DARBOUX INTEGRABILITY OF WAVE MAPS INTO 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

by

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ABSTRACT

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The harmonic map equations can be represented geometrically as an exterior differential system (EDS), $\mathcal{E}$. Using this representation we study the harmonic maps from 2D Minkowski space into 2D Riemannian manifolds. These are also known as wave maps. In this case, $\mathcal{E}$ is invariant under conformal transformations of Minkowski space. The quotient of $\mathcal{E}$ by these conformal transformations, $\bar{\mathcal{E}}$, is an $s = 0$ hyperbolic system.

The main result of our study is that the prolonged EDS, $\mathcal{E}^{(k)}$, is Darboux integrable if and only if the prolonged quotient EDS, $\bar{\mathcal{E}}^{(k+1)}$, is Darboux integrable. We also find invariants determining the Darboux integrability of both systems. Analyzing these invariants leads to three additional results. First, Darboux integrability of $\mathcal{E}$, without prolongation, requires that the range manifold have zero scalar curvature. Second, after one prolongation there are two inequivalent metrics for which $\mathcal{E}^{(1)}$ is Darboux integrable. Third, prolonging to $\mathcal{E}^{(2)}$ does not provide any further metrics with Darboux integrable wave maps.

(138 pages)

This thesis is dedicated to Jodie, for keeping me sane.

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CHAPTER 1

INTRODUCTION

1.1 Harmonic Maps

A harmonic map, $f : P \to Q$, between two pseudo-Riemannian manifolds, P and Q, with metrics g and h, is stationary for the "energy" Lagrangian, which, in coordinates, is

$$\Lambda = \frac{1}{2} g^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta} \sqrt{|\det g|} dx^1 \wedge \cdots \wedge dx^n. \tag{1.1}$$

This Lagrangian is a generalization of the free particle Lagrangian from classical mechanics,

$$\Lambda = \frac{1}{2} \left(\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right) dt. \tag{1.2}$$

The Lagrangian in equation (1.2) corresponds to the case $P = \mathbb{R}$, $Q = \mathbb{R}^3$, $g = dt^2$, and $h = dx^2 + dy^2 + dz^2$ in equation (1.1). In the Lagrangian formulation of mechanics, particles minimize the functional induced by Λ in equation (1.2). Therefore the path of a classical free particle is a harmonic map of the real line into 3D Euclidean space.

In differential geometry one of the main interests has been in harmonic maps between Riemannian manifolds. Examples of these include harmonic functions and minimal surfaces, while in the paper by Eells and Sampson [6], the gradient flow is used to find a harmonic representative of each homotopy class. This paper by Eells and Sampson is considered by many to be the beginning of the modern study of harmonic maps.

More interesting to theoretical physicists are harmonic maps from Minkowski space, these are referred to as wave maps by mathematicians, and as sigma models by theoretical physicists. These sigma models were introduced to describe low energy pion-nucleon interactions [8]. These maps are well known to be completely integrable when Q is a symmetric space [13], and to what extent they are Darboux integrable is a natural question. Further motivation for studying the Darboux integrability of these maps is provided by the paucity of examples of Darboux integrable EDS [2].

In our study we let ν be the natural volume form on P, induced by the metric g. We give the coordinate free formulation of the functional in equation (1.1)

$$\Lambda = \mathrm{tr}_g(f^* h)\nu. \tag{1.3}$$

For any coordinate chart this is exactly the Lagrangian above. Therefore we have the starting point for the formulation of a global EDS whose integral manifolds are graphs of maps stationary for this Lagrangian.

1.2 Thesis Structure

The main result of this thesis is to characterize Darboux integrability of the EDS for harmonic maps, $\mathcal{E}$, in terms of the Darboux integrability of its quotient system, $\bar{\mathcal{E}}$.

In chapter 2, we present the definitions necessary for our study and find the EDS, $\mathcal{E}$, for the harmonic map equations.

In chapter 3, we consider the case of wave maps into 2D Riemannian manifolds. We quotient our EDS, $\mathcal{E}$, by the conformal group and obtain an $s = 0$ hyperbolic system, $\bar{\mathcal{E}}$. Darboux integrability of $\mathcal{E}$ is given in terms of necessary and sufficient conditions on the quotient system, $\bar{\mathcal{E}}$. These conditions require that the wave maps have range manifolds (Q, h) with zero scalar curvature.

In chapter 4, we consider Darboux integrability of the prolonged system $\mathcal{E}^{(1)}$. In a manner analogous to chapter 3, we find necessary and sufficient conditions for Darboux integrability of $\mathcal{E}^{(1)}$ in terms of the curvature of the range manifold. These conditions are integrated to find the metrics for which $\mathcal{E}^{(1)}$ is Darboux integrable. These are

$$\frac{1}{1 \pm e^w}(dw^2 + dz^2). \tag{1.4}$$

In chapter 5, we find a coframe and structure equations for $\mathcal{E}^{(k)}$, the k^{th} prolongation of $\mathcal{E}$, adapted to the action of the conformal group. This coframe is used to find an invariant, denoted by F_k, which determines the Darboux integrability of $\mathcal{E}^{(k)}$ and its quotient, $\bar{\mathcal{E}}^{(k)}$. We give the following theorem to be proved in section 5.3.

Theorem 5.3.11: The non-negative integer k is the smallest integer such that $\mathcal{E}^{(k)}$ is Darboux integrable if and only if k is the smallest integer such that $F_{k+2} = 0$.

We compare this to the results in [3]. Then, in section 5.4, we prove the main result of the thesis.

Theorem 5.4.1: The non-negative integer k is the smallest integer such that $\mathcal{E}^{(k)}$ is Darboux integrable if and only if k is the smallest integer such that $\bar{\mathcal{E}}^{(k+1)}$ is Darboux integrable.

In chapter 6, the invariant from chapter 5 is used to reproduce the results of chapters 3 and 4. It is then shown that the second prolongation, $\mathcal{E}^{(2)}$, does not provide any new metrics of Q with Darboux integrable wave maps.

CHAPTER 2
PRELIMINARIES

2.1 Exterior Differential Systems

Here we present parts of the theory of exterior differential systems which will be needed for this thesis. Further details can be found in [4].

Definition 2.1. *An* ***exterior differential system****, $\mathcal{I}$, on a manifold, M, is an ideal, $\mathcal{I} \subset \Omega^*(M)$, which is closed with respect to exterior differentiation. Where $\Omega^i(M) = C^\infty$ sections of $\bigwedge^i T^*M$, and $\Omega^*(M) = \bigoplus \Omega^i(M)$.*
The ***Cauchy characteristic system****, $\mathcal{C} \subset TM$, of $\mathcal{I}$ is*

$$\mathcal{C}_x = \{X \in T_xM | X \lrcorner \mathcal{I}_x \subset \mathcal{I}_x\}. \tag{2.1}$$

An ***independence condition****, $\Omega \in \Omega^p(M)$ is a decomposable section of $\bigwedge^p \mathcal{C}^\perp$.*

An integral manifold of this system is an immersion $f : N \to M$ such that $f^*\mathcal{I} = 0$ and $f^*\Omega \neq 0$. Locally, these are the solutions to the partial differential equation that $\mathcal{I}$ encodes.

Next we define the prolongation of $\mathcal{I}$ with independence condition, Ω. This construction is equivalent to taking all the partial derivatives of our existing system of equations.

