = \psi$ and find

$$
[\sigma^{k-1}_{\Phi} \sigma^k_{\Phi} C](x^k, \bar{x}^{k-2}, u; \psi^{k+1}, \bar{\psi}^{k-1}, v; \psi, \bar{\psi}, w) = 0. 
$$

In terms of the decomposition (3.3), this equation implies that

$$
[\sigma^{k-1}_{\Phi} B](x^k, \bar{x}^{k-2}, u; \psi^{k+2}, \bar{\psi}^{k-2}, v, w) = 0,
$$

i.e., $B$ is independent of the Penrose fields $\Phi^{k-1}$. In a similar fashion, setting $\bar{\alpha} = \bar{\psi}$ leads to

$$
[\sigma^{k-1}_{\Phi} A](x^k, \bar{x}^{k-2}, u; \psi^k, \bar{\psi}^{k+2}, v, w) = 0.
$$

Analogous computations, which involve applying the derivatives $\sigma^k_{\Phi} \sigma^{k+1}_{\Phi}, \sigma^k_{\Phi} \sigma^{k+1}_{\Phi},$ and $\sigma^k_{\Phi} \sigma^{k+1}_{\Phi}$ to Eq. (2.69), yield

$$
[\sigma^{k-1}_{\Phi} D](x^k, \bar{x}^{k-2}, u; \psi^{k-2}, \bar{\psi}^{k+2}, v, w) = 0,
$$

$$
[\sigma^{k-1}_{\Phi} E](x^k, \bar{x}^{k-2}, u; \psi^k, \bar{\psi}^{k+2}, v, w) = 0
$$

and

$$
[\sigma^{k-1}_{\Phi} A](x^{k-2}, \bar{x}^k, u; \psi^k, \bar{\psi}^{k+2}, v, w) = 0,
$$

$$
[\sigma^{k-1}_{\Phi} B](x^{k-2}, \bar{x}^k, u; \psi^{k+2}, \bar{\psi}^{k-2}, v, w) = 0,
$$

$$
[\sigma^{k-1}_{\Phi} D](x^{k-2}, \bar{x}^k, u; \psi^{k-2}, \bar{\psi}^{k+2}, v, w) = 0,
$$

$$
[\sigma^{k-1}_{\Phi} E](x^{k-2}, \bar{x}^k, u; \psi^k, \bar{\psi}^{k+2}, v, w) = 0.
$$
Proposition 3.4: Let $C_{AA'}$ be a natural generalized symmetry of order $k > 1$. Then there is a natural $g$-valued function of order $k-1$

$$\Lambda = \Lambda(\Phi^1, \Phi^1, \ldots, \Phi^{k-1}, \Phi^{k-1})$$

such that, in the decompositions (3.3) and (3.4) for $C_{AA'}$, $G$, and $H$ are the gradients

$$G(\psi^k, \bar{\psi}^k, u, w) = [\partial_{\Phi}^{-1} \Lambda](\psi^k, \bar{\psi}^k, u, w),$$

$$H(\psi^{k-2}, \bar{\psi}^k, u, w) = [\partial_{\Phi}^{k-1} \Lambda](\psi^{k-2}, \bar{\psi}^k, u, w).$$

Proof: We begin by deriving the integrability conditions for Eq. (3.20) from the linearized equations (2.69). We return to Eq. (3.14), which, on account of Propositions 3.1 and 3.3, reduces to

$$[\partial_{\Phi}^{-1} G](\chi^k, \bar{\chi}^k, u; \psi^k, \bar{\psi}^k, u, w) = [\partial_{\Phi}^{k-1} G](\psi^k, \bar{\psi}^k, \chi^k, \bar{\chi}^k, u, w).$$

This is one of the integrability conditions needed to establish Eq. (3.20). The remaining integrability conditions

$$[\partial_{\Phi}^{-1} H](\chi^k-2, \bar{\chi}^k, u; \psi^k, \bar{\psi}^k, u, w) = [\partial_{\Phi}^{k-1} H](\psi^k, \bar{\psi}^k-2, \chi^k, u, w)$$

and

$$[\partial_{\Phi}^{-1} G](\chi^k-2, \bar{\chi}^k, u; \psi^k, \bar{\psi}^k, u, w) = [\partial_{\Phi}^{k-1} H](\psi^k, \bar{\psi}^k-2, \chi^k, u, w)$$

are obtained in an analogous manner from the equations resulting from applying $\partial_{\Phi}^{-1}$, $\partial_{\Phi}^{k-1}$, and $\partial_{\Phi}^{k-1}$ to Eq. (2.69).

