Evaluation of an Alternate Incompressible Energy-Enstrophy Turbulence Model

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A concise overview of turbulence modeling challenges is presented along with developmental concerns of traditional turbulence models. An alternate energy-enstrophy turbulence model developed by Dr. Warren Phillips is given followed by plans for closing the new model and evaluating the subsequent closure coefficients. This paper is effectively an overview of a PhD dissertation proposal and includes only a brief description of the project outline.

I. INTRODUCTION

Fluid mechanics has been a topic of study for hundreds of years and has fascinated many of the greatest minds of history. Many applications today are dependent on correctly understanding and predicting the motion of fluids and much research has been conducted to that end. The laws that govern fluid motion have been understood since the mid 1800s (Navier 1823; Stokes 1845) and the vector mathematics needed to fully analyze three-dimensional fluid mechanics was sufficiently understood only a few decades later (Hamilton 1844, 1853, 1866; Boyer 1989). Since that time, these laws of motion and mathematical properties have been studied by countless researchers, and much progress in the physical understanding of both laminar and turbulent flows has been achieved. However, the complexity presented in completely understanding and correctly predicting fluid mechanics leaves room for much improvement in the field.

Many analytical solutions exist for problems with simple geometries because the governing equations can be simplified and analytically applied. However, for more complex geometries and flow fields, the boundary conditions and vector equations are quite complex and computational means must be employed. Computational Fluid Dynamics (CFD) is a method of fluid flow calculation that uses a gridded domain to predict the flow field in a given geometry. This modeling method has grown in popularity as computational power has increased, making solutions to complex flow problems more readily achievable. However, even with modern computational power, accurate solutions to flow problems can take days or weeks to converge.

The difficulty of modeling fluid mechanics is greatly increased when turbulent flow is considered. Indeed, the most intriguing and complex flow solutions are those for turbulent flow fields. Analytical solutions for turbulent flow fields are much more difficult to develop than those for laminar flow fields and are only available for the most rudimentary geometries. Therefore, turbulent flow modeling is left almost entirely to the mercy of CFD.

The governing principles of mass and momentum conservation apply to turbulent flow and can be employed directly through CFD methods. However, the grid refinement required to capture the small scales of turbulence using CFD techniques is so extreme that solutions based on this technique can be fairly computationally expensive. Such CFD techniques are called Direct Numerical Simulation (DNS) methods and are currently employed mainly for relatively small domains in which the small scales of turbulence are large in comparison to the grid element size of the domain of interest. For a concise discussion of DNS, see Bernard and Wallace (2002a) or Wilcox (2006a). For a more extensive review of the subject including references to recent work, see Moin and Mahesh (1998).

In order to model the turbulent characteristics of larger geometries using CFD, supplementary relationships are commonly employed to the governing equations of fluid motion. These additional relationships should be based on the physics of turbulent motion to provide accurate simulations. The physical characteristics of turbulence have been studied in detail during the past century. Much of our current understanding of turbulence has been constructed through the outcomes of countless experiments and has resulted in much progress. Although these findings have greatly expanded our understanding of turbulent flow characteristics, they are somewhat preliminary in nature and have not proven to be fully representative of turbulent behavior. In other words, there is still much to be learned about turbulence. To this extent, notable authors have commented.

“I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is the turbulent motion of fluids. And about the former I am rather optimistic.” -Sir Horace Lamb (Goldstein 1969)

“Turbulence was probably invented by the Devil on the seventh day of Creation when the Good Lord wasn't looking.” -P. Bradshaw (1994)

“... less is known about the fine scale turbulence ... than about the structure of atomic nuclei. Lack of basic knowledge about turbulence is holding back
progress in fields as diverse as cosmology, meteorology, aeronautics and biomechanics.” -U. Frisch and S. Orszag (1990)

“Turbulence is the last great unsolved problem of classical physics. Remarks of this sort have been variously attributed to Sommerfeld, Einstein, and Feynman, although no one seems to know precise references, and searches of some likely sources have been unproductive. Of course, the allegation is a matter of fact, not much in need of support by a quotation from a distinguished author. However, it would be interesting to know when the matter was first recognized.” -P.J. Holmes, G. Berkooz, and J.L. Lumley (1996)

