Valuation and Asset Pricing in Infinite-Horizon Sequential Markets With Portfolio Constraints

Kevin X.D. Huang
Utah State University

Follow this and additional works at: https://digitalcommons.usu.edu/eri

Recommended Citation
https://digitalcommons.usu.edu/eri/185
VALUATION AND ASSET PRICING IN INFINITE-HORIZON SEQUENTIAL MARKETS WITH PORTFOLIO CONSTRAINTS

by

KEVIN X.D. HUANG

Department of Economics
Utah State University
3530 Old Main Hill
Logan, UT 84322-3530

February 2000
VALUATION AND ASSET PRICING IN INFINITE-HORIZON SEQUENTIAL MARKETS WITH PORTFOLIO CONSTRAINTS

Kevin X.D. Huang, Assistant Professor

Department of Economics
Utah State University
3530 Old Main Hill
Logan, UT 84322-3530

The analyses and views reported in this paper are those of the author(s). They are not necessarily endorsed by the Department of Economics or by Utah State University.

Utah State University is committed to the policy that all persons shall have equal access to its programs and employment without regard to race, color, creed, religion, national origin, sex, age, marital status, disability, public assistance status, veteran status, or sexual orientation.

Information on other titles in this series may be obtained from: Department of Economics, Utah State University, 3530 Old Main Hill, Logan, Utah 84322-3530.

Copyright © 2000 by Kevin X.D. Huang. All rights reserved. Readers may make verbatim copies of this document for noncommercial purposes by any means, provided that this copyright notice appears on all such copies.
We develop a theory of valuation of payoff streams in infinite-horizon sequential markets and discuss implications of this theory for equilibrium under various portfolio constraints. We study the nature of asset price bubbles in light of this theory. We show that there cannot be equilibrium price bubbles on asset in positive net supply under a transversality restriction. Our analysis extends the work by Huang and Werner [9] to stochastic settings with complete or incomplete markets.

*JEL classification: C61, D50, G10, G12*

Key words: valuation, asset price bubble, portfolio constraint.
VALUATION AND ASSET PRICING IN INFINITE-HORIZON SEQUENTIAL MARKETS WITH PORTFOLIO CONSTRAINTS*

1 Introduction

This paper is concerned with the issue of payoff valuation and asset pricing in an equilibrium model of sequential trading over an infinite horizon under constraints on portfolio strategies.

In the literature on sequential markets with portfolio constraints, the value of a stream of payoffs is usually measured by a functional that maps each payoff stream to the lowest price of a feasible portfolio strategy that generates it. Two important properties that the payoff pricing functional may or may not have are linearity and countable additivity. Linearity implies that valuation is additive, that is, the value of the sum of two payoff streams is equal to the sum of their values. Value additivity plays a fundamental role in the theory of financial markets. The well-known Modigliani-Miller theorem is one of its implications. Countable additivity implies a representation of the value of a stream of payoffs by an infinite sum of values of the payoff at each date.

Huang and Werner [9] demonstrate that, in the case of a single asset and no uncertainty, the payoff pricing functional is linear and countably additive on the set of nonnegative payoff streams if a constraint allows delayed market participation and does not restrict long positions in the asset and if there is no feasible arbitrage.

An arbitrage is a portfolio strategy that generates a positive payoff stream at a nonpositive price or a nonnegative payoff stream at a negative price, of which a Ponzi scheme is an example. A Ponzi scheme is a strategy of rolling over a debt forever and thereby never paying it back. LeRoy and Werner [11, pp. 33-37] distinguish two types of feasible arbitrage: unlimited feasible arbitrage that can be operated on an arbitrary scale and generally cannot exist in equilibrium. A limited feasible arbitrage is a feasible arbitrage that can only be operated on a finite scale and can potentially exist in equilibrium, depending on the form of a constraint.

In this paper we extend the work by Huang and Werner [9] to stochastic settings

---

*I am very grateful to Steve LeRoy, Manuel Santos, and Jan Werner for valuable advise and helpful comments. I have also benefited from comments and suggestions of Jerome Detemple, Ed Green, Narayana Kocherlakota, Ket Richter, and seminar participants at Minnesota, McGill, Midwest Economic Theory Meetings (Lansing), and Econometric society Meetings (New York). The usual disclaimer applies.
with possible incompleteness of markets. In Theorem 4.1 we show that if there is no feasible arbitrage and if a constraint allows delayed market participation and does not restrict positive net worth (when only a single asset is traded at a positive price, no restriction on positive net worth is equivalent to no restriction on long positions in the asset), then the payoff pricing functional is sublinear on the set of non-negative payoff streams. Sublinearity implies that valuation is subadditive, that is, the value of the sum of two payoff streams is no greater than the sum of their values. As illustrated by Example 4.1, the former can be strictly lower than the latter and the payoff pricing functional can be non-linear when markets are incomplete. We establish in Corollary 5.1 that when markets are complete the payoff pricing functional is linear as well as countably additive.

There are various constraints considered in the literature that satisfy the conditions of Theorem 4.1. Limited feasible arbitrage can exist in equilibria with some of the constraints but not in equilibria with others. Huang and Werner [9, Example 7.1] present an equilibrium of the former type; in this equilibrium, there exist limited feasible Ponzi schemes and the payoff pricing functional is not sublinear. We here observe that in the case of any equilibrium with any of the constraints considered by Hernandez and Santos [8] there exists no feasible arbitrage and, therefore, the payoff pricing functional is sublinear, and linear and countably additive when markets are complete.

A different type of constraint takes the form of a transversality restriction that requires expected net worth to be zero at infinity. Such transversality restriction is analyzed by Hernandez and Santos [7] and Magill and Quinzii [13], and is often considered in the modern finance and macroeconomics literatures. A remarkable feature of this type of constraint is that in the case of any equilibrium there exists no feasible arbitrage and the payoff pricing functional is linear and countably additive on the set of all payoff streams regardless of whether markets are complete or incomplete. We establish this result in Theorem 5.1.

Our theory of valuation can be applied to analyze the nature of equilibrium asset price bubbles. A price bubble occurs when the price of an asset differs from
the present value of the stream of future dividends to which the asset is a claim. The form of a constraint is crucial for the aforementioned valuation properties or lack thereof in the presence of a price bubble.¹ Huang and Werner [9] provide three examples of equilibrium price bubbles in which valuation is linear and countably additive, not sublinear, or linear but not countably additive. We here point out another possible case in light of Theorem 4.1 and Example 4.1: under a constraint considered by Hernandez and Santos [8] there are typically equilibria with price bubbles on infinitely-lived assets that are in zero net supply (see, e.g., Magill and Quinzii [14] and Huang and Werner [9]), so that equilibrium price bubbles may occur with sublinear but non-linear valuation in the case of incomplete markets.

