Simulating Logan Repayment by the Sinking Fund Method Sinking Fund Governed by a Sequence of Interest Rates

Placede Judicaelle Gangnang Fosso
Utah State University

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SIMULATING LOAN REPAYMENT BY THE SINKING FUND METHOD (SINKING FUND GOVERNED BY A SEQUENCE OF INTEREST RATES)

By:
Placède Judicaëlle Gangnang Fosso

A report submitted in partial fulfilment of the requirements for the degree of
MASTER OF SCIENCE in Statistics

Approved:

Dr Daniel Coster
Major Professor

Dr Christopher Corcoran
Committee Member

Dr Richard Cutler
Committee member

Utah State University
Logan, Utah

April 2012
ABSTRACT

SIMULATING LOAN REPAYMENT BY THE SINKING FUND METHOD
(Sinking fund governed by a sequence of interest rates)

by

Placede Gangnang fosso, Master of Science

Utah State University, 2012

Major Professor: Dr. Daniel C. Coster
Department: Mathematics and statistics

The sinking fund method is a way to repay a loan where the borrower pays the amount of interest accrued by the principal at the end of each time period and puts a certain amount in a sinking fund in order to repay the principal at the end of the loan. Usually, we assume that the interest rate on the sinking fund is the same during the entire time of the loan. In this study, we will depart from the usual assumptions and will look at different scenarios, including when changes of the interest rate on
the sinking fund follows a normal distribution, a uniform distribution and ARIMA processes.
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<tr>
<td>Table</td>
<td>Description</td>
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I would like to thank Dr. Daniel C. Coster, my advisor, for his guidance and patience. He has given me the support that my studies required.

I would also like to express thanks for all other helpful advice and suggestions from the professors and my colleagues in the Department of Mathematics and Statistics, Utah State University.

Placede G. Fosso
The sinking fund method is a way to structure a loan’s repayment; the borrower pays the interest on the loan periodically, but makes no partial payments on the loan amount. That is the payment made prior to the end of the loan term contains no principal. Because the borrower keeps current on the interest rate due, a single lump-sum payment will pay off the loan. In some cases, the borrower is required to accumulate the loan amount at the end of the loan term by making periodic deposits to a savings account, and then use the accumulated amount to erase the debt. We call the savings account used to accumulate the loan amount a sinking fund account. In the sinking fund method, the loan is governed by a sequence of interest rates \( \{i_t\} \) and the sinking fund is governed by a sequence of interest rates \( \{j_t\} \).

The sinking fund is widely used in negotiations with debentures where the issuer is obliged to create a sinking fund to pay the amount due at maturity.
to the holders of the debt. It is also used in several situations; when there is an expectation of future payments. Examples include: future business expansion, indemnities, etc.

This report has been produced in *Latex* and the analysis has been done in the computer programs *R* and *SAS*. 

---

Master of Statistics

Flacode GANGNANG FOSSO ©USU 2012
Chapter 2

METHODODOLOGY

2.1 Traditional Sinking Fund Method: “Zero Risk Case”

The traditional sinking fund method is structured such that all the payments are equal and made at the end of each period. The periodic interest rate $i$ charged by the lender on the loan is fixed. The borrower pays the amount $P = Li$ to the lender at the end of each period and deposits the amount $Q$ into a sinking fund earning the fixed interest rate $j$. Usually, $j < i$.

<table>
<thead>
<tr>
<th>Contributions</th>
<th>0</th>
<th>$P$</th>
<th>$P$</th>
<th>...</th>
<th>$P$</th>
<th>$P + R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>$n-1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.1: Payments to the lender

where $n$ is the number of periods of the payment, $R$ is the lump-sum obtained by withdrawing the total amount accumulated in the sinking fund at the end of $n$ periods. The cash flow of deposits in the sinking fund is:
Hence, the accumulated value in the sinking fund after $n$ periods of time is:

$$R = Q s_{n,i}$$  \hspace{1cm} (2.1)$$

The borrower’s total cash flow is:

$$L - P - Q - P - Q - P - Q - P - Q$$

The lender’s cash flow is:

$$-L + P + P + P + P + R$$

In order for the loan to be repaid we need to have:

$$L = Pa_{n,i} + R (1 + i)^{-n} = Pa_{n,i} + Q s_{n,j} (1 + i)^{-n}$$  \hspace{1cm} (2.2)$$

The principal is kept constant after each payment by setting $P = Li$ corresponding to the interest accrued by the loan in one period of time. Therefore, after each payment of $Li$, the principal in the loan is $L$. In this situation,

$$L = R = Q s_{n,j}$$  \hspace{1cm} (2.3)$$

<table>
<thead>
<tr>
<th>Contributions</th>
<th>0</th>
<th>$Q$</th>
<th>$Q$</th>
<th>...</th>
<th>$Q$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.2: Contributions into the sinking fund

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$L$</th>
<th>$-P - Q$</th>
<th>$-P - Q$</th>
<th>...</th>
<th>$-P - Q$</th>
<th>$-P - Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.3: Borrower’s total cash flow

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$-L$</th>
<th>$P$</th>
<th>$P$</th>
<th>...</th>
<th>$P$</th>
<th>$P + R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.4: Lender’s total cash flow
So,

\[ Q = \frac{L}{s_{n,j}} \]  

(2.4)

Then, (2.2) becomes:

\[ L = L a_{n,j} + L (1 + i)^{(-n)} \]  

(2.5)

This means a loan of \( L \) at an interest rate \( i \) is paid by a total of periodic payment of

\[ P + Q = Li + \frac{L}{s_{n,j}} \]  

(2.6)

each period of time. For

\[ a_{n,i\&j} = \frac{1}{i + \frac{1}{s_{n,j}}} \]  

(2.7)

\( a_{n,i\&j} \) is the annuity factor for a sinking fund loan under interest rates \( i \) and \( j \). Notice that:

\[ L = (P + Q) = (Li + \frac{L}{s_{n,j}}) a_{n,i\&j} \]  

(2.8)

If \( i = j \), we have that:

\[ a_{n,i\&j} = a_{n,\bar{n}} \]  

(2.9)

because

\[ \frac{1}{a_{n,\bar{n}}} = i + \frac{1}{s_{n,j}} \]  

(2.10)
From the point of view of the borrower, he is paying a loan under an actual interest rate of $i'$ where $i'$ is such that:

$$a_{n,i'} = \frac{1}{i + \frac{1}{s_{n,j}}}$$  \hspace{1cm} (2.11)

We have that as $i$ increases, $i'$ also increases; as $j$ increases, $i'$ decreases.

**Example of a loan paid with traditional sinking fund method**

We consider a loan of $10,000 to be paid back by the sinking fund method. The term of the loan is 10 years and the interest is to be repaid monthly at a nominal interest rate convertible monthly $i = 4\%$. The sinking fund account earns a nominal interest rate convertible monthly of 3%.

The borrower will pay the amount: $L_i = 10,000 \times \frac{0.04}{12} = 33.333$ each month to the lender and will deposit the amount $P$ such that $Ps_{n,j} = 10,000$. This gives us $P = 71.5607$.

At the end of 10 years, the accumulated amount in the sinking fund is $71.5607 \times s_{n,j} = 10000$, which corresponds to the amount of the loan. From the point of view of the borrower, the loan has an actual monthly interest rate of $i'$ where $i'$ is found using formula (2.11): $a_{n,i'} = \frac{1}{i + \frac{1}{s_{n,j}}}$.
This gives us $i' = 4.7595\%$.