Although the physical phenomenon of turbulence is not fully understood, the basic features of turbulent motion are known. First, turbulence is caused by inertial forces overpowering viscous forces within a fluid. At low Reynolds numbers, viscous forces dominate, producing laminar flow. However, as the Reynolds number increases, the inertial forces of the fluid increase and eventually overcome the viscous forces. At this point, velocity and pressure fluctuations develop and the flow becomes irregular. This leads to the second most fundamentally understood characteristic of turbulence. Turbulent flow is comprised of fluctuations in pressure, velocity, and temperature making it impossible to reproduce the exact fluctuations from consecutive experiments although the average flow field is recreated.

A definition for turbulence commonly accepted today was given by Hinze (1975):

“Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned.”

A definition that is perhaps more mathematically precise is that given by Phillips (2008a):

“Turbulent fluctuations are irregular variations in certain quantities of a flow field (such as pressure, temperature and velocity) that are not predictably repeatable from one experiment to another.”

The main difference between the two definitions is that Hinze’s definition does not specify the averaging method required to deduce the statistical information while Phillips’ definition specifies ensemble averaging as the suitable averaging method. It can be shown and is well understood that ensemble averaging must be used instead of spatial and temporal averaging in order to accurately distinguish the average flow field from turbulent fluctuations in unsteady common turbulent flow fields. For an overview on the subject see Bernard and Wallace (2002b). For a detailed discussion on averaging see Phillips (2008b) or Wilcox (2006b).

The complex nature of turbulent flow is not yet fully understood. However, research conducted during the past century has allowed for certain properties of mean turbulent flow to be quantified. Therefore, traditional governing equations of mean flow have been developed over time. The challenge of forming equations truly representative of mean turbulent flow has inspired much research that has resulted in varying degrees of success. The most widely-used resulting models for internal flows include the $k-e$ model based on the development of Jones and Launder (1972), and variations of the $k-\omega$ model originally developed by Kolmogorov (1942). Commonly used aerodynamic flow models include a model developed by Spalart and Allmaras (1992) and a model developed by Baldwin and Barth (1990).

Retracing the derivations of traditional turbulence modeling equations reveals seemingly minor yet possibly significant assumptions which have perhaps hindered the validity of the models in the past. Recent re-examination of these equations by Phillips (2008c) has led to alternative developments of the well-known and explored $k-e$, $k-\omega$, and $k-\zeta$ turbulence models. It is possible that the adjusted equations will produce more accurate results for turbulent flow calculations.

This paper includes the development of an alternate turbulence modeling approach suggested by Phillips (2008d), and an overview of the proposed research to be conducted by the author.

II. PHILLIPS TURBULENCE CLOSURE

A. Driving Concerns

Wilcox discusses three major concerns with the manner in which the traditional turbulence models previously discussed are derived and closed.

1. The dissipation per unit mass used in the traditional turbulence models is not the true dissipation of turbulent kinetic energy per unit mass (Wilcox 2006c).

2. The length scale used to close the traditional turbulence models is the length associated with the smaller turbulent eddies which have higher strain rates. However, the larger turbulent eddies are the energy-bearing eddies and are primarily responsible for the transport of turbulent-kinetic energy in a fluid flow field (Wilcox 2006d).

3. The traditional closing transport equations are developed by simple analogy and dimensional analysis and are not developed rigorously from the Navier-Stokes equations (Wilcox 2006d).
Phillips (2008e) echoes these concerns and adds two additional concerns to the list.

4. The traditional dissipation per unit mass actually includes a portion of the total molecular transport term. Therefore, the traditional turbulent molecular transport term neglects a portion of the molecular transport in the traditional turbulence models.

5. Because a portion of the molecular transport is neglected in the traditional turbulent molecular transport term, subsequent application of the Boussinesq hypothesis to the molecular turbulent transport is in question.