Santos and Woodford [15] provide sufficient conditions for there being no price bubbles in equilibrium with a constraint that net worth be limited by a pre-specified non-positive lower bound which may be time- and price-dependent. When markets are incomplete, there are more than one system of state prices to be used in calculating the present value of a payoff stream and of the dividend stream of an asset. If the present value of the aggregate endowment of goods is finite under all such systems, then there exists some system in terms of which there are no price bubbles on assets in positive net supply. Under an assumption of a sufficient degree of impatience about consumers’ preferences, the present value of the aggregate endowment being finite for some state-price system implies the non-existence of price bubbles on assets in positive net supply with respect to the same system.

We present in Theorem 6.1 a result about the non-existence of price bubbles in equilibrium with the aforementioned transversality restriction, which is not studied by Santos and Woodford [15]. We show that there are no price bubbles on assets in positive net supply regardless of the system of state prices chosen. Our result

¹Gilles and LeRoy [3][4] analyze the nature of price bubbles for an economy in which forward contracts for all dated and contingent future goods (assets) are traded in a single marketplace. The definition of an equilibrium in their framework does not involve any constraint on holding the forward contracts. Huang and Werner [9] study the relation between Gilles and LeRoy bubbles and bubbles in sequential equilibrium under various portfolio constraints. See also Gilles and LeRoy [5] for an alternative specification of market arrangement in modeling payoff bubbles.
does not depend on market completeness, endowments of goods, or consumers’ preferences.

Luttmer [12] shows that the payoff pricing functional under a convex cone constraint in a two-period market is sublinear if there is no feasible arbitrage. Santos and Woodford [15] study the problem of payoff valuation in infinite-horizon markets under a restriction that portfolio net worth be non-negative. They are primarily concerned with sufficient conditions for there being no equilibrium asset price bubbles. Huang and Werner [9] provide a systematic analysis of both the issue of payoff valuation and of asset pricing in infinite-horizon markets, and of their relations under various portfolio constraints. Yet they confine the analysis to an economy with no uncertainty. In this paper, we extend the work by Huang and Werner [9] to an economy with uncertainty. We characterize a class of portfolio constraints under which valuation is sublinear on the set of non-negative payoff streams in infinite-horizon markets in the absence of feasible arbitrage. The set of feasible portfolio strategies under such a constraint may not be a cone. Due to the open-ended nature of infinite horizon, the valuation may not be linear when markets are incomplete and the sublinear valuation may not hold for negative payoff streams (if otherwise markets were of finite horizon, then the valuation would be linear on the set of all payoff streams even if markets are incomplete). We prove that under a transversality constraint valuation is linear and countably additive on the set of all payoff streams regardless of whether markets are complete or incomplete, and there are no equilibrium price bubbles in positive net supply.

The rest of the paper is organized as follows. In Section 2 we introduce basic concepts of sequential markets and give a definition of equilibrium. In Section 3 we provide definitions of unlimited feasible arbitrage and limited feasible arbitrage. In Section 4 we prove our result of sublinear valuation, and use an example to illustrate the possible non-linearity of valuation in incomplete markets. Also, we investigate implications of this result for equilibrium under various constraints. In Section 5 we prove our results about linearity and countable additivity of valuation,
and discuss their implications for an equilibrium. In Section 6 we discuss the nature of equilibrium asset price bubbles in light of our valuation theory. Further, we prove our result about the non-existence of equilibrium price bubble. Section 7 contains remarks on extensions of our results. All proofs are contained in the Appendix.

2 The model

Time is discrete, begins at \( t = 0 \) and continues forever. Dynamic uncertainty is described by a set \( \Omega \) of states of the world and an increasing sequence \( \{N_t\}_{t=0}^{\infty} \) of finite partitions of \( \Omega \). We take \( N_0 = \{\Omega\} \) so that there is no uncertainty at date 0. This information structure can be modeled as an event tree \( D \) so that an element \( s^t \in N_t \) is referred to as a date-\( t \) event. For each \( s^t \), we denote by \( s^t_\prec \) its unique immediate preceding event if \( t \neq 0 \), \( \{s^t_+\} \) a finite set of its immediate succeeding events, \( D_{s^t} \) a subtree composed of \( s^t \) and all its succeeding events, and \( D_{s^t}\backslash\{s^t\} \) the subtree excluding its root.

There is a single commodity. Let \( C \) be a linear space of adapted scalar processes. The cone of non-negative processes in \( C \) is the consumption set and is denoted by \( C^+ \). There are \( J \) infinitely-lived assets that are traded at every date in exchange for consumption. Throughout \( d = \{d(s^t)\} \) and \( q = \{q(s^t)\} \) denote an \( \mathbb{R}^J \)-valued adapted process of dividends and of prices of the assets, respectively. To be specific, \( d_j(s^t) \) is a dividend paid before trade at \( s^t \) to the holder of one share of asset \( j \), \( q_j(s^t) \) is an ex-dividend price of \( j \), and thus \( q_j(s^{t+1}) + d_j(s^{t+1}) \) is the payoff of holding one share of \( j \) from \( s^t \) into \( s^{t+1} \in \{s^t_+\} \). We want to allow our analysis to be general enough to allow for assets that are not of limited liability. Therefore, neither dividends \( d \) nor prices \( q \) are presumed non-negative.

We say that markets are one-period complete at \( s^t \) at prices \( q \) if the rank of the one-period-payoff matrix \( \{q(s^{t+1}) + d(s^{t+1})\}_{s^{t+1} \in \{s^t_+\}} \) is equal to the number of immediate succeeding events of \( s^t \). Clearly, for markets to be one-period complete at \( s^t \) it is necessary that \( J \) is at least as large as the number of immediate succeeding events of \( s^t \). Markets are dynamically complete at \( q \) if they are one-period complete
in any event; otherwise, markets are incomplete.

There is a finite number of infinitely-lived agents. Each agent $h$ has the consumption possibility set $C_+$ and a preference $\succeq^h$ on $C_+$. The agent is endowed with a quantity of the consumption good $y^h \in C_+$ and a vector of assets $\bar{\theta}^h \in \mathbb{R}^J$ at $s^0$. The net supply of the assets is therefore $\bar{\theta} = \sum_h \bar{\theta}^h$. We assume that $\bar{\theta} \in \mathbb{R}^J_+$. The aggregate supply of the consumption good is given by $y(s^t) = \sum_h y^h(s^t) + d(s^t)'\bar{\theta}$ at $s^t$, for every $s^t$.

A portfolio strategy is described by an $\mathbb{R}^J$-valued adapted process $\theta = \{\theta(s^t)\}$, where $\theta(s^t)$ specifies the number of shares of assets held after trade in event $s^t$. An equilibrium model of sequential asset trading over an infinite horizon typically involves a constraint on portfolio strategies since otherwise there would generally not exist an optimal strategy for an agent who has a monotone preference over consumption bundles (see, e.g., Huang and Werner [9]). Let $\Theta^h$ be a set of feasible portfolio strategies available to an agent $h$.

