Improved Formulas for Synchrotron Radiation

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Acknowledgments: Mitchell Furst, Tom Lucatorto, Ping Shaw, Uwe Arp
Outline

• Background on synchrotron radiation
  ➢ 1\textsuperscript{st} & 2\textsuperscript{nd} generation only
  ➢ Radiometric utility
  ➢ Work at NIST (very cursory)

• Calculation of SR (other work)

• Calculation approach (this work)
  ➢ Coordinate systems
  ➢ Ultimate formula & conclusions

• “Fuzzing” effects & diffraction effects

• Conclusions
Synchrotron radiation—
Emitted by relativistic charged particles orbiting (accelerated) in magnetic fields

NIST SURF III Synchrotron / Measurement Hall
Schwinger formula (1949):

\[ P(\lambda, \gamma, \psi_0, \rho, \Delta \lambda, I_B, \Delta \theta, \Delta \psi) = \int_{\psi_0+\Delta \psi}^{\psi_0-\Delta \psi} \frac{2 e_0 \Delta \lambda \Delta \theta I_B \rho^2}{\varepsilon_0 \beta \lambda^4 \gamma^4} \left[ 1 + (\gamma \psi)^2 \right]^2 \]

\[ \times \left[ K_{2/3} \left( \xi(\lambda, \gamma) \right)^2 \frac{(\gamma \psi)^2}{1 + (\gamma \psi)^2} K_{1/3} \left( \xi(\lambda, \gamma) \right)^2 \right] d\psi \]

Legend:

- \( P \) = power
- \( \lambda \) = wavelength
- \( \gamma = E/(m_e c^2) \) = Electron energy
- \( \psi_0, \Delta \psi & \theta \) = collection angle info.
- \( I_B \) = Beam current
- \( \rho \) = Orbit radius (also \( R \))
- \( \xi = \lambda(1 + \gamma^2 \psi^2)^{3/2} / (2 \lambda_c) \) = Schwinger param.
- \( \lambda_c = 4 \pi \rho / (3 \gamma^3) \) = Critical wavelength
- \( K_\nu \) = Bessel fcn. (mod.)

Like a blackbody, a **calculable** primary source!

Why this work: assess diffraction effects and Schwinger formula
Application example: Calibration of deuterium lamps (source-based scale)

Synchrotron radiation-based irradiance calibration from 200 to 400 nm at the Synchrotron Ultraviolet Radiation Facility III

Ping-Shine Shaw, Uwe Arp, Robert D. Saunders, Dong-Joo Shin, Howard W. Yoon, Charles E. Gibson, Zhigang Li, Albert C. Parr, and Keith R. Lykke

A new facility for measuring irradiance in the UV was commissioned recently at the National Institute of Standards and Technology (NIST). The facility uses the calculable radiation from the Synchrotron Ultraviolet Radiation Facility as the primary standard. To measure the irradiance from a source under

Fig. 1. Schematic of the FICUS for synchrotron-radiation-based irradiance calibration.
BL-2: Large chamber and clean room
Recent UV/EUV Calibrations at SURF III: Missions and Collaborators

Collaborators: NOAA; NASA Goddard Space Flight Center; Laboratory for Atmospheric & Space Physics; Naval Research Lab; USC Space Flight Center; Jet Propulsion Laboratory.
Calculation of synchrotron radiation

Take $m^{th}$ frequency

$$k = m\omega_0 / c; \quad \omega = m\omega_0$$

$$J(r, t) = J(r)e^{-i\omega t}$$

Time-dependent current:

$$J(r, z, \theta, t) = j_0 \cdot \delta(z) \cdot \delta(r - R) \cdot \hat{\theta} \cdot \sum_{m'} \exp[i m' (\theta - \omega_0 t)]$$

Electric field:

$$E(r, t) = -e^{-i\omega t} \left( \frac{\mu_0 c k^2}{6\pi} \right) \times \left\{ \begin{array}{l} J(r - s) h_0^{(1)} (ks) \\ \int d^3s \left[ s^2 J(r - s) - 3s(s \cdot J(r - s)) \right] \left( \frac{h_2^{(1)} (ks)}{s^2} \right) \end{array} \right\}$$