Definition 2.2. *Let $\mathcal{I}, \Omega$ be as in definition (2.1). Then the* ***prolonged manifold*** *is*

$$M^{(1)} = G_p(\mathcal{I}) = \{(x, E) | E \subset T_xM, \dim(E) = p, \mathcal{I}|_E = 0, \Omega(E) \neq 0\}.$$

*If $\pi : M^{(1)} \to M$ is the projection $\pi(x, E) = x$, then the **prolonged EDS** is differentially generated by*

$$\mathcal{I}^{(1)}_{(x,E)} = \langle \pi^*(E^{\perp}) \rangle. \tag{2.2}$$

The independence condition becomes

$$\Omega^{(1)} = \pi^*\Omega. \tag{2.3}$$

The successive prolongations are then defined inductively with $\mathcal{I}^{(0)} = \mathcal{I}$ and $\mathcal{I}^{(k+1)} = (\mathcal{I}^{(k)})^{(1)}$. These systems all satisfy definition (2.1). Furthermore, $\mathcal{I}^{(1)}$ is always Pfaffian, and it can be shown that the integral manifolds of $\mathcal{I}$ and $\mathcal{I}^{(1)}$ are locally in one-to-one correspondence.

For this thesis we are interested in EDS with a special property known as Darboux integrability which will be defined below. In order to define Darboux integrability, we first define decomposability for EDS, see [2].

Definition 2.3. *An EDS $\mathcal{I}$, with characteristic system $\mathcal{C}$ on M is called **decomposable** of type $[p, q]$, where $p, q \geq 2$, if there exists a coframe*

$$\tilde{\theta}^1, \ldots, \tilde{\theta}^r, \hat{\sigma}^1, \ldots, \hat{\sigma}^p, \check{\sigma}^1, \ldots, \check{\sigma}^q \tag{2.4}$$

of $\mathcal{C}^{\perp}$ such that $\mathcal{I}$ is algebraically generated by 1-forms and 2-forms,

$$\mathcal{I} = \{\tilde{\theta}^1, \ldots, \tilde{\theta}^r, \hat{\Omega}^1, \ldots, \hat{\Omega}^s, \check{\Omega}^1, \ldots, \check{\Omega}^t\}, \tag{2.5}$$

where $s, t \geq 1$, $\hat{\Omega}^a \in \Omega^2(\hat{\sigma}^1, \ldots, \hat{\sigma}^p)$, and $\check{\Omega}^\alpha \in \Omega^2(\check{\sigma}^1, \ldots, \check{\sigma}^q)$. The differential systems algebraically generated by

(2.6)

$$\hat{\mathcal{V}} = \{\tilde{\theta}^1, \ldots, \tilde{\theta}^r, \hat{\sigma}^1, \ldots, \hat{\sigma}^p, \hat{\Omega}^1, \ldots, \hat{\Omega}^s\} \quad \text{and} \quad \check{\mathcal{V}} = \{\tilde{\theta}^1, \ldots, \tilde{\theta}^r, \check{\sigma}^1, \ldots, \check{\sigma}^q, \check{\Omega}^1, \ldots, \check{\Omega}^t\}$$

*are called the **associated singular differential systems** for $\mathcal{I}$ with respect to the decomposition (2.4).*

When the associated singular differential systems for $\mathcal{I}$ given in equation (2.6) are Pfaffian we give the following definition of Darboux integrability.

Definition 2.4. *An EDS, $\mathcal{I}$, on a manifold, M, with characteristic system $\mathcal{C}$ and associated singular differential systems $\hat{\mathcal{V}}, \check{\mathcal{V}}$ is called **Darboux integrable** if*

$$\hat{V}^\infty + \check{V} = \hat{V} + \check{V}^\infty = \mathcal{C}^\perp \tag{2.7}$$

and

$$\hat{V}^\infty \cap \check{V}^\infty = 0 \tag{2.8}$$

where $\hat{V}^\infty, \check{V}^\infty$ are the largest integrable subsystems of $\hat{V}, \check{V}$ respectively.

In our analysis of the Darboux integrability of the harmonic map equations we will use the theory of quotients of EDS [1]. We now give the appropriate definitions.

Definition 2.5. *Let $\mathcal{I}$ be an EDS on the manifold, M. A diffeomorphism $\phi : M \to M$ is a **symmetry** of $\mathcal{I}$ if $\phi^*\mathcal{I} \subset \mathcal{I}$. A **symmetry group** of $\mathcal{I}$, is a subgroup $G \subset$Diff(M) where each $g \in G$ is a symmetry of $\mathcal{I}$.*

We call the action of G on M regular if the quotient map $\mathfrak{q} : M \to M/G$ is a smooth submersion.

Definition 2.6. *For a symmetry group G of $\mathcal{I}$ acting regularly on the manifold M, the* ***quotient exterior differential system*** $\bar{\mathcal{I}} \subset \Omega^* (M/G)$ *is*

$$\bar{\mathcal{I}} = \{\bar{\theta} \in \Omega^* (M/G) \,|\mathfrak{q}^*\bar{\theta} \in \mathcal{I}\}. \tag{2.9}$$

2.2 The EDS for Harmonic Map from $\mathbb{R}^n$ to $\mathbb{R}^s$

The standard method of creating an EDS from a partial differential equation can be applied in the following way. The harmonic map equations in coordinates are

$$g^{ij}(y^\alpha_{ij} - \mathrm{L}^k_{ij} y^\alpha_k + y^\beta_i y^\gamma_j \Gamma^\alpha_{\beta\gamma}) = 0, \tag{2.10}$$

where L and Γ are the Christoffel symbols for g and h respectively. These s equations define a submanifold of $J^2(\mathbb{R}^n, \mathbb{R}^s)$ of dimension $n+s\begin{pmatrix} n+2 \\ 2 \end{pmatrix} - s$. The restriction of the contact ideal to this submanifold will give an EDS whose integral manifolds are graphs of harmonic maps.

The main drawback of this approach is that for a map between general manifolds P and Q, the equation (2.10) is only defined in a coordinate patch. In [5], a globally defined EDS for harmonic maps of Riemannian manifolds is given. A further advantage of this approach is that it works on the 1-jets, modulo the Cauchy characteristic for the EDS. With the modifications presented here this method can be applied to pseudo-Riemannian manifolds as well.

2.3 The Contact System on Hom(TP, TQ)

We wish to show that M =Hom(TP, TQ) can be identified with $J^1(P, Q)$ by finding the appropriate multi-contact system. Let the T_yQ-valued 1-form on Hom(TP, TQ),

$\tilde{\theta}$, be defined by

$$\tilde{\theta}(X) = (\pi_Q^M)_*(X) - L((\pi_P^M)_*(X)). \tag{2.11}$$

Working on the coordinate chart of $\mathrm{Hom}(TP, TQ)$,

$$U = \{(x, y, L) | x = (x^i), y = (y^\alpha), L(\partial_{x^i}) = y_i^\alpha \partial_{y^\alpha}\}, \tag{2.12}$$

let $X \in T_{(x,y,L)}M$ be given in coordinates by

$$X = X^i \partial_{x^i} + X^\alpha \partial_{y^\alpha} + X_i^\alpha \partial_{y_i^\alpha}. \tag{2.13}$$

Using equation (2.11), $(\pi_P^M)_*(X) = X^i \partial_{x^i}$, and $(\pi_Q^M)_*(X) = X^\alpha \partial_{y^\alpha}$ we get

$$\tilde{\theta}(X) = X^\alpha \partial_{y^\alpha} - L(X^i \partial_{x^i}) = (X^\alpha - y_i^\alpha X^i)\partial_{y^\alpha}, \tag{2.14}$$

thus

$$\tilde{\theta} = (dy^\alpha - y_i^\alpha dx^i) \otimes \partial_{y^\alpha}. \tag{2.15}$$

This identifies $\mathrm{Hom}(TP, TQ)$ with the jet bundle $J^1(P, Q)$, where $\tilde{\theta}$ defines the contact structure on $\mathrm{Hom}(TP, TQ)$.