### 2.2 Sinking fund governed by different sequences of interest rate and analysis

When performing analysis, we generally consider the case where all the payments are made at the end of each of the periods. $i$ is the periodic effective interest rate charged by the lender on the loan. At the end of each period, the borrower pays $P$ directly to the lender and deposits $Q$ into a fund earning a sequence of interest rates $j_t$. $j$ is the initial interest rate earned by the sinking fund. Usually, $j < i$. The cash flow to the principal is:

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$-L$</th>
<th>$Q$</th>
<th>$Q$</th>
<th>$\ldots$</th>
<th>$Q$</th>
<th>$Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$\ldots$</td>
<td>$n-1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.5: Cash flow to the principal

where $n$ is the number of periods of the payment, $R$ is the lump sum obtained by withdrawing the total accumulated in the sinking fund at the end of $n$ periods. The cash flow of deposits in the sinking fund is:

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$P$</th>
<th>$P$</th>
<th>$\ldots$</th>
<th>$P$</th>
<th>$P + R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$\ldots$</td>
<td>$n-1$</td>
</tr>
</tbody>
</table>

Table 2.6: Cash flow of the deposits in the sinking fund
Hence, the accumulated value in the sinking fund at time $n$ is:

$$R = Q \sum_{k=1}^{n} \prod_{t=k}^{n} (1 + j_t)$$  \hspace{1cm} (2.12)$$

The borrower’s total cash flow is:

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$L$</th>
<th>$-P - Q$</th>
<th>$-P - Q$</th>
<th>...</th>
<th>$-P - Q$</th>
<th>$-P - Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.7: Borrowers’s cash flow

The lender’s cash flow is:

<table>
<thead>
<tr>
<th>Contributions</th>
<th>$-L$</th>
<th>$P$</th>
<th>$P$</th>
<th>...</th>
<th>$P$</th>
<th>$P + R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>$n - 1$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 2.8: Lender’s cash flow

In order for the loan to be repaid, we need to have:

$$L = Pa_{n|\bar{r}} + R(1 + i)^{-n} = Pa_{n|\bar{r}} + Q \sum_{k=1}^{n} \prod_{t=k}^{n} (1 + j_t)^{-n}.$$  \hspace{1cm} (2.13)$$

2.3 Changes on the sinking fund rate follow the standard normal distribution

The loan is lent at a certain rate $i$; the borrower is required to pay the amount $li$ at the end of each year and to put the amount $P = \frac{L}{s_{n|\bar{r}}}$ in the sinking fund. $L$ is the amount of the loan, $n$ the time of the loan, and $j$ is
the rate on the sinking fund for the first year of the loan. For \( j \) governing the sinking fund during all the time of the loan, the amount in the sinking fund at the end of \( n \) time periods is \( L \). Also, after the first period of the loan, a random variable \( I \) following the standard normal distribution is added to \( j \) any period of time to characterize changes on the sinking fund rate. This implies that the change on the rate of the sinking fund follows a normal curve.

2.3.1 For \( n = 1 \) year

The time of the loan is 1 year and \( R \) the amount in the sinking fund at the end of the year is the same as \( L \) the amount of the loan. This means that \( E[R] = L \).

2.3.2 For \( n = 2 \) years

The time of the loan is 2 years; the amount in the sinking fund at the end of 2 years is \( R = Q(1 + j + I_1) + Q \); where \( I_1 \) is a random variable following the standard normal curve characterizing the change on the sinking fund rate over the second year. The expected value at the end of 2 years is \( E[R] = Q(1 + j) + Q = L \).
2.3.3 For $n = 3$ years

The loan is over 3 years. The amount in the sinking fund at the end of the time of the loan is $R = (Q(1 + j + I_1) + Q)(1 + j + I_1 + I_2) + Q$, where $I_1$ and $I_2$ are iid random variables following the standard normal curve characterizing changes on the sinking fund rate over the second and third years respectively. The expected value at the end of 3 years is $E[R] = Q(1 + j)^2 + Q(1 + j) + 2Q$.

1,000 simulated values of $R$ corresponding to this scenario have been generated for $L = 10,000$ and different values of $j$. The code for the $R$ function can be found in Appendix 1.

2.3.4 For $n = 4$ years

The loan is over 4 years. The amount in the sinking fund at the end of the time of the loan is:

$$R = ((Q (1 + j + I_1) + Q) (1 + j + I_1 + I_2) + Q) (1 + j + I_1 + I_2 + I_3) + Q;$$

where $I_1$, $I_2$ and $I_3$ are iid random variables following the standard normal distribution characterizing changes on the sinking fund rate over the second, third and fourth years respectively. The expected value at the end of 4 years
Methodology

is:

\[ E[R] = Q(1 + j)^3 + Q(1 + j)^2 + 1.04Q(1 + j) + 1.02Q. \]

As before, 1,000 simulated values of \( R \) corresponding to this scenario have been generated for \( L = 10,000 \) and different values of \( j \). The code for the \( R \) function can be found in Appendix 1. The corresponding means and standard deviations are presented in Table 2.9.

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>10001.86</td>
<td>10004.98</td>
<td>9990.502</td>
<td>9986.098</td>
<td>9995.05</td>
<td>10010.18</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>164.2927</td>
<td>173.9168</td>
<td>165.6999</td>
<td>170.8697</td>
<td>175.3449</td>
<td>178.7933</td>
</tr>
</tbody>
</table>

Table 2.9: Average and standard deviation when changes on the sinking fund rate follow a normal curve with no boundaries on \( j \) over 4 years.

The expressions of the expected values of \( R \) for loans over years one through four do not suggest a general analytical form of the expected value of the amount in the sinking fund when changes on the interest rate follow a standard normal curve; the variance of \( R \) is even more difficult to generalize. Also, the analytical expression includes the cases when the interest rate on the sinking fund is less than 0% and when it goes very high. These are unrealistic scenarios. Moreover, the expected values obtained by using simulations for \( n = 3 \) years and \( n = 4 \) years are fairly close to the expected amounts in the sinking fund using the analytical expressions. This is the reason that we will compute the expected amounts in the sinking fund and their corresponding standard deviations by simulations.
2.3.5 The time of the loan is undefined

We have a starting annual interest rate $j$ on the sinking fund and we assume that the interest rate on the sinking fund changes each month following the standard normal distribution. In the case that the interest rate takes a value less than 0%, we consider the annual interest rate as 0%; for interest rates greater than or equal to 12%, we consider the annual interest rate as 12%. We simulated 1000 values of $L$ and looked at the histogram, their mean and standard deviation. The results of our simulations can be found in Sections 3.1.2 and 3.1.3.

2.4 Changes on the sinking fund rate vary up or down by 0 to 3% every month

We have a starting annual interest rate $j$ on the sinking fund and we assume that the interest rate on the sinking fund changes each month following the uniform distribution on $[-3, 3]$. In the case that the interest rate takes a value less than 0%, we consider the annual interest rate as 0% and for interest rates greater than or equal to 12%, we consider the annual interest rate as 12%; We simulated 1000 values of $L$ and looked at the histogram their mean and standard deviation. The results of our simulations can be found in Section 3.2.
2.5 Changes on the sinking fund rate follow an ARIMA process

In order to pursue the aim of describing realistic situations, we will model the changes on the sinking fund rates as an ARIMA process. The starting interest rate on the sinking fund is still $j$ and $I_t$, the change on interest rate is an ARIMA process.