From these integrability conditions it is straightforward to verify that $\Lambda$ can be expressed as a natural function of order $k-1$ via

$$\Lambda = \int_0^1 dt \Phi_{A_1}^{\beta A_1} \cdots \Phi_{A_k}^{\alpha A_k} G_{\alpha A_1}^{\beta A_1} \cdots G_{\beta A_k}^{\alpha A_k} \phi_{A_1}^{\alpha A_1} \cdots \phi_{A_k}^{\alpha A_k} (\Phi^1, \Phi^1, \ldots, t\Phi^{k-1}, t\bar{\Phi}^{k-1})$$

$$+ \int_0^1 dt \Phi_{A_1}^{\beta A_1} \cdots \Phi_{A_k}^{\alpha A_k} H_{\alpha A_1}^{\beta A_1} \cdots H_{\beta A_k}^{\alpha A_k} (\Phi^1, \Phi^1, \ldots, t\Phi^{k-1}, t\bar{\Phi}^{k-1}).$$

Proposition 3.5: Let $C_{AA'}$ be a natural generalized symmetry of order $k$. Then there is a natural $g$-valued function $\Lambda = \Lambda(\Phi^1, \Phi^1, \ldots, \Phi^{k-1}, \Phi^{k-1})$ and a natural generalized symmetry $\hat{C}_{AA'}$ of order $k-1$ such that

$$C_{AA'} = \hat{C}_{AA'} + \nabla_{AA'} \Lambda.$$  

Proof: We choose $\Lambda$ as in Proposition 3.4 and define

$$\hat{C}_{AA'} = C_{AA'} - \nabla_{AA'} \Lambda.$$  

By Proposition 2.4 and linearity of the Eqs. (2.69), $\hat{C}_{AA'}$ is a generalized symmetry; by construction, $\partial_{\Phi}^{k-1} \hat{C}_{AA'}$ has the decomposition

$$[\partial_{\Phi}^{k-1} \hat{C}](\psi^{k+1}, \bar{\psi}^{k+1}, u, \alpha, \bar{\alpha}, w) = (\alpha, \psi) A(\psi^k, \bar{\psi}^{k-1}, \alpha, u, w) + (\bar{\alpha}, \bar{\psi}) B(\psi^{k+1}, \bar{\alpha}, \psi^{k-2}, u, w)$$

(3.27)
and \( \partial^k \tilde{C}_{AA'} \) has the decomposition
\[
\langle \partial^k \tilde{C} \rangle (\psi^{k-1}, \bar{\psi}^{k+1}, \psi; \alpha, \bar{\alpha}, w) = \langle \alpha, \psi \rangle D (\psi^{k-2}, \bar{\psi}^{k+1}, \alpha, v, w) + \langle \bar{\alpha}, \bar{\psi} \rangle E (\psi^{k-1} \alpha, \bar{\psi}^{k}, \psi, w).
\] (3.28)

We now show that the linearized equations (2.69) force \( A, B, D, \) and \( E \) to vanish, thus establishing Eq. (3.25). To this end, we consider the derivative of the linearized equations (2.69) with respect to \( \Phi^{k+1} \). We use the commutation relation (2.82) and Eq. (3.6) to find that
\[
2 \langle \alpha, \psi \rangle [\psi^{k+2}, \bar{\psi}^{k-2}, \psi; \psi, \bar{\psi}, w] = 0.
\] (3.29)

In this equation we set \( \alpha = \psi \) and substitute from (3.3) to obtain
\[
\langle \alpha, \alpha \rangle \psi^{k+2}, \bar{\psi}^{k-2}, \psi; \psi, \bar{\psi}, w = 0.
\] (3.30)

Similarly, setting \( \alpha = \bar{\psi} \) we obtain
\[
\langle \alpha, \alpha \rangle \psi^{k+2}, \bar{\psi}^{k-2}, \psi; \psi, \bar{\psi}, w = 0.
\] (3.31)