2.4 Lifting to the Frame Bundle

For $G = SO(n_1, n_2) \times O(s_1, s_2)$, $\mathcal{F}(P) \times \mathcal{F}(Q)$ is a principal G-bundle with base space $P \times Q$. Let G act on the product space

$$\mathcal{F} := \mathcal{F}(P) \times \mathcal{F}(Q) \times M_{n \times s}(\mathbb{R}) \tag{2.16}$$

on the right by

$$(x, u, y, v, p) * (a, b) = (x, ua, y, vb, b^{-1}p\,a). \tag{2.17}$$

Then the associated bundle, $M = \mathcal{F}(P) \times \mathcal{F}(Q) \times_G M_{n\times s}(\mathbb{R}) = \mathrm{Hom}(TP, TQ)$, has projection map $\pi^{\mathcal{F}}_M : \mathcal{F} \to M$

$$\pi^{\mathcal{F}}_M(x, u, y, v, p) = (x, y, vpu^{-1}). \tag{2.18}$$

In this section we will define a geometric coframe of $\mathcal{F}$, and using this coframe lift the contact structure of M to $\mathcal{F}$. We begin by lifting the canonical forms, connection forms, and curvature forms from the frame bundles, $\mathcal{F}(P)$ and $\mathcal{F}(Q)$, we get

$$\begin{aligned}
\omega &= \left(\pi^{\mathcal{F}}_{\mathcal{F}(P)}\right)^* \tilde{\omega} \\
\phi &= \left(\pi^{\mathcal{F}}_{\mathcal{F}(Q)}\right)^* \tilde{\phi} \\
\boldsymbol{\varpi} &= \left(\pi^{\mathcal{F}}_{\mathcal{F}(P)}\right)^* \tilde{\boldsymbol{\varpi}} \\
\boldsymbol{\varphi} &= \left(\pi^{\mathcal{F}}_{\mathcal{F}(Q)}\right)^* \tilde{\boldsymbol{\varphi}} \\
\boldsymbol{\Omega} &= \left(\pi^{\mathcal{F}}_{\mathcal{F}(P)}\right)^* \tilde{\boldsymbol{\Omega}} \\
\boldsymbol{\Phi} &= \left(\pi^{\mathcal{F}}_{\mathcal{F}(Q)}\right)^* \tilde{\boldsymbol{\Phi}}.
\end{aligned} \tag{2.19}$$

Lifting the structure equations (A.10) and (A.19) to $\mathcal{F}$ we get

$$\begin{aligned}
d\omega^i &= -\omega^i_j \wedge \omega^j & d\phi^\alpha &= -\phi^\alpha_\beta \wedge \phi^\beta \\
d\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \Omega^i_j & d\phi^\alpha_\beta &= -\phi^\alpha_\gamma \wedge \phi^\gamma_\beta + \Phi^\alpha_\beta,
\end{aligned} \tag{2.20}$$

for $\omega = e_i\omega^i$, $\phi = e_\alpha\phi^\alpha$,$\boldsymbol{\varpi} = \omega^i_j E^j_i$, $\boldsymbol{\varphi} = \phi^\alpha_\beta E^\beta_\alpha$, $\boldsymbol{\Omega} = \Omega^i_j E^j_i$, and $\boldsymbol{\Phi} = \Phi^\alpha_\beta E^\beta_\alpha$. Where e_i, e_α, E^j_i, and E^β_α, are the standard bases of $\mathbb{R}^n$, $\mathbb{R}^s$, $\mathfrak{gl}(n)$, and $\mathfrak{gl}(s)$ respectively.

We can complete the forms $\omega^i, \phi^\alpha, \omega^i_j, \phi^\alpha_\beta$ to a coframe of $\mathcal{F}$ with the forms dp^α_i, where p^α_i are the standard coordinates on $M_{n\times s}(\mathbb{R})$. However a more geometric coframe exists. To find this coframe let $(x, y) \mapsto (x, y, L_{(x,y)})$ be a cross-section of M. This defines a function $\tilde{p} : \mathcal{F}(P) \times \mathcal{F}(Q) \to M_{n\times s}(\mathbb{R})$ by

$$\tilde{p}(x, u, y, v) = v^{-1}L_{(x,y)}u. \tag{2.21}$$

This map is equivariant in the sense that,

$$\tilde{p}(x, ua, y, vb) = b^{-1}\tilde{p}(x, u, y, v)a. \tag{2.22}$$

That is, by equation (2.17), (2.22)

(2.23)

$$(x, u, y, v, \tilde{p}(x, u, y, v)) * (a, b) = (x, ua, y, vb, b^{-1}\tilde{p}(x, u, y, v)a) = (x, ua, y, vb, \tilde{p}(x, ua, y, vb)).$$

The exterior covariant derivative of $\tilde{p}$ is

$$\pi = D\tilde{p} = dp + \boldsymbol{\varphi} p - p\boldsymbol{\varpi}, \tag{2.24}$$

a $M_{n\times s}(\mathbb{R})$-valued 1-form. For $\pi = \pi^\alpha_i E^i_\alpha$, we get the $\mathbb{R}$-valued 1-forms,

$$\pi^\alpha_i = dp^\alpha_i + \phi^\alpha_\beta p^\beta_i - p^\alpha_j \omega^j_i. \tag{2.25}$$

This gives the coframe of $\mathcal{F}$

$$\{\omega^i, \phi^\alpha, \phi^\alpha_\beta, \omega^i_j, \pi^\alpha_i\}, \tag{2.26}$$

where $\omega^i, \phi^\alpha, \pi^\alpha_i$ are semi-basic to the projection $\pi^{\mathcal{F}}_M$ given in equation (2.18).

To complete the structure equations in (2.20) for the coframe in equation (2.26), we calculate

$$d\pi = d\boldsymbol{\varphi} p - \boldsymbol{\varphi} \wedge dp - dp \wedge \boldsymbol{\varpi} - p d\boldsymbol{\varpi}. \tag{2.27}$$

Substituting equation (2.24) and the structure equations, (2.20), gives

$$d\pi = -\boldsymbol{\varphi}\wedge\boldsymbol{\varphi}p + \boldsymbol{\Phi}p - \boldsymbol{\varphi}\wedge(\pi - \boldsymbol{\varphi}p + p\boldsymbol{\varpi}) - (\pi - \boldsymbol{\varphi}p + p\boldsymbol{\varpi})\wedge\boldsymbol{\varpi} + p\boldsymbol{\varpi}\wedge\boldsymbol{\varpi} - r\boldsymbol{\Omega}. \tag{2.28}$$

Simplifying gives

$$d\pi = \boldsymbol{\Phi}p - p\boldsymbol{\Omega} - \boldsymbol{\varphi}\wedge\pi - \pi\wedge\boldsymbol{\varpi} \tag{2.29}$$

or

$$d\pi_i^\alpha = \Phi_\beta^\alpha p_i^\beta - p_j^\alpha \Omega_i^j - \phi_\beta^\alpha \wedge \pi_i^\beta - \pi_j^\alpha \wedge \omega_i^j. \tag{2.30}$$

We now lift $\tilde{\theta}$ to $\mathcal{F}$ in terms of the coframe in equation (2.26). For $(x, u, y, v, p) \in \mathcal{F}$ let

$$\theta = v^{-1}(\pi_M^{\mathcal{F}})^*(\tilde{\theta}). \tag{2.31}$$

For $X \in T_{(x,u,y,v,p)}\mathcal{F}$ we get

$$v\theta(X) = \tilde{\theta}((\pi_M^{\mathcal{F}})_* X). \tag{2.32}$$

Using the definition of $\tilde{\theta}$ given in equation (2.11), this becomes

$$v\theta(X) = (\pi^M_Q)_*((\pi^{\mathfrak{F}}_M)_*X) - L|_{\pi^{\mathfrak{F}}_M(x,u,y,v,p)}((\pi^M_P)_*((\pi^{\mathfrak{F}}_M)_*X)). \tag{2.33}$$

Using equation (2.18), $\pi^M_Q \circ \pi^{\mathfrak{F}}_M = \pi^{\mathfrak{F}}_Q$, and $\pi^M_P \circ \pi^{\mathfrak{F}}_M = \pi^{\mathfrak{F}}_P$ this becomes

$$v\theta(X) = (\pi^{\mathfrak{F}}_Q)_*X - vpu^{-1}((\pi^{\mathfrak{F}}_P)_*X). \tag{2.34}$$