An equilibrium consists of an asset price process $q$, and a consumption and portfolio holding plan $(c^h, \theta^h)$ for each agent $h$, such that $(c^h, \theta^h)$ maximizes $\succeq^h$ subject to

\[
\begin{align*}
    c(s^0) + q(s^0)'\theta(s^0) &\leq y^h(s^0) + q(s^0)'\bar{\theta}^h, \\
    c(s^t) + q(s^t)'\theta(s^t) &\leq y^h(s^t) + [q(s^t) + d(s^t)]'\theta(s^t) \quad \text{for every } s^t \neq s^0, \\
    c &\in C_+ \quad \text{and} \quad \theta \in \Theta^h,
\end{align*}
\]

and markets for the consumption good and assets clear, that is

\[
\sum_h c^h(s^t) = y(s^t) \quad \text{and} \quad \sum_h \theta^h(s^t) = \bar{\theta} \quad \text{for every } s^t.
\]

Following the lead of Santos and Woodford [15] throughout we only consider the case of an equilibrium in which for every event $s^t$ at least one of the assets is traded at a positive (negative) price at $s^t$, has non-negative (non-positive) one-period payoffs at all $s^{t+1} \in \{s^t_+\}$, and has a positive (negative) one-period payoff at some $s^{t+1} \in \{s^t_+\}$; that is, in which there is always some way of carrying wealth into the future.
3 Limited and unlimited feasible arbitrage

Given a set $\Theta$ of feasible portfolio strategies, the payoff stream of a strategy $\theta \in \Theta$ is an adapted scalar process denoted by $z^\theta$ where

$$z^\theta(s^t) = [q(s^t) + d(s^t)]'\theta(s^t) - q(s^t)'\theta(s^t) \quad \text{for all } s^t \neq s^0. \quad (1)$$

A feasible arbitrage at prices $q$ is a feasible portfolio strategy $\theta$ such that

$$q(s^0)'\theta(s^0) \leq 0, \quad z^\theta(s^t) \geq 0 \quad \text{for all } s^t \neq s^0, \quad (2)$$

with at least one strict inequality. An example of a feasible arbitrage is a feasible Ponzi scheme. A feasible event-$s^t$ Ponzi scheme at $q$ is a feasible portfolio strategy $\theta$ such that $\theta(s^t) = 0$ for all $s^t \notin D_{st}$ and

$$q(s^t)'\theta(s^t) < 0, \quad z^\theta(s^t) = 0 \quad \text{for all } s^t \in D_{st}\backslash\{s^t\}. \quad \text{Another example of a feasible arbitrage is a feasible one-period arbitrage. A feasible one-period arbitrage in event $s^t$ at $q$ is a feasible portfolio strategy $\theta$ such that $\theta(s^t) = 0$ for all $s^t \neq s^t$ and }$$

$$q(s^t)'\theta(s^t) \leq 0, \quad [q(s^{t+1}) + d(s^{t+1})]'\theta(s^t) \geq 0 \quad \text{for all } s^{t+1} \in \{s^t\}, \quad$$

with at least one strict inequality.

There are two types of feasible arbitrage: unlimited feasible arbitrage and limited feasible arbitrage. An unlimited feasible arbitrage is a feasible arbitrage of which any positive multiple is a feasible arbitrage. A limited feasible arbitrage is any feasible arbitrage that is not an unlimited feasible arbitrage. In an equilibrium model of sequential asset trading with a portfolio constraint the distinction between the two types of feasible arbitrage is important. Unlimited feasible arbitrage generally cannot exist in equilibrium with monotone preferences since it is a feasible arbitrage that can be operated on an arbitrary scale. On the other hand, since a limited feasible arbitrage is a feasible arbitrage that can only be operated on a finite scale it can potentially exist in equilibrium depending on the form of the constraint (see LeRoy and Werner [11, p.33-37] and Huang and Werner [9]).
4 Sublinear valuation

As in Huang and Werner [9], the set of payoff streams of all portfolio strategies in \( \Theta \) is called the asset span, given by

\[ M = \{ z : z = z^\theta \text{ for some } \theta \in \Theta \}, \]

and the operator assigning to each payoff stream \( z \in M \) the lowest initial price of a feasible portfolio strategy that generates \( z \) is called the payoff pricing functional, defined as

\[ V(z) = \inf \{ q(s^0)'\theta(s^0) : \theta \in \Theta \text{ and } z^\theta = z \}. \]

Several important properties that the payoff pricing functional may or may not have are strict positivity, sublinearity, linearity, and countable additivity. If the payoff pricing functional is strictly positive, then there cannot exist any feasible arbitrage.

In this section we characterize portfolio constraints under which the payoff pricing functional is strictly positive and sublinear on the set of non-negative payoff streams when there exists no feasible arbitrage. The following two properties of the set \( \Theta \) of feasible portfolio strategies extend to stochastic settings the properties of delayed market participation and of unrestricted long holdings of an asset in Huang and Werner [9], respectively:

(D) for any \( \theta \in \Theta \) and every \( s^t \), if \( \tilde{\theta} \) is a portfolio strategy given by \( \tilde{\theta}(s^\tau) = \theta(s^\tau) \) for \( s^\tau \in D_{s^t} \) and \( \tilde{\theta}(s^\tau) = 0 \) for \( s^\tau \notin D_{s^t} \), then \( \tilde{\theta} \in \Theta \).

(UP) if \([q(s^t) + d(s^t)]'(s^t) \geq 0 \) for all \( s^t \neq s^0 \), then \( \theta \in \Theta \).

The first property says that modifying a feasible portfolio strategy by zero asset holdings off an arbitrary subtree leads to a feasible strategy; that is, an arbitrary finite delay in participating in the markets is allowed and hence (D). The second property says that if the pre-trade net worth of a portfolio strategy is non-negative in every event, then the strategy is feasible; in other words, positive net worth is unrestricted and hence (UP).
A strictly positive adapted scalar process \( \{a(s^t)\} \) that satisfies

\[
a(s^t)q(s^t) = \sum_{s^{t+1} \in \{s^t\}} a(s^{t+1})[q(s^{t+1}) + d(s^{t+1})]
\]  

at every \( s^t \) is referred to as a system of state prices. Denote by \( A \) the set of all such systems (upon a normalization). If \( A \) is non-empty, and if markets are dynamically complete at \( q \) then \( A \) is a singleton while if markets are not one-period complete in some event then \( A \) contains more than one system of state prices. A system of state prices provides a set of weights to be used in calculating the present value of an adapted payoff stream. Huang and Werner [9, Theorem 5.1] prove that, in the case of a single asset and no uncertainty, if the constraint allows delayed market participation and does not restrict long positions in the asset, then the absence of feasible arbitrage implies that \( A \) is a singleton and the value of a non-negative payoff stream measured by the payoff pricing functional is equal to the present value of the payoff stream.