We used the CD interest rates in the United States from December 1965 through September 2011 to build two ARIMA processes. One of them corresponds to the period without inflation and the other corresponds to the period with inflation. The data was obtained in the following website:

http://www.economagic.com/em-cgi/data.exe/fedbog/cd1m

2.5.1 ARIMA model on the CD interest rates for the period without inflation

In order to build the model corresponding to this period, we considered the data from December 1965 through May 1978. The “Best mode” has been chosen using SAS. The code can be found in Appendix 4.

Figure 2.1 of the CD rates from January 1965 through May 1978 shows that this times series is not stationary. In order to make our data a stationary
Figure 2.1: CD rates during period without inflation December 1965 - May 1978

process, we transformed the data by first order difference.
Figure 2.2: First order difference of CD rates during the period without inflation from December 1965 - May 1978

Figure 2.2 of the transformed data shows us that the transformed time series is fairly stationary with approximately constant variance and does not present any trend although some spikes do persist.
Figure 2.3: ACF of the transformed CD rates in the period of no inflation from December 1965 - May 1978

The ACF 2.3 and the PACF 2.4 clearly show several spikes. The spikes displayed in figures 2.3 and 2.4 suggest that we fit an ARMA model to the transformed data. We find the optimal model looking at the goodness of fits of several models.
Figure 2.4: PACF of the transformed CD rates in the period of no inflation from December 1965 - May 1978

We obtained the following model for the differenced time series:

\[
X_t = 0.33024X_{t-1} - 0.08989X_{t-6} - 0.1095X_{t-20} + 0.20483Z_{t-1} - 0.013256Z_{t-6} + 0.48454Z_{t-20}
\]  

(2.14)

where, \( Z_t \sim WN(0, 0.140623) \)

After fitting the model to the data, Figure 2.5 shows that the correlations among the lags are not significantly different from zero, that there is no identifiable pattern, and that there is constant variance. This represents the ACF of a white noise process. Thus, the residuals are approximately uncorrelated and are white noise. Therefore, we can conclude that our
The noise follows a Normal distribution

To build random variations for the interest rate on the sinking fund, we used the ARIMA model represented by the equation 2.14 to compute the fitted values \( \hat{X} \) and the corresponding residuals \( \hat{Z} \). We used the last 20 values (fitted values and residuals) to generate 500 values \( J_t \) in this way:

\[
\hat{X}_{132+t} = J_t; \quad t = 1, \ldots, 20
\]

\[
\hat{Z}_{132+t} = R_t; \quad t = 1, \ldots, 20
\]

\[
J_t = 0.33024J_{t-1} - 0.08989J_{t-6} - 0.1095J_{t-20} + 0.20483R_{t-1} - 0.013256R_{t-6} \\
+ 0.48454R_{t-20}; \quad t = 21 \ldots, 520,
\]
where the $R_t, t = 21, \ldots, 520$ were generated by a normal distribution with mean 0 and variance 0.140623. The 500 $J_t$ generated were used as changes of the interest rates on the sinking fund.

The noise follows a shifted Gamma distribution

To build random variations on the sinking fund’s interest rate in this case, we will follow the same process as previously; the only difference is that the $R_j, j = 21, \ldots, 520$ will be generated by a shifted Gamma distribution with parameters $Shape = 3$ and $Scale = 1$. These coefficients are arbitrarily chosen but are motivated by the fact that this Gamma distribution will have the variance of the Uniform distribution on (-3,3). The value of the shift is the mean of the generated sample.

2.5.2 ARIMA model for the CD interest rates for the period of inflation

In order to build the model for this period of time, we considered the data from January 1999 through September 2011. The “Best model” has been chosen using $SAS$. The code can be found in Appendix 4. Plot 2.6 of the CD rates from January 1965 through May 1978 shows that the time series is not stationary. In order to make our data a stationary process, we
Figure 2.6: CD rates in the period of no inflation from December 1965 - May 1978 transformed the data by first order difference.
Figure 2.7: First order difference of CD rates in the period with inflation from January 1999 - September 2011

Figure 2.7 of the transformed data presents us that the transformed time series is pretty well stationary with constant variance and does not present any trend although a spike does persist.
Figure 2.8: ACF of the transformed CD rates for the period with inflation January 1999 - September 2011.

The ACF 2.8 and the PACF 2.9 show clearly several picks.

The picks displayed in figures 2.8 and 2.9 suggest that we fit an ARMA model to the transformed data. We find the optimal model by looking at the goodness of fits of different models.
We obtained the following model for the differenced time series:

\[ X_t = -0.24674X_{t-1} + 0.27363X_{t-10} - 0.46482Z_{t-10}. \] (2.15)

where, \( Z_t \sim WN(0, 0.089764) \)
After fitting the model to the data, Figure 2.10 shows that the correlations among the lags are not significantly different from zero, that there is no identifiable pattern, and that there is constant variance. So Figure 2.10 is the ACF of a white noise process. Thus the residuals are approximately uncorrelated and are white noise. Therefore, we can conclude that our fitted model is appropriate.
The noise follows the standard normal distribution

To build random variations for the interest rate on the sinking fund, we used the ARIMA model represented by equation 2.15 to compute the fitted values $\hat{X}$ and the corresponding residuals $\hat{Z}$. We used the last 10 values (fitted values and residuals) to generate 500 values $J_t$ in this way:

$$\hat{X}_{139+t} = J_t; t = 1, \ldots, 10$$
$$\hat{Z}_{139+t} = R_t; t = 1, \ldots, 10$$
$$J_t = -0.24674J_{t-1} - 0.27363J_{t-10} - 0.46482R_{t-10}; t = 11 \ldots, 510$$

where the $R_t, t = 11, \ldots, 510$ are generated by a normal distribution with parameters 0 and 0.089764. The 500 generated $J_t$ will be used as changes of the interest rates on the sinking fund.

The noise follows a shifted Gamma distribution

To build random variations on the sinking fund’s interest rate in this case, we will follow the same process as previously; the only difference is that the $R_j; j = 11, \ldots, 510$ will be generated by a shifted Gamma distribution with parameters $shape = 3$ and $scale = 1$. These coefficients are arbitrarily chosen but are motivated by the fact that this Gamma distribution will have the variance of the Uniform distribution on (-3,3). The value of the shift is the mean of the generated observations.
In this chapter, we present and comment on the results of the simulations we ran throughout this work.

3.1 Changes on the sinking fund rate follow the standard normal curve

3.1.1 4-year loan

We will compare the expected amount in the sinking fund obtained analytically and by simulations for a loan over 4 years for different values of \( j \), the starting rate on the sinking fund. We assume that changes on the sinking fund rate follow the standard normal curve and have no boundaries. We will provide the standard deviations for the case when the expected values are found by simulations because analytical values of the standard deviations
are difficult to find as explained in Section 2.3.4. We generated 1000 simulated values of $L$.