These concerns affect the derivation of the turbulent-energy-transport equation, the derivation of the turbulent-dissipation-transport equation, and the accepted length and velocity scales. The following sections supply a concise overview of alternate forms of the kinetic-energy-transport equation and the dissipation-transport equation as well as an alternate turbulent length and frequency scale suggested by Phillips.

B. Turbulent Energy Transport

The turbulent-kinetic-energy transport equation can be derived from the specific Reynolds stress tensor. In contrast, the turbulent-energy transport equation can also be developed from the mechanical energy equation which is formed by taking the dot product of the velocity vector with the Navier-Stokes equations. Defining the total hydrostatic pressure as

\[ \hat{p} = p + \rho g_o Z + \frac{2}{3} \mu \nabla \cdot \mathbf{V} \]  

the Navier-Stokes equations can be written in vector form as

\[ \frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla \hat{p} + \nabla \cdot [2\mu \mathbf{S}(\mathbf{V})] + g_o Z \nabla \rho \]  

(2)

Taking the dot product of the velocity vector with this form of the Reynolds-averaged Navier-Stokes equations, rearranging, and using mathematical identities gives the mechanical energy equation for a Newtonian fluid

\[ \rho \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{V}^2 \right) \right] + \mathbf{V} \cdot \nabla \left( \frac{1}{2} \mathbf{V}^2 \right) = \nabla \cdot \left\{ \mu [\nabla (\mathbf{V}^2) + (\mathbf{V} \cdot \nabla) \mathbf{V}] \right\} - \nabla \cdot [\hat{p} - g_o Z \nabla \rho] - 2\mu \mathbf{S}(\mathbf{V}) \cdot \nabla \mathbf{S}(\mathbf{V}) \]  

(3)

The velocity and pressure at any given time in a turbulent flow can be expressed as the sum of the ensemble average of the property at that point and the fluctuating component. Using this notation, the velocity vector and pressure scalar at a point in the flow can be written

\[ \mathbf{V} = \overline{\mathbf{V}} + \tilde{\mathbf{V}} \]  

(4)

and

\[ p = \overline{p} + \tilde{p} \]  

(5)

It is important to note that the average of the fluctuating component is zero by definition.

\[ \overline{\mathbf{V}} = \overline{p} \equiv 0 \]  

(6)

Applying Eq. (4) and Eq. (5) to the general form of the mechanical energy equation and taking the ensemble average of the resulting formulation gives

\[ \rho \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{V}^2 \right) + k \right] = \mathbf{V} \cdot \nabla \left( \frac{1}{2} \mathbf{V}^2 + k \right) + \nabla \cdot \left[ \mu [\nabla (\mathbf{V}^2) + (\mathbf{V} \cdot \nabla) \mathbf{V}] \right] \] 

\[ - \nabla \cdot [\hat{p} - g_o Z \nabla \rho] - \mathbf{V} \cdot \hat{\mathbf{p}} - 2\mu \mathbf{S}(\mathbf{V}) \cdot \nabla \mathbf{S}(\mathbf{V}) \]  

(7)

where \( k \) is defined as the specific turbulent kinetic energy

\[ k = \frac{1}{2} \mathbf{V}^2 = \frac{1}{2} \left( \overline{V_x^2} + \overline{V_y^2} + \overline{V_z^2} \right) = -\frac{1}{2} \text{trace} \left( \frac{\nabla \mathbf{V}}{\rho} \right) \]  

(8)

and the two pressure terms are the mean and fluctuating hydrostatic pressure terms respectively

\[ \overline{p} = \overline{p} + \rho g_o Z + \frac{2}{3} \mu \nabla \cdot \overline{\mathbf{V}} \]  

(9)

\[ \tilde{p} = \tilde{p} + \frac{1}{3} \mu \nabla \cdot \tilde{\mathbf{V}} \]  

(10)

Using these definitions for pressure terms, the Reynolds-averaged Navier-Stokes equations in vector form can be written as

\[ \rho \left[ \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right] = \nabla \cdot \left[ \mu \nabla \mathbf{V} + \frac{2}{3} \mu \nabla \cdot \mathbf{V} \mathbf{I} \right] \]  