Magnetic field:

$$B(r, t) = -ie^{-i\omega t} \left( \frac{\mu_0 k}{4\pi} \right) \int d^3s \left\{ s \times J(r - s) \left( \frac{h_1^{(1)} (ks)}{s} \right) \right\}$$

Note: $r =$ point where field is found; $r-s =$ point(s) of current density.
Early development in synchrotron radiation theory

**Early calculation:**

**Early calculations motivated by radiative energy loss:**

**Refinement to Schwinger formula:**

**Further general discussion of synchrotron radiation fields:**
Coordinate systems—Cartesian
Coordinate systems—Elliptic

Coordinate $\nu$ constant on hyperbolae defined by foci

Coordinate $u$ constant on ellipses defined by foci

Focus
Coordinate systems—Oblate spheroidal

Rotate elliptic system through 360 degrees about minor axis. Angle $\theta = 3^{rd}$ coordinate.

Hyperbola becomes hyperboloid

Ellipse becomes spheroid

Foci trace out circle
Coordinate systems—Oblate spheroidal

*For synchrotron radiation, natural to have electron orbit circle traced by foci.*
Convenient to use mix of coordinates, depending:

\[ m \angle = \theta \mod 2\pi \]
At finite range, azimuthal angle difference between detector vs. relevant tangent point ≠ right angle.
Complete specification of coordinates:

Solid angle effect:
Complete specification of coordinates:

\[ s^2 = (S \sin \alpha)^2 + h^2 \]

**“source detector distance”**

\[ \tan \psi = \frac{h}{S \sin \alpha} \]

**“elevation angle”**
Integrating field over electron path, phase of integrand from \( J \) and \( h_t \). Near tangent in relativistic case, phase is nearly stationary (along path) at \( \theta = -\theta_0 = -\cos^{-1}(v/u) \):

\[
ks + m\theta = u \sin \theta_0 + m\theta_0 + (m - v)(\theta - \theta_0) + \frac{v}{6}(\theta - \theta_0)^3 + \ldots
\]

Schwinger trick
- Keep 3 nonzero Taylor terms in phase, 2 Taylor terms in all else
- Assume infinite distance to detector (optional)

\[
F = \int d\theta \ c(\theta) \exp[i\Psi(\theta)]
\]

\[
\approx \int_{-\infty}^{+\infty} d\phi \ (c_1 + c_2\phi) \exp[i(\Psi_0 + \Psi_1\phi + \Psi_3\phi^3)]
\]

\[\rightarrow \text{Fields = combinations of } K_{1/3} \text{ & } K_{2/3} \text{ (or } Ai \text{ & } Ai')\]

Real-time approach (more general) (P.-S. Shaw)

\[
\mathbf{E}(x, t) \sim \left[ \frac{\hat{s} \times [(\hat{s} - \beta) \times \dot{\beta}]}{(1 - \beta \cdot \hat{s})^3 s_{\text{ret}}} \right]
\]
For simple circular orbit, “natural” oblate spheroidal partners to $\theta$:

$$u = \frac{k}{2} \left[ \sqrt{(S + R)^2 + h^2} + \sqrt{(S - R)^2 + h^2} \right]$$

$$v = \frac{k}{2} \left[ \sqrt{(S + R)^2 + h^2} - \sqrt{(S - R)^2 + h^2} \right]$$

**Case $h = 0$:**

$$\Rightarrow u = kS; \ v = kR$$

Note reduction of 3D to fictitious 2D problem:

$$ks = k \sqrt{R^2 + S^2 - 2RS \cos \theta + h^2} = \sqrt{u^2 + v^2 - 2uv \cos \theta} = \sqrt{(u - ve^{i\theta})(u - ve^{-i\theta})}$$