<table>
<thead>
<tr>
<th>Q</th>
<th>2,426.24</th>
<th>2,390.27</th>
<th>2,354.90</th>
<th>2,320.12</th>
<th>2,285.51</th>
<th>2,252.81</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical Mean</td>
<td>10,147.51</td>
<td>10,146.28</td>
<td>10,145.06</td>
<td>10,143.85</td>
<td>10,142.64</td>
<td>10,142.44</td>
</tr>
<tr>
<td>Interest rate, $j$</td>
<td>2%</td>
<td>3%</td>
<td>4%</td>
<td>5%</td>
<td>6%</td>
<td>7%</td>
</tr>
<tr>
<td>Simulated mean</td>
<td>10,001.86</td>
<td>10,004.98</td>
<td>9,990.502</td>
<td>9,986.098</td>
<td>9,995.05</td>
<td>10,010.18</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>164.29</td>
<td>173.92</td>
<td>165.70</td>
<td>170.87</td>
<td>175.34</td>
<td>178.79</td>
</tr>
</tbody>
</table>

Table 3.1: Expected mean in the sinking fund derived analytically; by simulations and corresponding standard deviations for different values of $j$ loan over 4 years

![Histogram of lump-sums when changes on interest rate follow the Standard Normal curve and have no boundaries](image)

Figure 3.1: Histogram of lump-sums when changes on interest rate $\sim N(0,1)$ and have no boundaries: $n = 4$ years; starting $j = 3\%$; $L = 10,000$; 1000 simulated lump values
Results

Changes on interest rate of the sinking fund follow the Standard Normal curve and have no boundaries

Lump−sums over 4 years

Frequency

9700 9800 9900 10000 10100 10200 10300

0 50 100 150 200

Figure 3.2: Histogram of lump-sums when changes on interest rate $\sim N(0, 1)$ and have no boundaries: $n = 4$ years, starting $j = 4\%$; $L = 10,000$; 1000 simulated lump values

We observe that with standard normal changes to $j$ over 4 years provided there was no truncation on interest rates:

- Values of analytical means and simulated means are fairly close
- Standard deviations increase slightly with $j$
- On average, the lump-sum amount was close to the target of 10,000
- Variations in final lump-sum amounts is about the same whether the starting $j = 3\%$ or $j = 4\%$. 
3.1.2 10 years loan

We suppose that the changes on the sinking fund follow the standard normal curve and are between 0% and 12%. Tables 3.2 presents the means and standard deviations for different values of $j$ and for 10,000 as the principal. 3.3 presents the means and standard deviations for different values of $j$ for 100,000 as amounts of the principal.

- The amount of the loan is 10,000:

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>12,478.00</td>
<td>11,789.16</td>
<td>11,220.07</td>
<td>10,597.83</td>
<td>10,086.74</td>
<td>9,516.40</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1,199.96</td>
<td>1,115.26</td>
<td>1,082.20</td>
<td>988.24</td>
<td>986.19</td>
<td>917.37</td>
</tr>
</tbody>
</table>

Table 3.2: Averages and standard deviations when changes on the sinking fund rate follow the standard normal curve with boundaries on $j$: $n = 10$ years; $L = 10,000$
Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries. Lump sums over 10 years: n= 10 years; starting \( j = 3\% \); L= 10,000; 1000 simulated lump values

- The distribution of the final amounts in the sinking fund is right skewed. The histogram almost looks bimodal.
- The average final amounts in the sinking fund decreases as \( j \) increases.
- The variability decreases but not very fast as \( j \) increases.
Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries

Lump sums over 10 years

Frequency

9000 10000 11000 12000 13000 14000 15000

Figure 3.4: Histogram of lump-sums when changes on interest rate $\sim N(0, 1)$ and have boundaries: $n = 10$ years; starting $j = 4\%$; $L = 10,000$; 1000 simulated lump values

- The amount of the loan is 100,000:

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>126,260.3</td>
<td>119,230.3</td>
<td>113,735.9</td>
<td>106,475.1</td>
<td>102,600.5</td>
<td>96,406.58</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>18,229</td>
<td>16,943</td>
<td>16,168</td>
<td>15,422</td>
<td>14,384</td>
<td>13,690</td>
</tr>
</tbody>
</table>

Table 3.3: Averages and standard deviations when changes on the sinking fund rate follow the standard normal curve with boundaries on $j$: $n = 10$ years; $L = 100,000$
Results

Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries:

Lump - sums over 10 years

Frequency

100000 120000 140000 160000

0 20 40 60 80 100 120

Figure 3.5: Histogram of lump-sums when changes on interest rate $\sim N(0, 1)$ and have boundaries: $n = 10$ years; starting $j = 3\%$ ; $L = 100,000$; 1000 simulated lump values

Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries:

Lump - sums over 10 years

Frequency

80000 100000 120000 140000

0 20 40 60 80 100

Figure 3.6: Histogram of lump-sums when changes on interest rate $\sim N(0, 1)$ and have boundaries: $n = 10$ years; starting $j = 4\%$ ; $L = 100,000$; 1000 simulated lump values

We observe that scaling up the lump amounts does not really affect the final amount in the sinking fund; there is a linear effect on the final amount in the sinking fund. The standard deviations are 10 times bigger compared to when the target amount is 10,000.
3.1.3 30 years loan

We suppose that the changes on the sinking fund follow the standard normal curve and are between 0% and 12%. Tables 3.4 presents the means and standard deviations for different values of \( j \) and for 10,000 as the principal. Table 3.5 presents the means and standard deviations for different values of \( j \) for 100,000 as amounts of the principal.

- The amount of the loan is 10,000:

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>22,014.16</td>
<td>18,445.59</td>
<td>15,576.20</td>
<td>12,785.09</td>
<td>10,576.46</td>
<td>8,869.541</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>7,289.7</td>
<td>6,031.1</td>
<td>5,233.9</td>
<td>4,280.5</td>
<td>3,537.6</td>
<td>2,956.2</td>
</tr>
</tbody>
</table>

Table 3.4: Averages and standard deviations when changes on the sinking fund rate follow the standard normal curve with boundaries on \( j \); \( n = 30 \) years; \( L = 10,000 \)
Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries: 

Lump - sums over 30 years

Frequency

10000 20000 30000 40000

0 50 100 150 200 250 300

Figure 3.7: Histogram of lump-sums when changes on interest rate $\sim N(0, 1)$ and have boundaries: $n = 30$ years; starting $j = 3\%$; $L = 10,000$; 1000 simulated lump values

Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries: 

Lump - sums over 30 years

Frequency

5000 10000 15000 20000 25000 30000 35000

0 50 100 150

Figure 3.8: Histogram of lump-sums when changes on interest rate $\sim N(0, 1)$ and have boundaries: $n = 30$ years; starting $j = 4\%$; $L = 10,000$; 1000 simulated lump values

We observe that for changes on the interest rate following the standard normal curve over 30 years with a target amount of 10,000;

- Most of the time, the borrower makes a huge profit when the starting interest rate on the sinking fund is low with a relatively low risk.
- It would be difficult for the borrower to accumulate 10,000 in the sinking fund when the starting interest rate on the sinking fund is high.

- The distribution of the final amounts in the sinking fund is unimodal and right skewed.

- The variance decreases as $j$ increases.
• the amount of the loan is 100,000:

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>221.544.0</td>
<td>185.146.3</td>
<td>155.473.8</td>
<td>127.804.9</td>
<td>109.000.2</td>
<td>88.382.17</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>72.951.6</td>
<td>60.266.9</td>
<td>53.678</td>
<td>43.518.7</td>
<td>35.696</td>
<td>30.126.6</td>
</tr>
</tbody>
</table>

Table 3.5: Averages and standard deviations when changes on the sinking fund rate follow the standard normal curve with boundaries on $j$: $n = 30$ years; $L = 100,000$

Figure 3.9: Histogram of lump-sums when changes on interest rate $\sim N(0,1)$ and have boundaries: $n = 30$ years; starting $j = 3\%$; $L = 100,000$; 1000 simulated lump values
Changes on interest rate of the sinking fund follow the Standard Normal curve with boundaries

Lump−sums over 30 years

Frequency

50000 100000 150000 200000 250000 300000 350000

0 50 100 150

Figure 3.10: Histogram of lump-sums when changes on interest rate $\sim N(0,1)$ and have boundaries: $n = 30$ years; starting $j = 4\%$; $L = 100,000$; 1000 simulated lump values