(11)

Taking the dot product of the mean velocity vector with this form of the Reynolds-averaged Navier-Stokes equations, rearranging, and using mathematical identities gives the mean mechanical energy equation for a Newtonian fluid
Applying mathematical identities to this equation, subtracting the result from Eq. (7), and applying more mathematical identities gives the alternate form of the turbulent-energy-transport equation:

\[
\rho \left[ \frac{\partial k}{\partial t} + (\nabla \cdot \mathbf{V}) k \right] = 2\mu \left[ \mathbf{S}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) - \frac{2}{3} (\rho k + \mu \nabla \cdot \mathbf{V}) \nabla \cdot \mathbf{V} - \frac{2}{3} \nabla \cdot \left[ \mu \mathbf{S}(\mathbf{V}) \right] \right] + \rho \left[ (\nabla + (v + v_i/\sigma_z))^2 - \nabla \cdot \left[ \mu \mathbf{S}(\mathbf{V}) \right] \right]
\]

\[
\frac{\partial k}{\partial t} + (\nabla \cdot \mathbf{V}) k = 2\mu \left[ \mathbf{S}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) - \frac{2}{3} (\rho k + \mu \nabla \cdot \mathbf{V}) \nabla \cdot \mathbf{V} - \frac{2}{3} \nabla \cdot \left[ \mu \mathbf{S}(\mathbf{V}) \right] \right] + \rho \left[ (\nabla + (v + v_i/\sigma_z))^2 - \nabla \cdot \left[ \mu \mathbf{S}(\mathbf{V}) \right] \right]
\]

A close look at this equation reveals that the terms on the right-hand side are the true volumetric production, viscous dissipation, pressure dilation, molecular transport, and the volumetric turbulent transport of turbulent kinetic energy. Note that the only approximation used to develop this equation was that for a Newtonian fluid. Also note that the assumption of constant dynamic viscosity was not made in the development of this governing equation is done in traditional developments. Applying the Boussinesq hypothesis to this equation gives the Boussinesq-based turbulent-energy-transport equation.

Defining the dissipation per unit mass as

\[
\overline{\varepsilon} = 2\nu \left[ \mathbf{S}(\mathbf{V}) \cdot \mathbf{S}(\mathbf{V}) - \frac{2}{3} (\nabla \cdot \mathbf{V})^2 \right]
\]

and neglecting the pressure dilation term

\[
\rho \overline{\nabla \cdot \mathbf{V}} \approx 0
\]

gives a version of the turbulent-energy-transport equation that can be used in a traditional k-ε or k-ω model.
which is not available in the development of a turbulent-dissipation-transport equation.

C. Turbulent Length Scale
As previously mentioned, the length scale commonly employed in the development of traditional turbulence models is that associated with the turbulent dissipation. However, the larger turbulent eddies are the energy-bearing eddies and are responsible for the majority of the transport of turbulent energy in a flow. An alternate length scale suggested by Phillips (2008d) is formulated by examining the fluctuating vorticity of a flow field. It is important to note that the angular velocity of a fluid element is one-half the local vorticity. The mean kinetic energy per unit mass is traditionally defined as one-half the mean square of the translational velocity. Phillips suggests that the mean kinetic energy per unit mass can also be defined as one-half the mean square of the angular velocity multiplied by the square of some length scale. This length scale should be a length scale associated with the turbulent energy. Therefore, if 

\[ k = \frac{1}{2} \mathbf{V} \cdot \mathbf{V} = \frac{1}{2} \{ \mathbf{\nabla} \times \mathbf{V} \} \cdot \{ \mathbf{\nabla} \times \mathbf{V} \} \]  

(21)

where \( l_k \) is the length scale. From the definition of the fluctuating vorticity given in Eq. (20), Phillips defines this length scale as

\[ l_k = \frac{\sqrt{8k}}{\omega} \]  

(22)

D. Suggested \( k \)-Vorticity Model
Using Eq. (22) as the length scale in the definition of the turbulent viscosity, an alternate \( k-\omega \) model can be formed. Combining this new definition of turbulent viscosity with Eq. (18) gives the two fundamental equations of Phillips’ \( k-\omega \) model.