The following result generalizes Theorem 5.1 of Huang and Werner [9] to stochastic settings with possible incompleteness of markets.

**Theorem 4.1** Suppose that the set \( \Theta \) of feasible portfolio strategies satisfies properties (D) and (UP). There is no feasible arbitrage if and only if \( A \neq \emptyset \), and \( V \) is sublinear on the set of non-negative payoff streams and satisfies

\[
V(z) = \sup_{a \in A} \sum_{s^t \in \Theta \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t)
\]  

for every \( z \in M, \ z \geq 0 \).

In proving this theorem in the Appendix, we form a series of one-period linear programs and their dual problems, one pair for each event, and apply to each pair the duality theory of linear programming. We then apply a dynamic programming technique and a continuity property to the dual formulation. Finally, we connect the infinitely many one-period dual programs to form representation (4) of \( V \).
Theorem 4.1 says that if the constraint satisfies properties (D) and (UP) and if there is no feasible arbitrage, then $A$ is non-empty and the value of a non-negative payoff stream is equal to the highest present value of the payoff stream over the set of state-price systems. The payoff pricing functional is strictly positive and is sublinear on the set of non-negative payoff streams, but it may not be linear unless markets are dynamically complete. The potential non-linearity of valuation in incomplete markets is associated with the possibility that the present value of a payoff stream may differ for different systems of state prices. The following example illustrates this possibility.

**Example 4.1** Consider an event tree with uncertainty only at date $1$. Specifically, the root $s^0$ of the tree has two immediate successors, $\xi^1$ and $\eta^1$, and $\{\xi^t\} = \{\xi^{t+1}\}$ and $\{\eta^t\} = \{\eta^{t+1}\}$ are both singletons for all $t \geq 1$. A single infinitely-lived asset is traded at every date with the following price-dividend processes

\[
\begin{align*}
q(s^0) &= 1, \quad d(s^0) = 0, \\
q(\xi^t) &= x^{-t}, \quad d(\xi^t) = (x - 1)x^{-t}, \quad t \geq 1, \\
q(\eta^t) &= y^{-t}, \quad d(\eta^t) = (y - 1)y^{-t}, \quad t \geq 1,
\end{align*}
\]

for some $x, y \geq 1$. The set of state-price systems is non-empty and is given by

\[
A = \left\{ \left( \frac{a(\xi^t)}{a(s^0)} = \alpha, \quad \frac{a(\eta^t)}{a(s^0)} = 1 - \alpha \right) : 0 < \alpha < 1 \right\}.
\]

Consider the following portfolio constraint

\[
\theta(s^t) \geq 0 \quad \text{for all } s^t.
\]  

(5)

Given $d$ and $q$ specified above, constraint (5) satisfies properties (D) and (UP). Moreover, there exists no feasible arbitrage under (5). Theorem 4.1 implies that the payoff pricing functional $V$ is strictly positive and is sublinear on the set of non-negative payoff streams. We now show that $V$ is not linear. Consider non-negative payoff streams $z_1$, $z_2$ and $z_3$ defined by

\[
z_1(\xi^1) = 1, \quad z_1(\eta^1) = 0, \quad z_1(\xi^t) = z_1(\eta^t) = 0, \quad t \geq 1,
\]
\[ z_2(\xi^1) = 0, \; z_2(\eta^1) = 1, \; z_2(\xi^t) = z_2(\eta^t) = 0, \; t \geq 1, \]
\[ z_3(\xi^1) = 1, \; z_3(\eta^1) = 1, \; z_3(\xi^t) = z_3(\eta^t) = 0, \; t \geq 1, \]

and portfolio strategies \( \theta_1, \theta_2 \) and \( \theta_3 \) defined by

\[
\begin{align*}
\theta_1(s^0) &= 1, \; \theta_1(\xi^t) = 0, \; \theta_1(\eta^t) = y^t, \; t \geq 1, \\
\theta_2(s^0) &= 1, \; \theta_2(\xi^t) = x^t, \; \theta_2(\eta^t) = 0, \; t \geq 1, \\
\theta_3(s^0) &= 1, \; \theta_3(\xi^t) = 0, \; \theta_3(\eta^t) = 0, \; t \geq 1.
\end{align*}
\]

Since \( \theta_1, \theta_2 \) and \( \theta_3 \) all satisfy (5), they are feasible portfolio strategies; and since \( z^{\theta_1} = z_1, \; z^{\theta_2} = z_2 \) and \( z^{\theta_3} = z_3 \), these payoff streams are all in the asset span. Applying Theorem 4.1, we have

\[
\begin{align*}
V(z_1) &= \sup_{0<\alpha<1} \{ \alpha \cdot 1 + (1 - \alpha) \cdot 0 \} = 1, \\
V(z_2) &= \sup_{0<\alpha<1} \{ \alpha \cdot 0 + (1 - \alpha) \cdot 1 \} = 1, \\
V(z_3) &= \sup_{0<\alpha<1} \{ \alpha \cdot 1 + (1 - \alpha) \cdot 1 \} = 1.
\end{align*}
\]

Since \( z_1 + z_2 = z_3 \) and \( V(z_1) + V(z_2) > V(z_3) \), the payoff pricing functional \( V \) is not linear on the set of non-negative payoff streams.

This non-linearity of valuation occurs because markets are not one-period complete at the initial date. The present value of \( z_1 \) evaluated at the state-price system indexed by \( \alpha \) is equal to \( \alpha \) so that there is a continuum of the present value of \( z_1 \) according to different systems of state prices. Similarly, there is a continuum of the present value of \( z_2 \). The sum of the present values of \( z_1 \) and \( z_2 \) is constant and equal to 1 for any system of state prices. Also, in the case with \( x > 1 \) and \( y = 1 \) or with \( x = 1 \) and \( y > 1 \), the present value of the dividend stream of the asset at the initial date calculated using the state-price system indexed by \( \alpha \) is equal to \( \alpha \). Hence there is also a continuum of the present value of the dividend stream according to different systems of state prices. \( \square \)

We turn our attention now to implications of Theorem 4.1 for an equilibrium. Limited feasible arbitrage can exist in equilibria with some of the constraints but
not in equilibria with others. An example of an equilibrium of the former type can be found in Huang and Werner [9, Example 7.1]; in this example, there exist limited feasible Ponzi schemes and the payoff pricing functional is neither positive nor sublinear. On the contrary, in the case of any equilibrium with any of the constraints considered by Hernandez and Santos [8] there exists no feasible arbitrage and, therefore, the payoff pricing functional is strictly positive and sublinear. This class of constraints includes the eventually zero borrowing constraint, the constraint on borrowing against future wealth, the constraint on borrowing against future savings, and the almost finite-time debt-repayment constraint (see also Santos and Woodford [15] for the constraint on borrowing against future wealth).