We observe that:

- Scaling up the amount does not affect the final amounts in the sinking fund; there is a linear effect on the final amount in the sinking fund

- There is a linear effect on the standard deviation of the final amounts in the sinking fund when we make a comparison with the case where the principal amount is 10,000 and the loan over 30 years

- The distribution of the final amounts in the sinking fund is right skewed.
3.2 Changes on the sinking fund rate follow the Uniform Distribution on [-3,3]

3.2.1 10 years loan

- The amount of the loan is 10,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>12,468.75</td>
<td>11,818.71</td>
<td>11,244.46</td>
<td>10,636.09</td>
<td>10,111.78</td>
<td>9,534.424</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1,171.46</td>
<td>1,126.63</td>
<td>1,090.8</td>
<td>1,022.35</td>
<td>968.52</td>
<td>907.99</td>
</tr>
</tbody>
</table>

Table 3.6: Averages and standard deviations when changes on the sinking fund rate $\sim U[-3,3]$ with boundaries on $j$: $n = 10$ years; $L = 10,000$
Changes on interest rate of the sinking fund follow the Uniform distribution over (−3,3) curve with boundaries lump−sums over 10 years

Figure 3.11: Histogram of lump-sums when changes on interest rate $\sim U[-3,3]$ and have boundaries: $n=10$ years; starting $j = 3\%$; $L=10,000$; 1000 simulated lump values

- The average amounts in the sinking fund decreases as $j$ increases
- The standard deviation of the amount in the sinking fund decreases as $j$ increases

- The amount of the loan is 100,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>124278.0</td>
<td>119501.1</td>
<td>112377.6</td>
<td>106502.2</td>
<td>101143.0</td>
<td>95444.77</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>11872.56</td>
<td>11391.92</td>
<td>10792.99</td>
<td>10031.05</td>
<td>9817.09</td>
<td>9238.31</td>
</tr>
</tbody>
</table>

Table 3.7: Averages and standard deviations when changes on the sinking fund rate $\sim U[-3,3]$ with boundaries on $j$ : $n=10$ years; $L=100,000$
Results

Changes on interest rate of the sinking fund follow the Uniform distribution over \((-3,3)\) curve with boundaries

Figure 3.12: Histogram of lump-sums when changes on interest rate $\sim U[-3,3]$ and have boundaries: $n=10$ years; starting $j = 4\%$; $L=100,000$; 1000 simulated lump values

There is a linear effect on the average amounts in the sinking fund and their standard deviations. The patterns are similar for the case where the principal amount is 10,000.

3.2.2 30 years loan

- The amount of the loan is 10,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>20,787.10</td>
<td>17,683.75</td>
<td>14,938.40</td>
<td>12,404.61</td>
<td>10,251.36</td>
<td>8,429.453</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>4,069.64</td>
<td>3,498.87</td>
<td>2,962.77</td>
<td>2393.35</td>
<td>1,890.6</td>
<td>1,615.04</td>
</tr>
</tbody>
</table>

Table 3.8: Averages and standard deviations when changes on the sinking fund rate $\sim U[-3,3]$ with boundaries on $j$: $n=30$ years; $L=10,000$
Changes on interest rate of the sinking fund follow the Uniform distribution over \((-3,3)\) curve with boundaries.

- The average amounts in the sinking fund decreases as \(j\) increases
- The standard deviations of the amount in the sinking fund decreases as \(j\) increases

- The amount of the loan is 100,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>209.244.2</td>
<td>177.722.8</td>
<td>150.774.5</td>
<td>124.596.7</td>
<td>104.002</td>
<td>83.932.2</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>41.316.1</td>
<td>36.307.2</td>
<td>29.548.6</td>
<td>23.752.2</td>
<td>20.659.7</td>
<td>16.758.2</td>
</tr>
</tbody>
</table>

Table 3.9: Averages and standard deviations when changes on the sinking fund rate \(\sim U [-3,3]\) with boundaries on \(j\): \(n = 30\) years; \(L = 100,000\)
Results

Changes on interest rate of the sinking fund follow the Uniform distribution over (−3,3) curve with boundaries lump − sums over 30 years

Frequency

100000 150000 200000 250000
0 50 100 150 200 250

Figure 3.14: Histogram of lump-sums when changes on interest rate \( \sim U[−3, 3] \) and have no boundaries: \( n = 30 \) years; starting \( j = 4\% \); \( L = 100,000 \); 1000 simulated lump values

There is a linear effect on the average amounts in the sinking fund and their standard deviations. The patterns are similar for the case where the principal amount is 10,000.

3.3 Sinking fund during an period without inflation

3.3.1 The random shocks follow the Standard Normal distribution

10 years loan and the amount of the loan is 10,000
Table 3.10: Averages and standard deviations when changes on the sinking fund rate follow an ARIMA process in the period of no inflation: \( n = 10 \) years; \( L = 10,000 \); the random shocks \( \sim N(0,1) \)

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>10,050</td>
<td>10,025.36</td>
<td>10,030.81</td>
<td>10,061.45</td>
<td>10,062.28</td>
<td>10,061.82</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>273.33</td>
<td>265.25</td>
<td>283.96</td>
<td>292.89</td>
<td>278.81</td>
<td>296.64</td>
</tr>
</tbody>
</table>

Figure 3.15: Histogram of lump-sums when changes on interest rate follow an ARIMA process in the period of no inflation: \( n = 10 \) years; starting \( j = 3\% \); \( L = 10,000 \); 1000 simulated lump values; the random shocks \( \sim N(0,1) \); 1000 simulated lump values

We observe that:

- On average the amount in the sinking fund is enough to pay off the debt

- The standard deviations of the amounts in the sinking fund are pretty small (less than 3% of the amount of the principal) and increase slightly as the starting \( j \) increases.
Results

30 years loan and the amount of the loan is 100,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>104,893.5</td>
<td>104,534.1</td>
<td>105,891.7</td>
<td>106,744.4</td>
<td>107,340.4</td>
<td>109,007.9</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>26,967.5</td>
<td>28,442.2</td>
<td>30,472</td>
<td>32,817.6</td>
<td>34,952.6</td>
<td>36,166.08</td>
</tr>
</tbody>
</table>

Table 3.11: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process in the period without inflation: \( n = 30 \) years; \( L = 100,000 \); the random shocks \( \sim N(0,1) \)
Changes on interest rate of the sinking fund follow an ARIMA process in period of no inflation – The Residuals ~ N(0,1)

Figure 3.16: Histogram of lump-sums when changes on interest rate follow an ARIMA process in the period of no inflation: \( n = 30 \) years; Starting \( j = 4\% \); \( L = 100,000 \); 1000 simulated lump values; the random shocks \( \sim N(0,1) \); 1000 simulated lump values

There is a lot variability in the final amounts in the sinking fund.

### 3.3.2 The random shocks follow a Gamma distribution

We first simulated 1000 lump values without any boundaries on the variations of the interest on the sinking fund; that led us to obtain very high lump values. We decided to put an upper bound on the variations on the interest rates of the sinking fund. Interest rates higher than 12% would be considered as 12%.