\[ v_t = C_v \frac{k}{\omega} \]  

(23)

\[ \frac{\partial k}{\partial t} + (\mathbf{V} \cdot \nabla)k = 2v_\nu \mathbf{\tilde{S}}(\mathbf{V}) \cdot \mathbf{\tilde{S}}(\mathbf{V}) - \nu (\omega^2 + 4\nabla \cdot \{ \frac{1}{2} \nabla k - \nabla \cdot [v_\nu \mathbf{\tilde{S}}(\mathbf{V})] \}) + \nabla \cdot \{ (v + v_t/\sigma_k) \{ \frac{5}{3} \nabla k - 2\nabla \cdot [v_\nu \mathbf{\tilde{S}}(\mathbf{V})] \} \} \]  

(24)

where \( C_v \) and \( \sigma_k \) are closure constants.

The most important contributions of the Phillips development are first, the inclusion of the last term in Eq. (24) and second, the proposition of using the local energy-weighted turbulent length scale found in Eq. (22) to define the eddy viscosity. These contributions have not been included in other turbulence models and could potentially improve RANS turbulence modeling accuracy.

III. DISSERTATION PROJECT DESCRIPTION

A. Overview
Phillips’ development of the governing turbulent-energy-transport equation and turbulent-characteristic length scale provide a sound alternative to current turbulence models. In order to assess the validity of the Phillips development, the suggested model must be closed and evaluated. This includes considering various closure methods, determining appropriate values for the closure constants, and comparing numerical results to both experimental and numerical results of well-known cases. The proposed research will concentrate on evaluating methods for closing the Phillips vorticity-based turbulence model.

B. Closure Methods
Several options exist for closing the Phillips \( k \)-vorticity model. A short explanation of these methods is presented here.

1. RMS Turbulent Vorticity Closure
Perhaps the simplest approach to closing the Phillips vorticity model is to model the RMS turbulent vorticity in terms of the mean fluid velocity, the specific turbulent kinetic energy, and the turbulent eddy viscosity. The fluctuating vorticity RMS is defined by Eq. (20). By analogy with Eq. (24), a turbulent-vorticity transport equation can be obtained. This results in a model defined by

\[ v_t = C_v \frac{k}{\omega} \]  

(25)

\[ \frac{\partial k}{\partial t} + (\mathbf{V} \cdot \nabla)k = 2v_\nu \mathbf{\tilde{S}}(\mathbf{V}) \cdot \mathbf{\tilde{S}}(\mathbf{V}) - \nu (\omega^2 + 4\nabla \cdot \{ \frac{1}{2} \nabla k - \nabla \cdot [v_\nu \mathbf{\tilde{S}}(\mathbf{V})] \}) \]  

(26)

\[ + \nabla \cdot \{ (v + v_t/\sigma_k) \{ \frac{5}{3} \nabla k - 2\nabla \cdot [v_\nu \mathbf{\tilde{S}}(\mathbf{V})] \} \} \]

where \( C_v \), \( \sigma_k \), \( C_{\tilde{v}_1} \), \( C_{\tilde{v}_2} \), and \( \sigma_{\tilde{v}} \) are the closure constants for the model and need to be evaluated.

This formulation can be directly recast in terms of enstrophy. The enstrophy is defined as the solenoidal dissipation divided by the molecular viscosity. Additionally, the enstrophy is equal to the square of the fluctuating vorticity.

\[ \zeta \equiv \frac{\tilde{\omega}}{\nu} \equiv \omega^2 \]  

(28)
Using this change of variables, the formulation can be written

\[ \nu_t = C_\nu k / \sigma^{1/2} \]  