Factoring out v gives

$$v\theta(X) = v(v^{-1}((\pi^{\mathfrak{F}}_Q)_*X) - pu^{-1}((\pi^{\mathfrak{F}}_P)_*X). \tag{2.35}$$

This gives us

$$\theta(X) = (\phi - p\,\omega)(X). \tag{2.36}$$

Therefore θ is the $\mathbb{R}^s$-valued form $\phi - p\,\omega$, and the multi-contact system is

$$I = \{\phi^\alpha - p^\alpha_i \omega^i\}. \tag{2.37}$$

To find the structure equations for I we compute

$$d\theta = d\phi - dp \wedge \omega - pd\omega. \tag{2.38}$$

Substituting equation (2.24) and the structure equations, (2.20), gives

$$d\theta = -\boldsymbol{\varphi} \wedge \phi - (\pi - \boldsymbol{\varphi} p + p\boldsymbol{\varpi}) \wedge \omega + p\boldsymbol{\varpi} \wedge \omega. \tag{2.39}$$

Simplifying, we get

$$d\theta = -\pi \wedge \omega - \boldsymbol{\varphi} \wedge \theta \tag{2.40}$$

or

$$d\theta^\alpha = -\pi^\alpha_i \wedge \omega^i - \phi^\alpha_\beta \wedge \theta^\beta. \tag{2.41}$$

2.4.1 Pulling Back the Lagrangian to $\mathcal{F}$

Now we wish to define a function, $\lambda : M \to \mathbb{R}$ such that when pulled back to a graph of the open set $U \subset P$,

$$N_f = \{(x, f(x), f_*) | f : U \to Q\} \cong U \tag{2.42}$$

we get the energy density of f,

$$e(f) = \frac{1}{2}\mathrm{tr}_g(f^*h). \tag{2.43}$$

The energy Lagrangian will then be $\lambda\nu$, which can be pulled up to $\mathcal{F}$ and used to find the appropriate EDS. With this in mind we define the 2-form $\hat{h}$ on M for $X, Y \in T_{(x,y,L)}M$ as

$$\hat{h}(X, Y) = h\left(L\left(\left(\pi^M_P\right)_* X\right), L\left(\left(\pi^M_P\right)_* Y\right)\right). \tag{2.44}$$

We now show that pulled back to the graph of f, $\hat{h}$ becomes f^*h. For $\iota : N_f \to M$ and $X, Y \in T_{(x,f(x),f_*)}N_f$ we have

$$(\iota^*\hat{h})(X, Y) = \hat{h}(\iota_* X, \iota_* Y). \tag{2.45}$$

Using equation (2.44) this becomes

$$(\iota^*\hat{h})(X,Y) = h(L|_{(x,f(x),f_*)}((\pi_P^M)_*\iota_*X), L|_{(x,f(x),f_*)}((\pi_P^M)_*\iota_*Y)) \tag{2.46}$$
$$= h(f_*(\pi_P^M)_*\iota_*X, f_*(\pi_P^M)_*\iota_*Y)$$

Since $\pi_P^M \circ \iota = \pi_P^{N_f}$ this becomes

$$(\iota^*\hat{h})(X,Y) = f^*h((\pi_P^{N_f})_*X, (\pi_P^{N_f})_*Y) \tag{2.47}$$
$$= (\pi_P^{N_f})^* f^* h(X,Y).$$

For $\hat{g} = (\pi_P^M)^* g$, the energy density, λ, is

$$\lambda = \frac{1}{2}\mathrm{tr}_{\hat{g}}(\hat{h}). \tag{2.48}$$

The pullback of λ to $\mathfrak{F}$ is

$$(\pi_M^{\mathfrak{F}})^*\lambda = \frac{1}{2}\mathrm{tr}_{(\pi_M^{\mathfrak{F}})^*\hat{g}}\left((\pi_M^{\mathfrak{F}})^*\hat{h}\right). \tag{2.49}$$

In order to calculate a specific formula for λ we need to pullback $\hat{g}$ and $\hat{h}$ to $\mathfrak{F}$. From the definition (2.44), pulling $\hat{h}$ back to $\mathfrak{F}$, for $X, Y \in T_{(x,u,y,v,p)}\mathfrak{F}$ we get

(2.50)
$$(\pi_M^{\mathfrak{F}})^*\hat{h}(X,Y) = h(L|_{\pi_M^{\mathfrak{F}}(x,u,y,v,p)}((\pi_P^M)_*(\pi_M^{\mathfrak{F}})_*X), L|_{\pi_M^{\mathfrak{F}}(x,u,y,v,p)}((\pi_P^M)_*(\pi_M^{\mathfrak{F}})_*Y))$$
$$= h(vpu^{-1}((\pi_P^{\mathfrak{F}})_*X), vpu^{-1}((\pi_P^{\mathfrak{F}})_*Y))$$

Since $\omega = u^{-1} \circ \left(\pi_P^{\mathcal{F}}\right)_*$, this becomes

$$(2.51) \qquad \begin{aligned}(\pi_M^{\mathcal{F}})^* \hat{h}(X,Y) &= h(vp(\omega(X)), vp(\omega(Y))) \\ &= h(vp(e_i\omega^i(X)), vp(e_j\omega^j(Y))).\end{aligned}$$

Using linearity of p,v, and bilinearity of h, this becomes

$$(2.52) \qquad \begin{aligned}(\pi_M^{\mathcal{F}})^* \hat{h}(X,Y) &= h(vp(e_i), vp(e_j))\omega^i(X)\omega^j(Y) \\ &= h(v(p_i^\alpha e_\alpha), v(p_j^\beta e_\beta))\omega^i(X)\omega^j(Y) \\ &= h(v(e_\alpha), v(e_\beta))p_i^\alpha p_j^\beta \omega^i(X)\omega^j(Y).\end{aligned}$$

Finally, from the definition of $\mathcal{F}(Q)$, (A.11) we get

$$(2.53) \qquad (\pi_M^{\mathcal{F}})^* \hat{h}(X,Y) = \tilde{h}_{\alpha\beta} p_i^\alpha p_j^\beta \omega^i(X)\omega^j(Y).$$

This shows that $(\pi_M^{\mathcal{F}})^* \hat{h} = \tilde{h}_{\alpha\beta} p_i^\alpha p_j^\beta \omega^i \odot \omega^j$. Similarly we pull back $\hat{g}$, giving

$$(2.54) \qquad \begin{aligned}(\pi_M^{\mathcal{F}})^* \hat{g}(X,Y) &= g((\pi_P^M)_*(\pi_M^{\mathcal{F}})_*X, (\pi_P^M)_*(\pi_M^{\mathcal{F}})_*Y) \\ &= g((\pi_P^{\mathcal{F}})_*X, (\pi_P^{\mathcal{F}})_*Y) \\ &= g(uu^{-1}((\pi_P^{\mathcal{F}})_*X), uu^{-1}((\pi_P^{\mathcal{F}})_*Y)).\end{aligned}$$

As in equation (2.51) this becomes

$$(2.55) \qquad \begin{aligned}(\pi_M^{\mathcal{F}})^* \hat{g}(X,Y) &= g(u(\omega(X)), u(\omega(Y))) \\ &= g(u(e_i\omega^i(X)), u(e_j\omega^j(Y))).\end{aligned}$$

Since u is linear and g is bilinear, we get

$$(\pi_M^{\mathcal{F}})^*\hat{g}(X,Y) = g(u(e_i), u(e_j))\omega^i(X)\omega^j(Y). \tag{2.56}$$

Last we use the definition (A.1) to get

$$(\pi_M^{\mathcal{F}})^*\hat{g}(X,Y) = \tilde{g}_{ij}\omega^i(X)\omega^j(Y). \tag{2.57}$$

Therefore $(\pi_M^{\mathcal{F}})^*(\hat{g}) = \tilde{g}_{ij}\omega^i \odot \omega^j$, showing that the energy is

$$(\pi_M^{\mathcal{F}})^*\lambda = \frac{1}{2}\tilde{g}^{ij}p_i^\alpha p_j^\beta \tilde{h}_{\alpha\beta}, \tag{2.58}$$

where $\tilde{g}^{ij}$ is the inverse of $\tilde{g}$. That is $\tilde{g}^{ik}\tilde{g}_{kj} = \delta^i_j$.