10 years loan and the amount of the loan is 10,000
Table 3.12: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process in the period of no inflation with boundaries on $j: n = 10$ years; $L = 10,000$; the random shocks $\sim \text{Gamma}(\text{Shape} = 3, \text{Scale} = 1)$

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>12,300.23</td>
<td>11,678.52</td>
<td>11,067.82</td>
<td>10,491.24</td>
<td>9,952.857</td>
<td>9,430.735</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>714.079</td>
<td>662.558</td>
<td>642.3112</td>
<td>615.4577</td>
<td>567.4575</td>
<td>555.3329</td>
</tr>
</tbody>
</table>

Figure 3.17: Histogram of lump-sums when changes on interest rate follow an ARIMA process with boundaries in the period of no inflation: $n = 10$ years; starting $j = 4\%$; $L = 10,000$; the random shocks $\sim \text{Gamma}(3, 1)$; 1000 simulated lump values

On average, the borrower is benefiting from the right skewness of the Gamma distribution but the uncertainty is a bit higher compared to the case where the random shocks $\sim \text{N}(0, 1)$.

30 years loan and the amount of the loan is 100,000
Table 3.13: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process in the period of no inflation with boundaries on \( j \): \( n = 30 \) years; \( L = 100,000 \); the random shocks \( \sim \text{Gamma} (\text{Shape} = 3, \text{Scale} = 1) \)

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>196,881.5</td>
<td>166,161.3</td>
<td>140,737.9</td>
<td>115,590.4</td>
<td>96,885.28</td>
<td>79,664.44</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>24,085.51</td>
<td>20,598.91</td>
<td>16,674.03</td>
<td>13,243.39</td>
<td>11,681.71</td>
<td>9,769.105</td>
</tr>
</tbody>
</table>

Figure 3.18: Histogram of lump-sums when changes on interest rate follow an ARIMA process with boundaries in the period of no inflation: \( n = 30 \) years; Starting \( j = 4\% \); \( L = 100,000 \); the residuals \( \sim \text{Gamma} (3, 1) \); 1000 simulated lump values

3.4 Sinking fund during an inflation period

3.4.1 The random shocks follow the standard normal distribution

10 years loan and the amount of the loan is 10,000
### Results

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>10,017.87</td>
<td>10,029.99</td>
<td>10,030.81</td>
<td>10,038.89</td>
<td>10,050.03</td>
<td>10,060.58</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>97.86</td>
<td>104.87</td>
<td>103.02</td>
<td>105.77</td>
<td>107.49</td>
<td>110.67</td>
</tr>
</tbody>
</table>

Table 3.14: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process in the period of inflation: \( n = 10 \) years; \( L = 10,000 \); the random shocks \( \sim N(0,1) \)

![Histogram of lump-sums when changes on interest rate follow an ARIMA process in the period of inflation: n = 10 years; starting \( j = 4\% \); L = 10,000; the random shocks \( \sim N(0,1) \); 1000 simulated lump values](image)

Figure 3.19: Histogram of lump-sums when changes on interest rate follow an ARIMA process in the period of inflation: \( n = 10 \) years; starting \( j = 4\% \); \( L = 10,000 \); the random shocks \( \sim N(0,1) \); 1000 simulated lump values

On average, the amounts in the sinking fund at the end of 10 years is enough to pay off the loan. The standard deviations are about 1% of the amount of the loan meaning the risk that the borrower does not get enough money at the end of 10 years is relatively low.

30 years loan and the amount of the loan is 100,000
Table 3.15: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process in the period of inflation: \( n = 30 \) years; \( L = 100,000 \); the random shocks \( \sim N(0,1) \)

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>100,304.0</td>
<td>100,252.9</td>
<td>100,438.0</td>
<td>100,393.2</td>
<td>100,961.7</td>
<td>101,083.7</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>5,019.47</td>
<td>5,324.57</td>
<td>5,558.73</td>
<td>5,849.06</td>
<td>6,091.62</td>
<td>6,165.75</td>
</tr>
</tbody>
</table>

Figure 3.20: Histogram of lump-sums when changes on interest rate follow an ARIMA process in the period of inflation: \( n = 30 \) years; starting \( j = 3\% \); \( L = 100,000 \); the random shocks \( \sim N(0,1) \); 1000 simulated lump values

The amounts in the sinking fund is on average enough to pay off the loan with standard deviations of about 5% the amount of the loan.

3.4.2 The random shocks follow a Gamma distribution

As for the period without inflation, we will put an upper bound on the variations on the interest rates of the sinking fund. Interest rates higher than 12% would be considered as 12%.
10 years loan and the amount of the loan is 10,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>12,163.04</td>
<td>11,530.85</td>
<td>10,952.36</td>
<td>10,384.05</td>
<td>9,827.843</td>
<td>9,297.757</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>682.5762</td>
<td>709.3229</td>
<td>647.8475</td>
<td>602.3469</td>
<td>561.6924</td>
<td>528.9065</td>
</tr>
</tbody>
</table>

Table 3.16: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process with boundaries on $j$ in the period of inflation with boundaries on $j$: $n = 10$ years; $L = 10,000$; the random shocks $\sim \text{Gamma}(\text{Shape} = 3, \text{Scale} = 1)$
Change on interest rate of the sinking fund follows an ARIMA process in period of inflation
The residuals follow a Gamma distribution
Lump−sums over 10 years

Figure 3.21: Histogram of lump−sums when changes on interest rate follow an ARIMA process with boundaries in the period of inflation: n = 10 years; starting \( j = 4\% \); \( L = 10,000 \); the random shocks \( \sim Gamma(3, 1) \); 1000 simulated lump values

The amount in the sinking fund at the end of 10 years is on average higher than the target amount; 10,000. The risk for the borrower to not accumulate sufficient amount to repay the loan is higher compared to the similar scenario with the random shocks following \( N(0, 1) \).

### 30 years loan and the amount of the loan is 100,000

<table>
<thead>
<tr>
<th>Interest rates</th>
<th>2%</th>
<th>3%</th>
<th>4%</th>
<th>5%</th>
<th>6%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulated mean</td>
<td>187,708.0</td>
<td>159,057.9</td>
<td>134,015.3</td>
<td>110,593.7</td>
<td>92,235.1</td>
<td>75,334.5</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>22,051.76</td>
<td>18,267.91</td>
<td>15,385.41</td>
<td>12,711.97</td>
<td>10,720.11</td>
<td>8,675.424</td>
</tr>
</tbody>
</table>

Table 3.17: Averages and standard deviations when changes on the sinking fund rate follows an ARIMA process with boundaries on \( j \) in period of inflation with boundaries on \( j \): n = 30 years; \( L = 100,000 \); the random shocks \( \sim Gamma(Shape = 3, Scale = 1) \)
Change on interest rate of the sinking fund follows an ARIMA process in period of inflation. The residuals follow a Gamma distribution.

Lump-sums over 30 years

Figure 3.22: Histogram of lump-sums when changes on interest rate follow an ARIMA process with boundaries in period of inflation: \( n = 30 \) years; starting \( j = 4\% \); \( L = 100,000 \); the random shocks \( \sim Gamma(3,1) \) - 1000 simulated lump values

The amount in the sinking fund at the end of 30 years decreases as \( j \) increases. There exists some variability in these amount; meaning there is a considerable risk that the final amount in the sinking is not enough to pay off the debt.
From our study, we conclude that, in general, when changes on the interest of the sinking fund follow the standard normal distribution or the uniform distribution between -3 and 3, as the starting interest rate in the sinking fund increases, the expected amount in the sinking fund decreases, but has less variability. For high values of $j$, the borrower could easily not have enough money to pay off the loan. When the changes on the interest rate of the sinking fund follow an ARIMA process, the target amount is reached on average, but there could be a lot of variability, in these amounts meaning that the risk for the borrower to not have sufficient funds to pay off the loan could be very high. The situations profitable to the borrower depend not only on the starting interest of the sinking fund and if the average accumulated amount in the sinking fund would be enough to pay off the debt; but also on the risk (quantified by the standard deviation of the lump sums) that presents each situation.
There are some limitations to our study. Other than the cases where the changes on the interest rate follow ARIMA processes, our study assumed independence, which is not realistic. In addition, our study did not model the inflation when we built the ARIMA process corresponding to the inflation period.