(29)

\[
\begin{align*}
\frac{\partial k}{\partial t} + (\bar{V} \cdot \nabla) k &= 2\nu_t \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) - \nu (\zeta + 4\nu \cdot \{ \frac{1}{3} \nabla k - \nabla \cdot \{ \nu_t \tilde{S}(\bar{V}) \} \}) \\
&\quad + \nabla \cdot (\nu + \nu_t / \sigma_k) (\frac{4}{3} \nabla k - 2\nu \cdot \{ \nu_t \tilde{S}(\bar{V}) \}) \\
&\quad + \frac{\partial \zeta}{\partial t} + \bar{V} \cdot \nabla \zeta
\end{align*}
\]

(30)

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + \bar{V} \cdot \nabla \zeta &= 4C_{\omega 1} \nu_t \frac{\zeta}{k} \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) - 2C_{\omega 2} \sigma^{3/2} \\
&\quad + \nabla \cdot (\nu + \nu_t / \sigma_\zeta) \nabla \zeta \cdot \nabla \zeta / \zeta
\end{align*}
\]

(31)

where the closure coefficients are the same as those needed for Eqs. (25)–(27).

2. Solenoidal Dissipation Closure

Another approach to closing the formulation is to model the solenoidal dissipation by relating it to the enstrophy. A traditional modeled version of the turbulent dissipation is given by

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + \bar{V} \cdot \nabla \zeta &= 2C_{\omega 1} \nu_t \frac{\zeta}{k} \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) \\
&\quad - C_{\omega 2} \frac{\zeta^2}{k} + \nabla \cdot (\nu + \nu_t / \sigma_\zeta) \nabla \zeta
\end{align*}
\]

(32)

Through direct analogy with this traditional turbulent dissipation equation, a transport equation for the solenoidal dissipation can be written

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + \bar{V} \cdot \nabla \zeta &= 2C_{\omega 1} \nu_t \frac{\zeta}{k} \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) \\
&\quad - C_{\omega 2} \frac{\zeta^2}{k} + \nabla \cdot (\nu + \nu_t / \sigma_\zeta) \nabla \zeta
\end{align*}
\]

(33)

Using the relations given in Eq. (28), the solenoidal dissipation transport equation can be recast in terms of the enstrophy. This results in the model

\[ \nu_t = C_\nu k / \sigma^{1/2} \]  

(34)

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} + \bar{V} \cdot \nabla \zeta &= 2C_{\omega 1} \nu_t \frac{\zeta}{k} \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) \\
&\quad - C_{\omega 2} \frac{\zeta^2}{k} + \nabla \cdot (\nu + \nu_t / \sigma_\zeta) \nabla \zeta
\end{align*}
\]

(35)

where \( C_\nu, \sigma, C_{\omega 1}, C_{\omega 2}, C_{\omega 3}, \) and \( \sigma_\zeta \) are the closure constants for the model and may need to be reevaluated because the solenoidal dissipation, \( \hat{\varepsilon} \), differs from the traditional definition of dissipation, \( \varepsilon \).

3. DNS Solenoidal Dissipation Closure

Another possibility to closing the Phillips vorticity-based model is to use the results of a recently developed DNS-based solenoidal-dissipation model by Kreuzinger, Friedrich, and Gatski (2006) that has provided good agreement with DNS results. The solenoidal-dissipation-transport equation for incompressible flow is given in this study as

\[
\begin{align*}
\frac{\partial \hat{\varepsilon}}{\partial t} + \bar{V} \cdot \nabla \hat{\varepsilon} &= 2C_{\omega 1} \nu_t \frac{\hat{\varepsilon}}{k} \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) \\
&\quad - C_{\omega 2} \frac{\hat{\varepsilon}^2}{k} + \nabla \cdot [(\nu + \nu_t / \sigma_\zeta) \nabla \hat{\varepsilon}]
\end{align*}
\]

(37)

Eq. (37) can be recast using the change of variables from solenoidal dissipation to enstrophy, and the resulting formulation is written as

\[ \nu_t = C_\nu k / \sigma^{1/2} \]  