These equations imply that \( A \) and \( B \) are independent of the Penrose fields \( \Phi^{l+1} \) for \( l = 1, \ldots, k - 2 \). To see this, let us consider the spinor \( A \). If we assume \( A \) is a natural spinor of order \( l \), then the derivative of Eq. (3.31) with respect to \( \Phi^{l+1} \) becomes, after using the commutation relation (2.81)
\[
\langle \chi, \psi \rangle (\bar{\chi}, \bar{\psi}) \partial^l A (\chi^{l+1}, \bar{\chi}^{l-1}, \psi^{k}; \psi, \bar{\psi}, w) = 0,
\] (3.32)

which implies
\[
\partial^l A = 0.
\] (3.33)

A simple induction argument then shows that \( A \) is independent of all the Penrose fields \( \Phi^{l} \) for \( l = 1, \ldots, k - 2 \). An identical argument establishes that \( B \) is independent of \( \Phi^{l} \) for \( l = 1, \ldots, k - 2 \). In a similar fashion we can show that \( D \) is independent of the Penrose fields \( \Phi^{l} \) and \( \Phi^{l} \), for \( l = 1, \ldots, k - 2 \). We conclude that \( A \) and \( B \) are \( SL(2, \mathbb{C}) \) invariant spinors constructed solely from the \( \epsilon \) spinors. But there are no \( SL(2, \mathbb{C}) \) invariant spinors with the rank and symmetry of \( A \) or \( B \) built solely from the \( \epsilon \) spinors, so \( A \) and \( B \) must vanish.

If we differentiate the linearized equations for \( \tilde{C} \) with respect to \( \Phi^{k+1} \), a similar line of reasoning shows that \( D \) and \( E \) must also vanish.

We can now classify all natural generalized symmetries of the Yang–Mills equations.

**Theorem 3.6:** Let
\[
C_a = C_a (x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \cdots b_k})
\]
be a \( k \)th-order natural generalized symmetry of the Yang–Mills equations. Then there is a natural \( g \)-valued function
\[
\Lambda = \Lambda (x, A_a, A_{a,b_1}, \ldots, A_{a,b_1 \cdots b_{k-1}})
\]
such that, modulo the field equations
\[
C_a = \nabla_a \Lambda.
\]
Proof: From Proposition 3.5 we have that every generalized symmetry of order \( k \) differs from a symmetry of order \( k - 1 \) by a generalized gauge symmetry. By induction, every generalized symmetry of order \( k \) differs from a gauge symmetry by a generalized symmetry of order 1, which we denote by \( C^{(1)}_{AA'} \). From Proposition 3.1 and 3.3, we can apply Eq. (3.29) to \( C^{(1)}_{AA'} \). From the discussion following Eq. (3.29) we conclude that \( C^{(1)}_{AA'} \) is in fact independent of the Penrose fields and is thus an \( SL(2,\mathbb{C}) \) invariant spinor of type (1,1) constructed from the \( \epsilon \) spinors. But there are no such spinors, as can be seen, for example, by noting that such a spinor would define a Lorentz invariant vector field. And so it follows that \( C^{(1)}_{AA'} = 0 \).

IV. DISCUSSION

We have shown that all natural generalized symmetries of the Yang–Mills equations are generalized gauge symmetries. These symmetries are variational, but give rise to trivial conservation laws. These results lend support to Vinogradov’s argument, even for systems with degenerate symbol, in the following sense. While the generalized gauge symmetries are nontrivial generalized symmetries; they arise precisely from the degeneracy of the symbol for the Yang–Mills equations. Modulo the symmetries coming from the degeneracy directions of the symbol, there are no natural generalized symmetries.

In order to extend our results to all generalized symmetries of the Yang–Mills equations we will have to drop the requirements (2.22) and (2.23). Thus we must consider solutions of the linearized equations (2.16) which are (i) not gauge covariant, and (ii) not Poincaré covariant, i.e., \( C_a \) is now allowed to be any function of the coordinates (2.29) or, better yet, the coordinates (2.63). In the gravitational case,\(^9\) the generalizations analogous to (i) and (ii) lead to no new types of symmetries. Preliminary computations imply that (i) is unlikely to lead to any new symmetries also in the Yang–Mills case for similar reasons to those found in Ref. 9. On the other hand, the relaxation of Poincaré invariance may lead to new, nontrivial symmetries (beyond those of Proposition 2.2). Indeed, the putative generalized symmetries can be constructed using the conformal Killing vectors admitted by the underlying Minkowski space–time, and this significantly changes the analysis beginning with Proposition 3.5. We will present the complete symmetry analysis elsewhere.

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5While the infinitesimal transformations are local in the fields and their derivatives, the corresponding finite transformations need not be local.