To calculate the pullback of the volume form we operate on $X_1, \ldots, X_n \in T_{(x,u,y,v,p)}\mathcal{F}$,

$$\begin{aligned}(\pi_P^{\mathcal{F}})^*(\nu)(X_1, \ldots, X_n) &= \nu((\pi_P^{\mathcal{F}})_*X_1, \ldots, (\pi_P^{\mathcal{F}})_*X_n) \\ &= \nu(u(\omega(X_1)), \ldots, u(\omega(X_n))).\end{aligned} \tag{2.59}$$

Since u is linear and ν is n-linear this becomes

$$(\pi_P^{\mathcal{F}})^*(\nu)(X_1, \ldots, X_n) = \nu(u(e_{i_1}), \ldots, u(e_{i_n}))\omega^{i_1}(X_1)\ldots\omega^{i_n}(X_n). \tag{2.60}$$

Using $\nu(u(e_1), \ldots, u(e_n)) = 1$, from equation (A.1), and that ν is alternating, we conclude that

$$(\pi_P^{\mathcal{F}})^*(\nu) = \omega^1 \wedge \cdots \wedge \omega^n. \tag{2.61}$$

Letting $\mathfrak{w} = \omega^1 \wedge \cdots \wedge \omega^n$, the Lagrangian is

$$\tilde{\Lambda} = \frac{1}{2}\tilde{g}^{ij} p_i^\alpha p_j^\beta \tilde{h}_{\alpha\beta}\mathfrak{w}. \tag{2.62}$$

2.4.2 The Euler-Lagrange EDS for Harmonic Maps

The Lagrangian in (2.62) is not unique. The multi-contact system, I has the property that $\iota^* I = 0$ for a graph, $\iota : N_f \to \mathcal{F}$. Therefore any form Λ with the property

$$\Lambda = \tilde{\Lambda} \mod I \tag{2.63}$$

will satisfy $\iota^*\Lambda = e(f)\nu$. In [5], section 4.2.2, it is shown that if there exists Λ satisfying (2.63) and

$$d\Lambda = \theta^\alpha \wedge \Psi_\alpha, \tag{2.64}$$

for some n-forms Ψ_α, then the system algebraically generated by

$$\mathcal{E}_\Lambda = \{\theta^\alpha, d\theta^\alpha, \Psi_\alpha\} \tag{2.65}$$

has integral manifolds that are stationary for the functional induced by $\tilde{\Lambda}$. Furthermore it is shown that

$$\Lambda = \tilde{\Lambda} + \theta^\alpha \wedge \left(\frac{\partial\tilde{\lambda}}{\partial p_i^\alpha}\mathfrak{w}_{(i)}\right), \tag{2.66}$$

satisfies (2.64). Where $\tilde{\lambda} = (\pi_M^{\mathcal{F}})^*\lambda$ and $\mathfrak{w}_{(i)} = w_i \lrcorner\, \mathfrak{w}$, for w_i dual to ω^i in the coframe (2.26). This addition, $\theta^\alpha \wedge \left(\frac{\partial\tilde{\lambda}}{\partial p_i^\alpha}\mathfrak{w}_{(i)}\right)$, is called the boundary term.

Adding the boundary term, $\tilde{g}^{ij}p_j^\beta\tilde{h}_{\alpha\beta}\theta^\alpha\wedge\mathfrak{w}_{(i)}$, we get

$$\Lambda = \frac{1}{2}\tilde{g}^{ij}p_i^\alpha p_j^\beta\tilde{h}_{\alpha\beta}\mathfrak{w} + \tilde{g}^{ij}p_j^\beta\tilde{h}_{\alpha\beta}\theta^\alpha\wedge\mathfrak{w}_{(i)}. \tag{2.67}$$

We now compute $d\mathfrak{w}$ and $d\mathfrak{w}_{(i)}$ to be used in the calculation of $d\Lambda$.

$$\begin{aligned} d\mathfrak{w} &= \sum_{i=1}^{n}(-1)^{i-1}\omega^1\wedge\cdots\wedge\omega^{i-1}\wedge d\omega^i\wedge\omega^{i+1}\wedge\cdots\wedge\omega^n \\ &= \sum_{i=1}^{n}(-1)^{i-1}d\omega^i\wedge\omega^1\wedge\cdots\wedge\omega^{i-1}\wedge\omega^{i+1}\wedge\cdots\wedge\omega^n \\ &= d\omega^i\wedge\mathfrak{w}_{(i)}. \end{aligned} \tag{2.68}$$

Substituting the structure equations, (2.20), we get

$$d\mathfrak{w} = -\omega_j^i\wedge\omega^j\wedge\mathfrak{w}_{(i)}. \tag{2.69}$$

Using $\omega^j\wedge\mathfrak{w}_{(i)} = \delta_i^j\mathfrak{w}$, this becomes

$$d\mathfrak{w} = -\delta_i^j\omega_j^i\wedge\mathfrak{w}. \tag{2.70}$$

Since ω_j^i is $\mathfrak{so}(n_1,n_2)$-valued, and therefore trace-free, we have

$$d\mathfrak{w} = 0. \tag{2.71}$$

Similarly,

$$d\mathfrak{w}_{(i)} = d\omega^j\wedge\mathfrak{w}_{(ij)}, \tag{2.72}$$

where $\mathfrak{w}_{(ij)} = w_j \lrcorner\, \mathfrak{w}_{(i)}$. Substituting the structure equations, (2.20), gives

$$d\mathfrak{w}_{(i)} = -\omega^j_k \wedge \omega^k \wedge \mathfrak{w}_{(ij)} \tag{2.73}$$

Using the equation $\omega^k \wedge \mathfrak{w}_{(ij)} = \delta^k_j \mathfrak{w}_{(i)} - \delta^k_i \mathfrak{w}_{(j)}$, this becomes

$$\begin{aligned} d\mathfrak{w}_{(i)} &= -\omega^j_k \wedge (\delta^k_j \mathfrak{w}_{(i)} - \delta^k_i \mathfrak{w}_{(j)}) \\ &= -\delta^k_j \omega^j_k \wedge \mathfrak{w}_{(i)} + \delta^k_i \omega^j_k \wedge \mathfrak{w}_{(j)} \\ &= \omega^j_i \wedge \mathfrak{w}_{(j)}. \end{aligned} \tag{2.74}$$

Again, the last step uses that ω^j_k is trace-free.