5 Linear and countably additive valuation

As illustrated by Example 4.1, the payoff pricing functional in Theorem 4.1 may not be linear on the set of non-negative payoff streams. An exception is the case when markets are dynamically complete. The following result extends Theorem 5.1 of Huang and Werner [9] to stochastic settings with complete markets.

**Corollary 5.1** Suppose that markets are dynamically complete and the set $\Theta$ of feasible portfolio strategies satisfies properties (D) and (UP). There is no feasible arbitrage if and only if there is a unique system of state prices $a$, and $V$ is linear and countably additive on the set of non-negative payoff streams and satisfies

$$V(z) = \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t)$$

for every $z \in M, z \geq 0$.

Equation (6) displays valuation according to the present value rule. Thus the payoff pricing functional is linear and countably additive on the set of non-negative payoff streams in dynamically complete markets if there is no feasible arbitrage and if the set of feasible portfolio strategies satisfies properties (D) and (UP).
We present now a transversality constraint under which the payoff pricing functional is linear and countably additive regardless of whether markets are complete or incomplete. Suppose that $A \neq \emptyset$ and consider the following constraint:

For every $s \in D$

$$\lim_{t \to \infty} \sum_{s^t \in D_s \cap N_t} a(s^t) q(s^t)' \theta(s^t) = 0 \quad \text{for every } a \in A.$$ (7)

Constraint (7) requires that the sum of the present value of net worth over each information partition on every subtree be zero at infinity. This type of transversality constraint is studied by Hernandez and Santos [7], Magill and Quinzii [13], and Florenzano and Gourdel [2], and is often considered in the modern finance and macroeconomics literatures.

**Theorem 5.1** Suppose that $A \neq \emptyset$ and the set $\Theta$ of feasible portfolio strategies is given by (7). Then there is no feasible arbitrage, and $V$ is linear and countably additive and satisfies

$$V(z) = \sum_{t=1}^{\infty} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} z(s^t)$$ (8)

for every $z \in M$ and every $a \in A$.

The payoff pricing functional in Theorem 5.1 is linear and countably additive despite possible incompleteness of markets, and the present value of a payoff stream is independent of the system of state prices used in calculating the present value. It is worth noting that the theorem holds for all payoff streams in the asset span, and not as Theorem 4.1 and Corollary 5.1 only for non-negative payoff streams.

In any equilibrium with transversality constraint (7) there exists no feasible arbitrage and Theorem 5.1 applies. Therefore, linear and countably additive valuation holds in equilibrium even if markets are incomplete.

## 6 Asset price bubbles

As in Santos and Woodford [15] and Huang and Werner [9], a price bubble occurs when the price of an asset differs from the present value of its future dividend
stream calculated using a state-price system $a \in A$. For use later, we derive now a relation between an asset’s price bubble at a given date and the price of the asset at infinity. Using (3) recursively we obtain for every asset $j$, every event $s^t$, and every date $r > t$ that

$$q_j(s^t) = \sum_{\tau=t+1}^{r} \sum_{s^\tau \in D_{s^\tau \cap N_r}} \frac{a(s^\tau)}{a(s^t)} d_j(s^\tau) + \sum_{s^r \in D_{s^r \cap N_r}} \frac{a(s^r)}{a(s^t)} q_j(s^r). \quad (9)$$

Assume that the two terms on the right-hand side of (9) have well-defined finite limits as $r$ goes to infinity. Taking $r \to \infty$ in (9) leads to

$$q_j(s^t) = \sum_{s^r \in D_{s^r \setminus \{s^t\}}} \frac{a(s^r)}{a(s^t)} d_j(s^r) + \lim_{r \to \infty} \sum_{s^r \in D_{s^r \cap N_r}} \frac{a(s^r)}{a(s^t)} q_j(s^r). \quad (10)$$

In (10), the difference between the left-hand side (the price of $j$ at $s^t$) and the first term on the right-hand side (the present value of future dividends of $j$ with respect to $a$) is called the price bubble of $j$ at $s^t$ with respect to $a$. Denote it by $\sigma_j(s^t; a)$. Equation (10) implies that

$$\sigma_j(s^t; a) = \lim_{r \to \infty} \sum_{s^r \in D_{s^r \cap N_r}} \frac{a(s^r)}{a(s^t)} q_j(s^r). \quad (11)$$

The form of a constraint is crucial for the nature of equilibrium asset price bubbles. Huang and Werner [9] provide three examples of equilibrium price bubbles in which valuation is linear and countably additive, not sublinear, or linear but not countably additive. We here observe another possibility in light of Theorem 4.1 and Example 4.1. If the net supply of an (infinitely-lived) asset is zero, then there are typically equilibria with price bubbles on that asset under a constraint considered by Hernandez and Santos [8] (see, e.g., Magill and Quinzii [14] and Huang and Werner [9]). Therefore, equilibrium price bubbles may occur when valuation is sublinear but not linear in the case of incomplete markets.

When markets are incomplete, there are more than one system of state prices to be used in calculating the present value of the dividend stream of an asset. Santos and Woodford [15] provide sufficient conditions for there being, in a certain sense, no price bubbles in equilibrium with a constraint that net worth be limited
by a pre-specified non-positive lower bound. If the present value of the aggregate endowment of goods is finite under all state-price systems and if the net supply of an asset is positive, then there exists some system in terms of which there is no price bubble on that asset. Under an assumption of a sufficient degree of impatience about consumers' preferences they prove a stronger result: if the present value of the aggregate endowment is finite for some state-price system and if the net supply of an asset is positive, then there is no price bubble on that asset with respect to the same system.

We present now a result about the non-existence of asset price bubbles in equilibrium with transversality constraint (7), which is not studied by Santos and Woodford [15]. We show that there is no price bubble on an asset in positive net supply regardless of the aggregate endowment of the consumption good, of the system of state prices used in calculating the present value of the dividend stream of the asset, and of the agents' preferences.

**Theorem 6.1** Consider an equilibrium \( \{q, (c^h, \theta^h)_h\} \) with \( q \geq 0 \) \( (q \leq 0) \) in which the portfolio constraints for each agent are of the form (7). Then \( \sigma_j(s^t; a) = 0 \) for every event \( s^t \), every state-price system \( a \in A \), and every asset \( j \) that is in positive net supply (i.e., for which \( \theta_j > 0 \)).