(38)

\[
\begin{align*}
\frac{\partial \hat{\varepsilon}}{\partial t} + \bar{V} \cdot \nabla \hat{\varepsilon} &= 2C_{\omega 1} \nu_t \frac{\hat{\varepsilon}}{k} \tilde{S}(\bar{V}) \cdot \bar{S}(\bar{V}) \\
&\quad - \nu (\zeta + 4\nu \cdot \{ \frac{1}{3} \nabla k - \nabla \cdot \{ \nu_t \tilde{S}(\bar{V}) \} \}) \\
&\quad + \nabla \cdot [(\nu + \nu_t / \sigma_\zeta) \{ \frac{\zeta}{3} \nabla k - 2\nu \cdot \{ \nu_t \tilde{S}(\bar{V}) \} \}]
\end{align*}
\]

(39)

where \( C_\nu, \sigma, C_{\omega 1}, C_{\omega 2}, C_{\omega 3}, \) and \( \sigma_\zeta \) could possibly be taken from those given in the study.
4. General Enstrophy Closure

A close look at closure methods 1 through 3 suggests a general model which would encompass the three models. This formulation is written

\[ \nu_t = C_v k^{3/2} \]  

(41)

\[ \frac{\partial k}{\partial t} + (\nabla \cdot \nabla) k + 2\nu \tilde{\nabla} (\nabla) \tilde{\nabla} = 2\nu [S(\nabla) \cdot S(\nabla) - \nu (\zeta + 4\nu \cdot \frac{1}{3} \nabla k - \nabla \cdot [\nu S(\nabla)])] \]  

(42)

\[ + \nabla \cdot ((\nu + \nu / \sigma_k) \frac{1}{3} \nabla k - 2\nabla \cdot [\nu S(\nabla)]) ] \]

\[ \frac{\partial \zeta}{\partial t} + \nabla \cdot \nabla \zeta \]

(43)

\[ = C_v \nu_t \frac{\zeta}{k} \tilde{\nabla} (\nabla) \tilde{\nabla} - C_2 \zeta^{3/2} \]

\[ - C_3 \frac{\nu}{k} (\zeta + \nu \cdot \frac{1}{3} \nabla k - \nabla \cdot [\nu S(\nabla)]) ]^2 \]

\[ + \nabla \cdot ((\nu + \nu / \sigma_k) \nabla \zeta) - C_4 (\nu + \nu / \sigma_k) \nabla \zeta \cdot \nabla \zeta / \zeta \]

where \( C_v, \sigma_k, \sigma_\zeta, \) and \( C_1 \) through \( C_7 \) are the closure constants for the model and need to be evaluated.

The proposed research will concentrate on evaluating the closure coefficients for the General Enstrophy Closure Model given above. The validity of the first three closure methods will then be assessed as they are special cases of the General Enstrophy Closure Model with some predetermined coefficients. It is not yet apparent which closure methods will be the most promising.

C. Closure Coefficient Evaluation

Closure coefficients for the resulting turbulence model can be obtained several ways. If the closing equations for the model are taken from existing closing equations, the existing closure coefficients can be used as an initial guess for the closure coefficients of the resulting model. However, it cannot be expected that these closure coefficients will be entirely viable for the new model presented here because the Phillips vorticity-based turbulence model is founded on a length scale and turbulent-energy equation which differ from the traditional length scale and turbulent-energy equation. Therefore, most closure coefficients in the proposed turbulence models will need to be evaluated.

Analytical methods are often used to evaluate closure coefficients. In the ideal situation, the coefficients can be evaluated by isolating each term of the equations and comparing the isolated terms to physical aspects of the flow. The terms in the governing equations do not appear in a completely isolated form in nature, so empirical relations are often developed by employing wind tunnel tests, DNS simulations, or other computational methods that simulate isolated characteristics of turbulent flow. The model developer is free to choose the actual flows used to evaluate the closure coefficients. However, the resulting model will be tuned to produce more accurate solutions for flows that are similar to those flows used to calculate the closing coefficients.

IV. CONCLUSION

The focus of the proposed research is to evaluate methods to closing the Phillips vorticity-based turbulence model. The completion of this project will in no way be dependent on the success or failure of the new turbulence model to provide a more accurate approach to modeling turbulence. The proposed research is an evaluation of the Phillips vorticity-based turbulence model, not a validation.

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