From the structure equations, (2.20), we get

$$d\Lambda = \tilde{g}^{ij} p^\alpha_i dp^\beta_j \tilde{h}_{\alpha\beta} \wedge \mathfrak{w} + \tilde{g}^{ij} dp^\beta_j \tilde{h}_{\alpha\beta} \wedge \theta^\alpha \wedge \mathfrak{w}_{(i)} + \tilde{g}^{ij} p^\beta_j \tilde{h}_{\alpha\beta} d\theta^\alpha \wedge \mathfrak{w}_{(i)} - \tilde{g}^{ij} p^\beta_j \tilde{h}_{\alpha\beta} \theta^\alpha \wedge d\mathfrak{w}_{(i)} \tag{2.75}$$

Using $dp^\beta_j = (\pi^\beta_j - \phi^\beta_\gamma p^\gamma_j + p^\beta_k \omega^k_j)$, and equations (2.71),(2.74), and (2.41) this becomes

$$\begin{aligned} d\Lambda = \tilde{g}^{ij} p^\alpha_i (\pi^\beta_j - \phi^\beta_\gamma p^\gamma_j + p^\beta_k \omega^k_j) \tilde{h}_{\alpha\beta} \wedge \mathfrak{w} + \tilde{g}^{ij} (\pi^\beta_j - \phi^\beta_\gamma p^\gamma_j + p^\beta_k \omega^k_j) \tilde{h}_{\alpha\beta} \wedge \theta^\alpha \wedge \mathfrak{w}_{(i)} \\ + \tilde{g}^{ij} p^\beta_j \tilde{h}_{\alpha\beta} (-\pi^\alpha_k \wedge \omega^k - \phi^\alpha_\gamma \wedge \theta^\gamma) \wedge \mathfrak{w}_{(i)} - \tilde{g}^{ij} p^\beta_j \tilde{h}_{\alpha\beta} \theta^\alpha \wedge \omega^k_i \wedge \mathfrak{w}_{(k)}. \end{aligned} \tag{2.76}$$

Distributing we get

$$\begin{aligned} d\Lambda = &- \tilde{g}^{ij} p^\alpha_i p^\gamma_j \phi_{\alpha\gamma} \wedge \mathfrak{w} + p^\alpha_i p^\beta_k \tilde{h}_{\alpha\beta} \omega^{ik} \wedge \mathfrak{w} + \tilde{g}^{ij} \tilde{h}_{\alpha\beta} \pi^\beta_j \wedge \theta^\alpha \wedge \mathfrak{w}_{(i)} - \tilde{g}^{ij} p^\gamma_j \phi_{\alpha\gamma} \wedge \theta^\alpha \wedge \mathfrak{w}_{(i)} \\ &+ p^\beta_k \tilde{h}_{\alpha\beta} \omega^{ik} \wedge \theta^\alpha \wedge \mathfrak{w}_{(i)} - \tilde{g}^{ij} p^\beta_j \phi_{\beta\gamma} \wedge \theta^\gamma \wedge \mathfrak{w}_{(i)} + p^\beta_j \tilde{h}_{\alpha\beta} \omega^{jk} \wedge \theta^\alpha \wedge \mathfrak{w}_{(k)}, \end{aligned} \tag{2.77}$$

where $\tilde{g}^{ij}\omega_j^k = \omega^{ik}$ and $\tilde{h}_{\alpha\beta}\phi_\gamma^\beta = \phi_{\alpha\gamma}$. Last we use that ω^{ij} and $\phi_{\alpha\beta}$ are skew-symmetric to get

$$d\Lambda = -\theta^\alpha \wedge (\tilde{g}^{ij}\tilde{h}_{\alpha\beta}\pi_i^\beta \wedge \mathfrak{w}_{(j)}). \tag{2.78}$$

The EDS whose integral manifolds are those which are stationary for the functional $\int_P \Lambda$ is

$$\mathcal{E}_\Lambda = \{\theta^\alpha, \pi_i^\alpha \wedge \omega^i, \tilde{g}^{ij}\tilde{h}_{\alpha\beta}\pi_i^\beta \wedge \mathfrak{w}_{(j)}\}. \tag{2.79}$$

This is the obvious generalization of the system in [5]. This system is semi-basic with respect to the action of $SO(n_1, n_2) \times O(s_1, s_2)$ on $\mathcal{F}$. Therefore the Cauchy characteristic of $\mathcal{E}_\Lambda$ contains the infinitesimal generators of this action. In fact, this is the full characteristic system.

2.4.3 Noether's Theorem

Let the section of $T\mathcal{F}$, Z, be an infinitesimal symmetry of Λ. That is

$$Z(\Lambda) = 0. \tag{2.80}$$

If $\Pi = d\Lambda$, and Λ satisfies (2.64), then we have

$$Z \lrcorner \Pi = (Z \lrcorner \theta^\alpha) \wedge \Psi_\alpha - \theta^\alpha \wedge (Z \lrcorner \Psi_\alpha). \tag{2.81}$$

This is in $\mathcal{E}_\Lambda$. From Cartan's formula for the Lie derivative we have

$$d(Z \lrcorner \Pi) = Z(\Pi) - Z \lrcorner d\Pi = dZ(\Lambda) = 0. \tag{2.82}$$

Therefore $Z \lrcorner \Pi$ is a closed form in $\mathcal{E}_\Lambda$. In [5] it shows

$$d(-Z \lrcorner \Lambda) = -Z(\Lambda) + Z \lrcorner d\Lambda = Z \lrcorner \Pi. \tag{2.83}$$

Therefore $-Z \lrcorner \Lambda$ is the conserved density corresponding to the symmetry Z.

2.4.4 Conformal Invariance for $n = 2$

Let g_1 and g_2 be conformally equivalent, that is

$$g_2 = e^{2\mu} g_1. \tag{2.84}$$

Under this assumption, if $\tilde{\omega}^i$ is an orthonormal frame for g_1, then $e^\mu \tilde{\omega}^i$ is an orthonormal frame for g_2. Therefore, the volume form for g_1 is

$$\nu_1 = \tilde{\omega}^1 \wedge \cdots \wedge \tilde{\omega}^n, \tag{2.85}$$

and the volume form for g_2 is

$$\nu_2 = (e^\mu \tilde{\omega}^1) \wedge \cdots \wedge (e^\mu \tilde{\omega}^n) = e^{n\mu} \nu_1. \tag{2.86}$$

These are then considered pulled back to $\mathcal{F}$, with $\mathcal{F}(P)$ being the orthonormal frame bundle for g_1. That is

$$(\pi_P^{\mathcal{F}})^*(g_1) = \tilde{g}_{ij} \omega^i \odot \omega^j, \tag{2.87}$$

and

$$(\pi_P^{\mathcal{F}})^*(\nu_1) = \mathfrak{w}. \tag{2.88}$$

We now pullback g_2 and its volume form, ν_2. Using equation (2.84), we have

$$(\pi_P^{\mathcal{F}})^*(g_2) = (\pi_P^{\mathcal{F}})^*(e^{2\mu}g_1) = e^{2\mu}\tilde{g}_{ij}\omega^i \odot \omega^j. \tag{2.89}$$

Similarly, equation (2.86) gives

$$(\pi_P^{\mathcal{F}})^*(\nu_2) = (\pi_P^{\mathcal{F}})^*(e^{n\mu}\nu_1) = e^{n\mu}\mathfrak{w}. \tag{2.90}$$

This shows that

$$\tilde{\Lambda}_2 = e^{(n-2)\mu}\tilde{\Lambda}_1, \tag{2.91}$$

and thus for $n = 2$ the energy Lagrangian is conformally invariant.

CHAPTER 3

THE EDS FOR WAVE MAPS INTO 2D RIEMANNIAN MANIFOLDS

In this chapter we restrict our study to harmonic maps from 2D Minkowski space into a 2D Riemannian manifold. These are also known as 2D wave maps or σ-models. The EDS for these maps is invariant under conformal transformations of the domain manifold. We use this invariance to compute the corresponding quotient EDS $\bar{\mathcal{E}}$ by choosing a coframe adapted to the infinitesimal generators. The condition of Darboux integrability of $\mathcal{E}$ is written as necessary and sufficient conditions in terms of properties of $\bar{\mathcal{E}}$. For these conditions to be satisfied the target Riemannian manifold must have zero scalar curvature.