According to Theorem 6.1, in any equilibrium with transversality constraint (7) there is no price bubble on an asset in positive net supply no matter which of the systems of state prices is chosen, since the price of the asset is equal to the present value of its future dividend stream calculated by using any such system. In the case that all assets are in positive net supply there are unambiguously no equilibrium price bubbles even if markets are incomplete.

It is important to emphasize that our result in Theorem 6.1 does not depend on market completeness, endowments of goods, or agents' preferences.

We present now an example of an equilibrium as of Theorem 6.1.

**Example 6.1** Consider an economy with no uncertainty, so that the information structure is described by a sequence of dates, \( t = 0, 1, 2, \ldots \). There is a single
infinitely-lived asset specified by the dividend process \( d_t = 1 - \beta, \ t \geq 0 \), with \( \beta \in (0, 1) \). The economy consists of two infinitely-lived agents, \( h = 1, 2 \), with the same preferences represented by the utility function

\[
U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t),
\]

where \( u() \) is a bounded, strictly increasing, strictly concave, and continuously differentiable function, defined on \( \mathbb{R}_+ \). Agent 1's initial endowment of the asset is \( \bar{\theta}^1 = 1 \), while agent 2's initial endowment is \( \bar{\theta}^2 = 0 \). The net supply of the asset is therefore \( \bar{\theta} = 1 \). Endowments of the consumption good are given by

\[
y_t^1 = \gamma - 1, \quad y_t^2 = \gamma + \beta, \ t = 0, 2, 4, \ldots,
\]

\[
y_t^1 = \gamma + \beta, \quad y_t^2 = \gamma - 1, \ t = 1, 3, 5, \ldots,
\]

where \( \gamma \geq 1 \). Consider the following price process of the asset

\[
q_t = \beta, \ t \geq 0. \tag{12}
\]

Under price process (12) there exists a unique system of state prices given by

\[
a_t = \beta^t, \ t \geq 0. \tag{13}
\]

The following processes of consumption and asset holdings and price process (12) describe an equilibrium for this economy in which the portfolio constraints for each agent are of the form (7)

\[
c_t^1 = c_t^2 = \gamma, \ t \geq 0,
\]

\[
\theta_t^1 = 0, \ t = 0, 2, 4, \ldots,
\]

\[
\theta_t^1 = 1, \ t = 1, 3, 5, \ldots,
\]

\[
\theta_t^2 = 1 - \theta_t^1, \ t \geq 0.
\]

To verify that these prices and allocations are indeed an equilibrium note that markets for good and assets clear, and the budget constraints and transversality
condition (7) are both satisfied. It remains to show that the consumption and asset holding plans are optimal for the agents. One can verify that

$$\beta^t u'(c^h_t) q_t = \beta^{t+1} u'(c^h_{t+1})(q_{t+1} + d_{t+1})$$

holds for all $t \geq 0$ and for $h = 1, 2$. Therefore, it is suboptimal for an agent to increase his consumption at one date by reducing the consumption at another date. Neither can the agent increase his consumption at some date without reducing the consumption at any other date. To see that this latter statement is true, consider the strategy of reducing the agent’s asset holdings by $\epsilon$ shares at some date $t$ to finance his extra consumption at that date. To avoid reduction in his future consumption while at the same time satisfy the budget constraints, the agent has to reduce his asset holdings by at least $\epsilon \beta^{-\tau}$ shares at date $t+\tau$, for every $\tau \geq 1$. This strategy, however, violates transversality condition (7).

Using state-price system (13) we can calculate at any date $t$ the present value of the future dividend stream of the asset given by

$$\sum_{\tau=t+1}^{\infty} \frac{a_{\tau}}{a_t} d_{\tau} = (1 - \beta) \sum_{\tau=1}^{\infty} \beta^\tau = \beta,$$

which is equal to $q_t$. Thus there is no price bubble on the asset in this equilibrium. $\square$

It is worth remarking that even when the conditions of Theorem 6.1 are satisfied, equilibrium price bubble may not be excluded on an asset in zero net supply. To see this possibility, note that Example 4.3 of Santos and Woodford [15] can be recast as an example of an equilibrium under transversality constraint (7) with the same prices and allocations of consumption and asset holdings. In this example, the conditions of Theorem 6.1 are satisfied and there exists no feasible arbitrage, but there is a price bubble on a perpetual bond that is in zero net supply. Here the price bubble occurs with linear and countably additive valuation.
7 Extensions

In this paper we have provided a systematic analysis of the issue of valuation of payoff streams in an equilibrium model of sequential asset trading over an infinite horizon under various portfolio constraints and possible incompleteness of markets. We investigate the nature of equilibrium asset price bubbles in light of this analysis. We show that there cannot be price bubbles on assets in positive net supply in any equilibrium under a transversality restriction.

To help exposition, we have restricted our analysis to a simple model with a single consumption good, a finite number of repeatedly-traded infinitely-lived assets, and a finite number of infinitely-lived agents. In this section we discuss the extensions of our results along these directions.

All of our results extend to economies with multiple consumption goods. Observe that with multiple goods completeness or incompleteness of markets is still a property of a particular equilibrium, but it is now dependent on both the prices of assets and of goods (see, e.g., Santos and Woodford [15]).

Our results also extend to economies with more sophisticated asset structures. We can allow the issuance of new assets and the liquidation of existing ones at any date to allow for infinitely many assets in the economy while maintaining the assumption that only finitely many assets are traded at each given date (see, e.g., Santos and Woodford [15]). All of our results stand up to such a generalization. Moreover, we can even allow for the issuance or retirement of existing assets at any date so as to have a time-varying net supply of assets. Our theory of valuation of Sections 4 and 5 in general, and Theorems 4.1 and 5.1 in particular, do not depend on the supply of assets and continue to hold under circumstances like those. Theorem 6.1 applies to all assets in asymptotically positive net supply under such circumstances.

Finally, neither our theory of valuation of Sections 4 and 5 nor the main results, Theorems 4.1, 5.1 and 6.1, depend on the number of agents. Therefore, they apply to economies with infinitely many agents such as overlapping generations
economies (see Huang and Werner [9] for alternative specifications of the set of feasible portfolio strategies in overlapping generations economies).

8 Appendix

Proof of Theorem 4.1: If \( A \neq \emptyset \) and equation (4) holds for every \( z \in M, z \geq 0 \), then the payoff pricing functional \( V \) is strictly positive. The absence of feasible arbitrage follows from strict positivity of \( V \).

Assume now that there exists no feasible arbitrage. Thus there is no feasible one-period arbitrage in particular. By virtue of the Farkas-Stiemke lemma (see, e.g., LeRoy and Werner [11, p.58]), we have \( A \neq \emptyset \) since \( \Theta \) satisfies property (UP).