We assumed that the amount paid by the borrower at the end of each period was fixed; it may be interesting to look at scenarios where the changes on the interest rate of the loan is also a random variable that follows the same or different distribution as the distribution for changes on interest rate of the sinking fund.
Bibliography


Appendices
Appendix 1: R Function for a loan over 3 or 4 years, change on the sinking fund: follows a standard normal curve and there is no boundaries on the sinking fund rates

# Model of a stochastic process for an annual sinking fund rate
# that varies up or down each year

# L is the amount of the loan
# n is the time of the loan in years
# i is the interest rate governing the loan in %
# j is the interest rate earns by the sinking fund in % at the time of the loan
# p is the amount to put in the sinking fund every year
# R is the amount in the sinking fund at the end of n years

# The borrower will give the amount L*i at end of each period to
the lender and will put the amount \( q \) in a sinking fund earning \# 
a interest rate \( j \). If \( j \) does not change, the accumulated amount \# 
in the sinking fund at the end of the time of the loan will be \# 
\( K = L \). 

The interest rate on the sinking fund follows a standard normal 
curve \( n = 3 \) or \( 4 \) years. There are boundaries on \( j \)

\[
lumpstdn34=function(L,n,j) 
{
  s=((1+(j/100))^n)-1)/(j/100)
  q=L/s
  R=rep(0,1000)
  L=rep(0,n)
  for(m in 1:1000)
  {
    L[1]=q
    for (k in 1:(n-1))# Amount in the sinking fund at the end of 
      # the first year
    
  
}
l = j + rnorm(1)

L[k + 1] = L[k] * (1 + (l/100)) + q # Amount in the sinking fund at
# the end of the (k+1)th period

} of the
R[m] = L[n]

return(list(R, mean(R), sd(R), hist(R, xlab="Lump-sums", main="changes on interest rate of the sinking fund follow the
    standard normal curve and have no boundaries")))

}
Appendix 2: R Function for a loan over n years, change on the sinking fund follows a standard normal curve and on the sinking fund rates are between 0% and 12%

# Changes of the interest rate on the sinking follow the standard normal curve

lumpstdn=function(L,n,j)
{
  s=(((1+(j/1200))^(n*12))-1)/(j/1200)
  q=L/s
  R=rep(0,1000)
  J=rep(0,(n*12))
  j=j+rnorm(1)

  for(m in 1:1000)
{ 
    J[1]=q*(1+(j/1200))
    for (k in 1:((n*12)-1))
    {
        j=j+rnorm(1)
        if (j>=0)
        {
            if(j>12)
            {
                j=12
                J[k+1]=(J[k]+q)*(1+(1/100))
            }
            else{
                J[k+1]=(J[k]+q)*(1+(j/1200))
            }
        }
        else {
            j=0
            J[k+1]=(J[k]+q)
        }
    } else { 
        J[k+1]=(J[k]+q)*(1+(j/1200))
    }
} else { 
    R[m]=J[(n*12)]
return(list(R, mean(R), sd(R), hist(R, xlab="Lump-sums", main="Changes on interest rate of the sinking fund follow the normal curve")))
Appendix 3: R Function for a loan over n years, change on the sinking fund follows a Uniform distribution on $[-3, 3]$ and on the sinking fund rates are between 0% and 12%

# Interest rate on the sinking fund varies up or down by chut 0-3% each year

lumpunif=function(L,n,j)
{
  s=(((1+(j/1200))^(n*12))-1)/(j/1200)
  q=L/s
  R=rep(0,1000)
  J=rep(0,(n*12))
  j=j+sample(-3:3,1)

  for(m in 1:1000)
  {
  

\[ J[1] = q \times (1 + \frac{j}{1200}) \]

for (k in 1:((n*12)-1))
{
    j=j+sample(-3:3,1)
    if (j>0)
    {
        if(j>12)
        {
            j=12
            J[k+1]=(J[k]+q) \times (1 + \frac{1}{100})
        } else{
            J[k+1]=(J[k]+q) \times (1 + \frac{j}{1200})
        }
    } else {
        j=0
        J[k+1]=(J[k]+q)
    }
}

R[m]=J[(n*12)]

return(list(R,mean(R),sd(R),hist(R, xlab="lump-sums", main="Interest rate on the sinking fund"))
varies up or down by chut 0-3% each year "))))

}
Appendix 4: Time series identification

options pageno=1 nodate;
*libname 'C:\placede';
run;

odsrtffile="C:\placede\identifytimeseries2.rtf";
odsgraphicson;
title1 'Identification of parameters for several time series';

procimport datafile='C:\placede\cdrates.xls'
out= cdratesdbms=xls replace;
GETNAMES=YES;
run;
procimport datafile='C:\placede\cdrates.xls'
out= cdrates2 dbms=xls replace;
SHEET="cdrates2";
GETNAMES=YES;
run;
procimport datafile='C:\place\cdrates.xls'
out= cdrates3 dbms=xls replace;
SHEET="cdrates3";
GETNAMES=YES;
run;

title2"Identifying order of full raw time series";
procari data = cdrates plots(unpack);
identifyvar = value;
run;

title2"Identifying order of raw time series in time of no inflation"
procari data = cdrates2 plots(unpack);
identifyvar = value;
run;

title2"Identifying order of raw time series in time of inflation";
procari data = cdrates3 plots(unpack);
identifyvar = value;
run;

title2"Identifying order of differenced full time series";
procarimadata = cdrates plots(unpack);
identifyvar = value(1);

run;

title2"Identifying order of differenced time series in time of no inflation ";
procarimadata = cdrates2 plots(unpack);
identifyvar = value(1);

run;

title2"Identifying order of differenced time series in time of inflation";
procarimadata = cdrates3 plots(unpack);
identifyvar = value(1);
run;
title2 "Identifying order of differenced at different lags for the full time series";
procarimadata = cdrates plots(unpack);
identifyvar = value(1);
estimatep=9q=9plot;
estimate p=(1,3,6,7,9) q=(1,3,6,7,9) plot;

/* estimate p=(1,8) q=9 plot;
estimate p=(1,6) q=9 plot;
estimate p=(1,2) q=9 plot;
estimate p=1 q=9 plot;
estimate p=8 q=9 plot;
estimate p=8 plot;
estimate q=9 plot;

run;
ods graphics off;
ods rtf close;
*/
title2"Identifying order of differenced at different lags time series in time of no inflation ";
procarimadata = cdrates2 plots(unpack);
identifyvar = value(1);
estimatep=(1,6,12) q=(1,6,12) plot;
estimateq=(1,6,12) p=(1,12) plot ;
estimatep=(12,20) q=(1,12) plot;
estimatep=(12,20) q=(1,6,12) plot;
estimatep=(1,12,20) q=(1,6,12) plot;
run;
/*
estimate p=6 plot;
estimate p=12 plot;
estimate q=(1,6,12)plot;
estimate q=6 plot;
estimate q=(1,12) p=12 plot;
*/

title2"Identifying order of differenced time series at different lags in time of inflation";
procarimadata = cdrates3 plots(unpack);
identifyvar = value(1);
estimatep=(1,10) q=(1,10) plot;
estimatep=(1) q=(1,10) plot;
estimatep=(1,10) q=(1) plot;
estimatep=(1) q=(1) plot;
estimatep=(10) plot;
estimateq=(10) plot;
/*
estimate p=(10) plot;
estimate p=1 q=(10) plot;
estimate p=(10) q=1 plot;
estimate q=1 plot;
estimate q=(10) plot;
*/
run;

odsgraphicsoff;
odsrtfclose;
Appendix 5: R Function for a loan over n years, change on the sinking fund follows an ARIMA process in time of no inflation