3.1 The Wave Map Differential Equations

Choosing coordinates (x^1, x^2) on P and (y^1, y^2) on Q such that $g = dx^1 \odot dx^2$ and $h = e^{2\rho(y^1,y^2)}(dy^1 \odot dy^1 + dy^2 \odot dy^2)$, equation (2.10) becomes

$$y^1_{12} = -\frac{\partial \rho}{\partial y^1}(y^1_1 y^1_2 - y^2_1 y^2_2) - \frac{\partial \rho}{\partial y^2}(y^1_1 y^2_2 + y^1_2 y^2_1), \tag{3.1}$$

$$y^2_{12} = -\frac{\partial \rho}{\partial y^1}(y^1_1 y^2_2 + y^1_2 y^2_1) + \frac{\partial \rho}{\partial y^2}(y^1_1 y^1_2 - y^2_1 y^2_2). \tag{3.2}$$

As stated in section 2.2, rather than finding an EDS for these equations, defined $J^2(\mathbb{R}^{1,1}, \mathbb{R}^2)$, we will use the global EDS presented in section 2.4.

3.2 EDS Setup

We now restrict the computations from chapter 2 to the case $P = \mathbb{R}^{1,1}$(Minkowski space), $s = 2$, $\tilde{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\tilde{h} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $P = \mathbb{R}^{1,1}$ has a global oriented orthonormal coframe, the lift to $\mathcal{F}(P)$ is not necessary. With these assumptions the manifold $\mathcal{F}$ in equation (2.16) is

$$\mathcal{F} = \mathbb{R}^{1,1} \times \mathcal{F}(Q) \times M_{2\times 2}(\mathbb{R}), \tag{3.3}$$

and the structure equations in (2.20) are

$$\begin{aligned} d\omega^1 &= d\omega^2 = 0, \\ d\phi^1 &= -\phi^1_2 \wedge \phi^2, \\ d\phi^2 &= \phi^1_2 \wedge \phi^1, \\ d\phi^1_2 &= \kappa \phi^1 \wedge \phi^2, \end{aligned} \tag{3.4}$$

where ϕ^1_2 is the connection form on Q and κ is the Gauss curvature. The contact system from equation (2.37) is

$$I = \{\theta^1, \theta^2\}, \tag{3.5}$$

where

$$\begin{aligned} \theta^1 &= \phi^1 - p^1_1\omega^1 - p^1_2\omega^2, \\ \theta^2 &= \phi^2 - p^2_1\omega^1 - p^2_2\omega^2. \end{aligned} \tag{3.6}$$

From equation (2.41) we get the structure equations

$$\begin{aligned} d\theta^1 &= -\phi^1_2 \wedge \theta^2 - \pi^1_1 \wedge \omega^1 - \pi^1_2 \wedge \omega^2, \\ d\theta^2 &= \phi^1_2 \wedge \theta^1 - \pi^2_1 \wedge \omega^1 - \pi^2_2 \wedge \omega^2 \end{aligned} \tag{3.7}$$

where, from equation (2.25),

$$
\begin{aligned}
\pi^1_1 &= dp^1_1 + p^2_1\phi^1_2, \\
\pi^1_2 &= dp^1_2 + p^2_2\phi^1_2, \\
\pi^2_1 &= dp^2_1 - p^1_1\phi^1_2, \\
\pi^2_2 &= dp^2_2 - p^1_2\phi^1_2.
\end{aligned}
\tag{3.8}
$$

The energy Lagrangian in equation (2.62) is

$$\tilde{\Lambda} = (p^1_1 p^1_2 + p^2_1 p^2_2)\omega^1 \wedge \omega^2. \tag{3.9}$$

Adding the boundary term as in equation (2.67) gives

$$\Lambda = (p^1_1 p^1_2 + p^2_1 p^2_2)\omega^1 \wedge \omega^2 - p^1_1\theta^1 \wedge \omega^1 + p^1_2\theta^1 \wedge \omega^2 - p^2_1\theta^2 \wedge \omega^1 + p^2_2\theta^2 \wedge \omega^2. \tag{3.10}$$

The Poincare-Cartan form in equation (2.78) is

$$d\Lambda = \theta^1 \wedge (\pi^1_1 \wedge \omega^1 - \pi^1_2 \wedge \omega^2) + \theta^2 \wedge (\pi^2_1 \wedge \omega^1 - \pi^2_2 \wedge \omega^2). \tag{3.11}$$

Thus the Euler-Lagrange system from equation (2.79) is

$$\mathcal{E} = \{\theta^1, \theta^2, \pi^1_1 \wedge \omega^1, \pi^1_2 \wedge \omega^2, \pi^2_1 \wedge \omega^1, \pi^2_2 \wedge \omega^2\}, \tag{3.12}$$

with independence condition $\omega^1 \wedge \omega^2 \neq 0$. The Cauchy characteristic is the infinitesimal generator of the action of $O(2)$ on $\mathcal{F}$,

$$\mathcal{C} = \{\partial_{\phi^1_2}\}. \tag{3.13}$$

Using definition (2.3) and the structure equations (3.7), $\mathcal{E}$ is decomposable of type $[3,3]$, with associated singular Pfaffian systems

$$\begin{aligned} \hat{V} &= \{\theta^1, \theta^2, \pi^1_2, \pi^2_2, \omega^2\}, \\ \check{V} &= \{\theta^1, \theta^2, \pi^1_1, \pi^2_1, \omega^1\}. \end{aligned} \tag{3.14}$$

(We note the in general the system in equation (2.79) is not generated by 1-forms and 2-forms, and therefore cannot be decomposable.)

3.3 Conformal Group Action and the Quotient Manifold

From equation (2.91) the Lagrangian in equation (3.9) is invariant with respect to the group Conf($\mathbb{R}^{1,1}$), conformal transformations of Minkowski space. Since this action prolonged to $\mathcal{F}$ preserves the contact ideal I as well, it must also preserve $\mathcal{E}$ by its definition in equation (2.65). With the goal of computing the quotient of $\mathcal{E}$ by Conf($\mathbb{R}^{1,1}$), we calculate Conf($\mathbb{R}^{1,1}$) and its action on $\mathcal{F}$. Let x^1, x^2 be coordinates on $\mathbb{R}^{1,1}$ such that

$$\begin{aligned} dx^1 &= \omega^1, \\ dx^2 &= \omega^2. \end{aligned} \tag{3.15}$$

Then we have $g = dx^1 \odot dx^2$. For a diffeomorphism of $\mathbb{R}^{1,1}$

$$(x^1, x^2) \mapsto (F(x^1, x^2), G(x^1, x^2)) \tag{3.16}$$

to be conformal it must satisfy

$$dF \odot dG = H(x^1, x^2) dx^1 \odot dx^2, \tag{3.17}$$

for some nowhere-vanishing function H. Expanding equation (3.17) we get

(3.18)
$$(F_1G_1)dx^1 \odot dx^1 + (F_1G_2 + F_2G_1)dx^1 \odot dx^2 + (F_2G_2)dx^2 \odot dx^2 = H(x^1, x^2)dx^1 \odot dx^2.$$

This has two solutions,

(3.19)
$$F_2 = G_1 = 0, F_1 \neq 0, G_2 \neq 0 \qquad \text{and} \qquad F_1 = G_2 = 0, F_2 \neq 0, G_1 \neq 0,$$

showing that $\mathrm{Conf}(\mathbb{R}^{1,1}) = \mathbb{Z}_2 \ltimes \mathcal{G}$. Where the oriented conformal group, $\mathcal{G} = \mathrm{Conf}^+(\mathbb{R}^{1,1})$ acts on $\mathbb{R}^{1,1}$ by

(3.20)
$$(F, G) * (x^1, x^2) = (F(x^1), G(x^2)),$$

and the non-identity element of $\mathbb{Z}_2$ acts on $\mathbb{R}^{1,1}$ by

(3.21)
$$(x^1, x^2) \mapsto (x^2, x^1).$$

The action of $\mathcal{G}$ from equation (3.20) can be prolonged to $\mathcal{F}$ as

(3.22)
$$(x^1, x^2, y, v, p^1_1, p^2_1, p^1_2, p^2_2) \mapsto \left(F(x^1), G(x^2), y, v, \frac{p^1_1}{F'(x^1)}, \frac{p^2_1}{F'(x^1)}, \frac{p^1_2}{G'(x^2)}, \frac{p^2_2}{G'(x^2)} \right).$$

(For the infinitessimal version of this calculation see (3.34).)