To prove (4), fix \( z \in M, z \geq 0 \). We first show that

\[ q(s^t)'\theta(s^t) \geq 0 \]  

(14)

for every \( s^t \), and for every feasible portfolio strategy \( \theta \) that generates \( z \). Suppose, by contradiction, that \( q(s^t)'\theta(s^t) < 0 \) for some \( s^t \). If \( t = 0 \), then \( \theta \) is a feasible arbitrage which contradicts the assumption that there exists no feasible arbitrage. Suppose that \( t > 0 \) and consider a portfolio strategy \( \tilde{\theta} \) given by \( \tilde{\theta}(s^\tau) = \theta(s^\tau) \) for \( s^\tau \in D_{s^t} \) and \( \tilde{\theta}(s^\tau) = 0 \) elsewhere. Since \( \Theta \) satisfies property (D), \( \tilde{\theta} \) is a feasible portfolio strategy. Its payoff is equal to \( -q(s^t)'\theta(s^t) \) at \( s^t \), is equal to \( z \) on \( D_{s^t} \backslash \{s^t\} \), and is equal to zero elsewhere. Further, its date-0 price is zero. Since \( z \geq 0 \) and \( -q(s^t)'\theta(s^t) > 0 \), \( \tilde{\theta} \) is a feasible arbitrage. A contradiction.

We now use (14) to show that

\[ V(z) \geq \sup_{a \in A} \sum_{s^t \in D_{s^t}} \frac{a(s^t)}{a(s^0)} z(s^t). \]  

(15)

Let \( \theta \) be a feasible portfolio strategy with \( z^\theta = z \), and choose an arbitrary \( a \in A \). Multiplying both sides of (1) by \( a(s^t) \), summing over all \( s^t \) from \( t = 1 \) to some \( \tau \geq 1 \), and using (3), we obtain

\[ (s^0)'\theta(s^0) = \sum_{t=1}^{\tau} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} q(s^t) + \sum_{s^\tau \in N_\tau} \frac{a(s^\tau)}{a(s^0)} q(s^\tau)'\theta(s^\tau) \geq \sum_{t=1}^{\tau} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} z(s^t), \]
where the inequality follows from (14). Taking \( \tau \to \infty \) on the right-hand side of the inequality leads to

\[
q(s^0)'\theta(s^0) \geq \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).
\]

In the above inequality, taking supremum on the right-hand side over all state-price system \( a \in A \), and taking infimum on the left-hand side over all feasible portfolio strategy that generates \( z \), we obtain (15).

We next use (14) and a stochastic duality technique to show that

\[
V(z) \leq \sup_{a \in A} \sum_{s^t \in D \setminus \{s^0\}} \frac{a(s^t)}{a(s^0)} z(s^t).
\] (16)

Note that the right-hand side of (16) is finite, as demonstrated by the analysis in the previous paragraph. Consider for each \( s^t \) the following minimization problem

\[
\min_{\theta(s^t)} \quad q(s^t)'\theta(s^t)
\]

s.t. \( [q(s^{t+1}) + d(s^{t+1})]'\theta(s^t) \geq \sup_{a \in A} \sum_{s^\tau \in D_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau), \ s^{t+1} \in \{s^t\}, \)

and its dual problem

\[
\max_{\{a(s^{t+1})\}} \sum_{s^{t+1} \in \{s^t\}} \left[ \alpha(s^{t+1}) \sup_{a \in A} \sum_{s^\tau \in D_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau) \right]
\]

s.t. \( \sum_{s^{t+1} \in \{s^t\}} \alpha(s^{t+1})[q(s^{t+1}) + d(s^{t+1})] = q(s^t), \ a(s^{t+1}) \geq 0, \ s^{t+1} \in \{s^t\}. \)

The constraint in the dual problem has a feasible solution since \( A \neq \emptyset \). Using (14) we can show, similarly as in proving (15) above, that any feasible portfolio strategy \( \theta \) that generates \( z \) induces a portfolio \( \theta(s^t) \) at \( s^t \) which is a feasible solution of the constraint in the primal problem. By the duality theory of linear programming, both the primal problem and the dual problem have finite optimal solutions and the values of their optimal objectives are equal. Since \( A \neq \emptyset \) and the objective in the dual problem is continuous in \( \alpha(s^{t+1}) \), the dual problem can be rewritten as

\[
\sup_{\{a(s^{t+1})\}} \sum_{s^{t+1} \in \{s^t\}} \left[ \alpha(s^{t+1}) \sup_{a \in A} \sum_{s^\tau \in D_{s^{t+1}}} \frac{a(s^\tau)}{a(s^{t+1})} z(s^\tau) \right]
\]
The value of the optimal objective in the above program is equal to

$$
\sup_{s^t+1 \in \{s^t+1\}} \left[ \frac{a(s^t+1)}{a(s^t)} \sum_{s^t \in D_s \setminus \{s^t\}} \frac{a(s^t)}{a(s^t+1)} z(s^t) \right],
$$

where the outer supremum is taken over all state prices \(\{a(s^t+1)/a(s^t)\}\) consistent with (3). By a dynamic programming argument, the above value is equal to

$$
\sup_{a \in A} \sum_{s^t \in D_s \setminus \{s^t\}} \frac{a(s^t)}{a(s^t)} z(s^t).
$$

Therefore, there exists a portfolio strategy \(\theta\) such that

$$
q(s^t)' \theta(s^t) = \sup_{a \in A} \sum_{s^t \in D_s \setminus \{s^t\}} \frac{a(s^t)}{a(s^t)} z(s^t) \quad \text{for all } s^t, \tag{17}
$$

$$
[q(s^t) + d(s^t)]' \theta(s^t) \geq \sup_{a \in A} \sum_{s^t \in D_s \setminus \{s^t\}} \frac{a(s^t)}{a(s^t)} z(s^t) \geq 0 \quad \text{for all } s^t \neq s^0, \tag{18}
$$

where the second inequality in (18) holds since \(z \geq 0\). Note that (17) in particular implies that the initial price of \(\theta\) is equal to

$$
q(s^0)' \theta(s^0) = \sup_{a \in A} \sum_{s^t \in D_s \setminus \{s^0\}} \frac{a(s^0)}{a(s^t)} z(s^t), \tag{19}
$$

and (17) and (18) together imply that the payoff stream of \(\theta\) is given by

$$
z^\theta(s^t) = [q(s^t) + d(s^t)]' \theta(s^t) - q(s^t)' \theta(s^t) \geq z(s^t) \quad \text{for all } s^t \neq s^0. \tag{20}
$$

Since there is always some way of carrying wealth into the future, (18), (19) and (20) imply that it is possible to modify \(\theta\) to obtain a portfolio strategy \(\hat{\theta}\) such that

$$
\hat{\theta}(s^0) = \theta(s^0), \quad z^\hat{\theta} = z, \tag{21}
$$

$$
[q(s^t) + d(s^t)]' \hat{\theta}(s^t) \geq [q(s^t) + d(s^t)]' \theta(s^t) \geq 0 \quad \text{for all } s^t \neq s^0. \tag{22}
$$