- The residuals $\sim N(0, 1)$

# Changes of the interest rate on the sinking fund follow an ARIMA model with x obtained by simulation of an ARIMA process.

```r
y=read.table("cdrates2.txt", head=TRUE, sep = "\t")
y1=y[,2]
Z1=y[,3]
E1=y[,4]

phi1=0.33024
phi12 = -0.08989
phi20 = -0.1095
```
\[
\theta_1 = 0.20483 \\
\theta_6 = -0.013256 \\
\theta_{12} = 0.48454
\]

\[
lump\text{parimano}in\text{f} = \text{function}(L, n, j) \\
\{
\begin{align*}
  s &= ((1 + (j/1200))^{n*12}) - 1)/(j/1200) \\
  q &= L/s \\
  R &= \text{rep}(0, 1000) \\
  J &= \text{rep}(0, (n*12)) \\
  Z2 &= \text{rep}(0, 500) \\
  Z1 &= \text{rev}(Z1) \\
  Z &= c(Z1[1:20], Z2) \\
  E1 &= \text{rev}(E1)
\end{align*}
\]

\[
\text{for}(m \text{ in } 1:1000) \\
\{
\begin{align*}
  E2 &= \text{rnorm}(500, 0, 0.140623)
\end{align*}
\]

E=c(E1[1:20], E2)

for(i in 21:520){
    Z[i]= phi1*Z[i-1] + phi12*Z[i-12] + phi20*Z[i-20] +
            E[i] + theta1*E[i-1] + theta6*E[i-6] + theta12*E[i-12]
}

x=Z[(521-(n*12)):520] - mean(Z[(521-(n*12)):520])

j=j+x[1]
J[1]=q*(1+(j/1200))

for(k in 1:(n*12)-1)
{
    j=j+x[k+1]
    J[k+1]=(J[k]+q)*(1+j/1200)
}

R[m]=J[(n*12)]

return(list(R, mean(R), sd(R), hist(R, xlab="Lump-sums", main="change on interest rate of the"))
sinking fund follows an ARIMA process in period of 
no inflation ")))}
}

• The residuals $\sim \text{Gamma}(3, 1)$

lumparimag=function(l,n,j)
{
  s=(((1+(j/1200))^(n*12))-1)/(j/1200)
  q=1/s
  R=rep(0,1000)
  J=rep(0,(n*12))

  Z2=rep(0,500)
  Z1=rev(Z1)
  Z=c(Z1[1:20],Z2)
  E1=rev(E1)

  for(m in 1:1000)
  {
    a=rgamma(500,shape=3,scale=1)  # alpha=3, lambda=1
    E2=a-mean(a)
\[ E = c(E_1[1:20], E_2) \]

\[
\text{for}(i \text{ in } 21:520) \{
Z[i] = \phi_1 Z[i-1] + \phi_{12} Z[i-12] + \phi_{20} Z[i-20] + E[i] + \theta_1 E[i-1] + \theta_{6} E[i-6] + \theta_{12} E[i-12]
\}
\]

\[ x = Z[(521-(n*12)):520] - \text{mean}(Z[(521-(n*12)):520]) \]

\[ j = j + x[1] \]

\[ J[1] = q \times (1 + (j/1200)) \]

\[
\text{for} (k \text{ in } 1:((n*12)-1))
\{ 
\quad j = j + x[k+1] 
\quad \text{if} \ (j \geq 0)
\quad \{ 
\quad \quad \text{if}(j > 12)
\quad \quad \{ 
\quad \quad \quad j = 12 
\quad \quad \quad J[k+1] = (J[k] + q) \times (1 + (1/100)) 
\quad \quad \} \ \text{else}
\quad \quad \{ 
\quad \quad \quad J[k+1] = (J[k] + q) \times (1 + (j/1200)) 
\quad \quad \} 
\} 
\]
else {
    j=0
    J[k+1]=(J[k]+q)
}

R[m]=J[(n*12)]

return(list(R,mean(R),sd(R),
hist(R,xlab="Lump-sums over 10 years",
main="Changes on the interest rate of the sinking fund
follow an ARIMA process in period of no inflation
The residuals follow a Gamma distribution")))
Appendix 6: Changes on the interest rate of the sinking fund follow an ARIMA process during the inflation period

- The residuals $\sim N(0,1)$

```r
y=read.table("cdrates3.txt", head=TRUE, sep = "\t")
y1=y[,2]
Z1=y[,3]
E1=y[,4]

phi1= -0.24674
phi10= 0.27363
theta10= -0.46482

lumparimainf=function(l,n,j)
{
  s=(((1+(j/1200))^(n*12))-1)/(j/1200)
}
```
q=1/s
R=rep(0,1000)
J=rep(0,(n*12))

Z2=rep(0,500)
Z1=rev(Z1)
Z=c(Z1[1:10],Z2)
E1=rev(E1)

for(m in 1:1000)
{
  E2=rnorm(500,0,0.089764)
  E=c(E1[1:10],E2)

  for(i in 11:510)
  {
    Z[i]=phi1*Z[i-1]+phi10*Z[i-10]+E[i]+theta10*E[i-10]
  }
  x=Z[(511-(n*12)):510]-mean(Z[(511-(n*12)):510])

  j=j+x[1]
  J[1]=q*(1+(j/1200))
for (k in 1:((n*12)-1))
{
    j=j+x[k+1]
    J[k+1]=(J[k]+q)*(1+j/1200)
}
R[m]=J[(n*12)]

return(list(R,mean(R),sd(R),hist(R,xlab="Lump-sums",main="Changes on the interest rate of the sinking fund follow an ARIMA process in period of inflation
The residuals follow a Normal distribution")))

• The residuals $\sim $ Gamma (3, 1)

lumparimag2=function(L,n,j)
{
    s=(((1+(j/1200))^(n*12))-1)/(j/1200)
    q=L/s
R=rep(0,1000)

J=rep(0,(n*12))

Z2=rep(0,500)
Z1=rev(Z1)
Z=c(Z1[1:10],Z2)
E1=rev(E1)

for(m in 1:1000){
  a=rgamma(500,shape=3,scale=1)  # alpha=3, lambda=1
  E2=a-mean(a)
  E=c(E1[1:10],E2)
  for(i in 11:510){
    Z[i]=phi1*Z[i-1]+phi10*Z[i-10]+E[i]+theta10*E[i-10]
  }
  x=Z[(511-(n*12)):510]-mean(Z[(511-(n*12)):510])
  j=j+x[1]
  J[1]=q*(1+(j/1200))}
for (k in 1:((n*12)-1))
{
    j=j+x[k+1]
    if (j>=0)
    {
        if(j>12)
        {
            j=12
            J[k+1]=(J[k]+q)*(1+(1/100))
        } else{
            J[k+1]=(J[k]+q)*(1+(j/1200))
        }
    } else { j=0 J[k+1]=(J[k]+q) }
}
R[m]=J[(n*12)]
return(list(R,mean(R),sd(R),hist(R,xlab="Lump-sums",
main="Change on interest rate of the sinking fund follows
an ARIMA process in period of inflation ")))
}