Graf’s addition theorem gives the following expansion:

$$h_l^{(1)}(ks) = h_l^{(1)} \left( (u^2 + v^2 - 2uv \cos \theta)^{1/2} \right)$$

$$= \left( \frac{2}{\pi ks} \right)^{1/2} H_{l+1/2}^{(1)} \left( (u^2 + v^2 - 2uv \cos \theta)^{1/2} \right)$$

$$= \left( \frac{2}{\pi ks} \right)^{1/2} \left( \frac{u - ve^{-i\theta}}{u - ve^{+i\theta}} \right)^{l/2+1/4} \sum_{k=-\infty}^{+\infty} H_{l+1/2+k}^{(1)}(u)J_k(v)e^{-ik\theta}$$

$\Rightarrow$ Allows integration over $\theta$ of each term, $v/u$-type geometric progression of terms converges the sum quickly.
Outcomes of new approach of calculation:

- Graf’s formula facilitates convenient “exact” calculation
  - Debye’s asymptotic formula for $H$ is helpful (very large argument & large order)
  - Olver’s asymptotic formula for $J$ is helpful (large order, argument very near order)
  - Asymptotic expansion using $\text{Ai}$ and $\text{Ai'}$, but with a slightly different argument
  - Calculations can be done extremely quickly

- Analysis of expansion suggests Schwinger results deviates at higher orders in $m^{-1/3}$
Role of fuzzing of beam (horizontal, vertical and orbital tilt spread):

\[
\begin{align*}
\sigma(x_s) &= 0.173 \text{ cm} \\
\sigma(z_s) &= 0.085 \text{ cm} \\
\sigma(s_x) &= 0.000653 \text{ rad} \\
\sigma(s_z) &= 0.000782 \text{ rad}
\end{align*}
\]

Apparent elevation angle along tangent:

\[
\Psi_{el} = \frac{z - z_s}{d_s} - s_z
\]

Image of beam:

Credit: Uwe Arp
Preliminary calculation of radiation (for diffraction effects):

Approximate radiation fields (main beam):

\[
\begin{align*}
\left\{ E_x \right\} & \approx \frac{1}{d_s} \left\{ \psi \, K_{2/3} (n \psi^3 / 3) \right\} \\
\left\{ E_z \right\} & \equiv \frac{1}{d_s} \left\{ i\Psi_{el} \, K_{1/3} (n \psi^3 / 3) \right\} \\
& \exp \left[ ik \left( \frac{(x-x_s)^2 + (z-z_s)^2}{2d_s} \right) \right] e^{-i\omega t}
\end{align*}
\]

\[\psi = \sqrt{1/\gamma^2 + \Psi_{el}^2}\]

Apertures along SURF III Beamline 2:

<table>
<thead>
<tr>
<th>Diameter (mm)</th>
<th>Dist. from tangent (m)</th>
<th>Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.71</td>
<td>0.414</td>
<td>Flux in central region</td>
</tr>
<tr>
<td>22.86</td>
<td>2.11</td>
<td>“</td>
</tr>
<tr>
<td>22.86</td>
<td>4.82</td>
<td>“</td>
</tr>
<tr>
<td>13.00 (main aperture)</td>
<td>10.42</td>
<td>“ + fringed beam waist</td>
</tr>
</tbody>
</table>

Kirchhoff diffraction integral (Gaussian optics):

\[
E_i(x_d, d_s + d_d, z_d) \approx \frac{1}{i\lambda d_d} \int_A dx \, dz \, E_i(x, d_s, z) \exp \left[ ik \left( \frac{(x-x_d)^2 + (z-z_d)^2}{2d_d} \right) \right]
\]

Appropriately treated, aperture effects can be chiefly additive.
Irradiance profile ($\lambda=334$ nm):
Profile with vertical polarizer (locates orbit plane):
Varying fuzz (extrapolating orbital tilt variation) changes total flux (shown for 334 nm):
Conclusion

• SURF III is available for calibrations

• We are improving on the Schwinger formula
  ➢ Important at longer \( \lambda \)

• We are correcting for diffraction
  ➢ To matter in the future, also longer \( \lambda \)

Thank you!