In light of (22), \(\hat{\theta}\) is a feasible portfolio strategy since \(\Theta\) satisfies property (UP). Equations (19) and (21) then establish (16). Combining (15) and (16) gives (4).
Since (4) holds for every $z \in M$, $z \geq 0$, $V$ is strictly positive and is sublinear on the set of non-negative payoff streams. This completes the proof. □

Proof of Corollary 5.1: Note that the completeness of markets and the non-emptiness of $A$ imply that $A$ is a singleton. The proof then follows directly from Theorem 4.1. □

Proof of Theorem 5.1: We first prove (8). Fix $z \in M$ and $a \in A$. Let $\theta$ be a feasible portfolio strategy with $z^0 = z$. Multiplying both sides of (1) by $a(s^t)$ and summing over all $s^t$ from $t = 1$ to some $\tau \geq 1$, and using (3), we obtain

$$\sum_{t=1}^{\tau} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} z(s^t) = q(s^0)\theta(s^0) - \sum_{s^\tau \in N_\tau} \frac{a(s^\tau)}{a(s^0)} q(s^\tau)\theta(s^\tau).$$

(23)

Given that $\theta$ satisfies (7), the two terms on the right-hand side of (23) have well-defined finite limits as $\tau$ goes to infinity and, therefore, so does the term on the left-hand side. Taking $\tau \to \infty$ on both sides of (23) leads to

$$\sum_{t=1}^{\infty} \sum_{s^t \in N_t} \frac{a(s^t)}{a(s^0)} z(s^t) = q(s^0)\theta(s^0) - \lim_{\tau \to \infty} \sum_{s^\tau \in N_\tau} \frac{a(s^\tau)}{a(s^0)} q(s^\tau)\theta(s^\tau) = q(s^0)\theta(s^0),$$

(24)

where the second equality in (24) holds since $\theta$ satisfies (7). Taking infimum on the right-most side of (24) over all feasible portfolio strategy $\theta$ that generates $z$, we obtain equation (8). Since (8) holds for every $z \in M$ and every $a \in A$, $V$ is strictly positive, linear, and countably additive. The absence of feasible arbitrage follows from strict positivity of $V$. □

Proof of Theorem 6.1: Fix $s^t \in D$ and $a \in A$. Since for each agent $h$ the portfolio holding plan $\theta^h$ is feasible under (7), we have

$$\lim_{\tau \to \infty} \sum_{s^\tau \in D_s \cap N_t} a(s^\tau)q(s^\tau)^t\theta^h(s^\tau) = 0.$$ 

(25)

Summing the right-hand side of (25) over all agent $h$, and using the market clearing condition for assets ($\sum_h \theta^h(s^\tau) = \bar{\theta}$ for every $s^\tau$), we obtain

$$\lim_{\tau \to \infty} \sum_{s^\tau \in D_s \cap N_t} a(s^\tau)q(s^\tau)^t \bar{\theta} = 0.$$ 

(26)
Given that \( a > 0, \ q \geq 0 \ (q \leq 0) \), and \( \bar{\theta} \geq 0 \), (26) implies that

\[
\bar{\theta}_j \left[ \lim_{r \to \infty} \sum_{s' \in D_{st} \cap N_s} a(s') q_j(s') \right] = 0 \quad \text{for every asset } j.
\]  

(27)

Substituting (11) into (27), and using the fact that \( a(s^t) > 0 \), we get

\[
\bar{\theta}_j \sigma_j(s^t; a) = 0 \quad \text{for every asset } j.
\]

(28)

Equation (28) implies that \( \sigma_j(s^t; a) = 0 \) for every asset \( j \) for which \( \bar{\theta}_j > 0 \). This completes the proof. \( \square \)
References


Valuation and asset pricing in infinite-horizon sequential markets with portfolio constraints*

Kevin X.D. Huang

Department of Economics, Utah State University, 3530 Old Main Hill, Logan, UT 84322, USA (e-mail: khuang@b202.usu.edu)

February 2000

Summary. We develop a theory of valuation of payoff streams in infinite-horizon sequential markets and discuss implications of this theory for equilibrium under various portfolio constraints. We study the nature of asset price bubbles in light of this theory. We show that there cannot be equilibrium price bubbles on assets in positive net supply under a transversality restriction. Our analysis extends the work by Huang and Werner [9] to stochastic settings with complete or incomplete markets.

Keywords and Phrases: Valuation, Asset price bubble, Portfolio

JEL Classification Numbers: C61, D50, G10, G12.

* I am very grateful to Steve LeRoy, Manuel Santos, and Jan Werner for valuable advice and helpful comments. I have also benefited from comments and suggestions of Jerome Detemple, Ed Green, Narayana Kocherlakota, Ket Richter, and seminar participants at Minnesota, McGill, Midwest Economic Theory Meetings (Lansing), and Econometric Society Meetings (New York). The usual disclaimer applies.
1 Introduction

This paper is concerned with the issue of payoff valuation and asset pricing in an equilibrium model of sequential trading over an infinite horizon under constraints on portfolio strategies.

In the literature on sequential markets with portfolio constraints, the value of a stream of payoffs is usually measured by a functional that maps each payoff stream to the lowest price of a feasible portfolio strategy that generates it. Two important properties that the payoff pricing functional may or may not have are linearity and countable additivity. Linearity implies that valuation is additive, that is, the value of the sum of two payoff streams is equal to the sum of their values. Value additivity plays a fundamental role in the theory of financial markets. The well-known Modigliani-Miller theorem is one of its implications. Countable additivity implies a representation of the value of a stream of payoffs by an infinite sum of values of the payoff at each date.

Huang and Werner [9] demonstrate that, in the case of a single asset and no uncertainty, the payoff pricing functional is linear and countably additive on the set of non-negative payoff streams if a constraint allows delayed market participation and does not restrict long positions in the asset and if there is no feasible arbitrage.

An arbitrage is a portfolio strategy that generates a positive payoff stream at a non-positive price or a non-negative payoff stream at a negative price, of which a Ponzi scheme is an example. A Ponzi scheme is a strategy of rolling over a debt forever and thereby never paying it back. LeRoy and Werner [11, p.33-37] distinguish two types of feasible arbitrage: unlimited feasible arbitrage and limited feasible arbitrage. An unlimited feasible arbitrage is a feasible arbitrage that can be operated on an arbitrary scale and generally cannot exist in equilibrium. A limited feasible arbitrage is a feasible arbitrage that can only be operated on a finite scale and can potentially exist in equilibrium depending on the form of a constraint.

In this paper we extend the work by Huang and Werner [9] to stochastic settings