Equation (3.22) allows us to compute the orbit of $\mathcal{G}$ through the point $(x_0, y_0, v_0, \tilde{p}) \in \mathcal{F}$ to be

$$\mathcal{O}_{(x_0,y_0,v_0,\tilde{p})} = \{(x, y_0, v_0, p) | x \in \mathbb{R}^{1,1}, p^1_1 = \lambda\tilde{p}^1_1, p^2_1 = \lambda\tilde{p}^2_1, p^1_2 = \mu\tilde{p}^1_2, p^2_2 = \mu\tilde{p}^2_2 \text{ for } \lambda, \mu \neq 0\}$$

These orbits are sub-maximal when $p^1_1 = p^2_1 = 0$ or $p^1_2 = p^2_2 = 0$.

Theorem 3.1. *The group $\mathcal{G}$ acts regularly on the open set*

$$U = \mathbb{R}^{1,1} \times \mathcal{F}(Q) \times (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0\}) \subset \mathcal{F}, \tag{3.23}$$

and the quotient of U by $\mathcal{G}$ is

$$\bar{U} = \mathcal{F}(Q) \times S^1 \times S^1. \tag{3.24}$$

Proof. Defining the map $\mathfrak{q} : U \to \mathcal{F}(Q) \times S^1 \times S^1$ by

$$\mathfrak{q}(x, y, v, p) = \left[y = y, v = v, s_0 = \arctan\left(\frac{p_1^2}{p_1^1}\right), t_0 = \arctan\left(\frac{p_2^2}{p_2^1}\right)\right], \tag{3.25}$$

we see that the level sets of $\mathfrak{q}$ coincide with the orbits of $\mathcal{G}$. Therefore the image of $\mathfrak{q}$ is the quotient.

□

The diffeomorphism of $\mathbb{R}^{1,1}$ in equation (3.21) extends to a diffeomorphism $\psi : \mathcal{F} \to \mathcal{F}$ as

$$\psi(x^1, x^2, y, v, p_1^1, p_1^2, p_2^1, p_2^2) = (x^2, x^1, y, v, p_2^1, p_2^2, p_1^1, p_1^2). \tag{3.26}$$

The function ψ acts on the forms in equations (3.6), (3.8), and (3.15) as

$$
\begin{aligned}
\psi^*(\omega^1) &= \omega^2, \\
\psi^*(\omega^2) &= \omega^1, \\
\psi^*(\theta^1) &= \theta^1, \\
\psi^*(\theta^2) &= \theta^2, \\
\psi^*(\pi^1_1) &= \pi^1_2, \\
\psi^*(\pi^2_1) &= \pi^2_2, \\
\psi^*(\pi^1_2) &= \pi^1_1, \\
\psi^*(\pi^2_2) &= \pi^2_1.
\end{aligned}
\tag{3.27}
$$

From equation (3.27) the corresponding actions on the EDS $\mathcal{E}$ and its associated singular differential systems are

$$
\begin{aligned}
\psi^*(\mathcal{E}) &= \mathcal{E}, \\
\psi^*(\hat{V}) &= \check{V}, \\
\psi^*(\check{V}) &= \hat{V}.
\end{aligned}
\tag{3.28}
$$

Therefore ψ is an involution of the pair $\{\hat{V}, \check{V}\}$.

3.4 The Quotient of the EDS by $\mathbf{Conf}^+(\mathbb{R}^{1,1})$

From now on we consider the EDS $\mathcal{E}$ in equation (3.12) on the open set U given in equation 3.23. In order to compute the quotient of $\mathcal{E}$ we will need the infinitesimal generators of $\mathcal{G}$. Let

$$
Z = f\partial_{x^1} + g\partial_{x^2},
\tag{3.29}
$$

and calculate,

$$Z(\omega^1 \odot \omega^2) = Z(\omega^1) \odot \omega^2 + \omega^1 \odot Z(\omega^2) \\ = df \odot \omega^2 + \omega^2 \odot dg. \tag{3.30}$$

For a conformal transformation, $Z(\omega^1 \odot \omega^2) = \lambda \omega^1 \odot \omega^2$. This implies, as expected, $f = f(x^1)$ and $g = g(x^2)$. We lift this to an action on $\mathcal{F}$ which preserves I by setting

$$\tilde{Z} = Z + a_1 \partial_{\phi^1} + a_2 \partial_{\phi^2} + a_3 \partial_{p_1^1} + a_4 \partial_{p_2^1} + a_5 \partial_{p_1^2} + a_6 \partial_{p_2^2}, \tag{3.31}$$

where

$$Z = f(x^1)\partial_{x^1} + g(x^2)\partial_{x^2}, \tag{3.32}$$

and the functions $a_i : \mathcal{F} \to \mathbb{R}$, are of the form $\partial_{\phi_2^1} \lrcorner\, da_i = 0$. Computing

$$\begin{array}{rcl} \tilde{Z}(\theta^1) & = & da_1 + a_2 \phi_2^1 - (p_1^1 f' + a_3)\omega^1 - (p_2^1 g' + a_4)\omega^2, \\ \tilde{Z}(\theta^2) & = & da_2 + a_1 \phi_2^1 - (p_1^2 f' + a_5)\omega^1 - (p_2^2 g' + a_6)\omega^2, \end{array} \tag{3.33}$$

and forcing the result to be in I gives

$$Z^{(0)} = f\partial_{x^1} + g\partial_{x^2} - f' p_1^1 \partial_{p_1^1} - g' p_2^1 \partial_{p_2^1} - f' p_1^2 \partial_{p_1^2} - g' p_2^2 \partial_{p_2^2}. \tag{3.34}$$

The integrable distribution generated by $Z^{(0)}$ in equation (3.34) is then

$$\mathbf{\Gamma} = \{\partial_{x^1}, \partial_{x^2}, p_1^1 \partial_{p_1^1} + p_1^2 \partial_{p_1^2}, p_2^1 \partial_{p_2^1} + p_2^2 \partial_{p_2^2}\}. \tag{3.35}$$

The first step in computing the quotient is to find a coframe as in proposition B.34. We start with the semi-basic forms

$$\mathbf{\Gamma}^{\perp} = \left\{ \tilde{\sigma}, \tilde{\tau}, \tilde{\tilde{\eta}}_1, \tilde{\tilde{\xi}}_1, \phi^1_2 \right\}, \tag{3.36}$$

where

$$\begin{array}{rcl} \tilde{\sigma} & = & -\omega^1 - \frac{p^2_2}{p^1_1p^2_2-p^1_2p^2_1}\theta^1 + \frac{p^1_2}{p^1_1p^2_2-p^1_2p^2_1}\theta^2 \quad = \quad \frac{p^1_2}{p^1_1p^2_2-p^1_2p^2_1}\phi^2 - \frac{p^2_2}{p^1_1p^2_2-p^1_2p^2_1}\phi^1, \\ \tilde{\tau} & = & -\omega^2 + \frac{p^2_1}{p^1_1p^2_2-p^1_2p^2_1}\theta^1 - \frac{p^1_1}{p^1_1p^2_2-p^1_2p^2_1}\theta^2 \quad = \quad \frac{p^2_1}{p^1_1p^2_2-p^1_2p^2_1}\phi^1 - \frac{p^1_1}{p^1_1p^2_2-p^1_2p^2_1}\phi^2, \\ \tilde{\tilde{\eta}}_1 & = & p^1_2\pi^2_2 - p^2_2\pi^1_2 \quad = \quad p^1_2dp^2_2 - p^2_2dp^1_2 - \left((p^1_2)^2 + (p^2_2)^2\right)\phi^1_2, \\ \tilde{\tilde{\xi}}_1 & = & p^1_1\pi^2_1 - p^2_1\pi^1_1 \quad = \quad p^1_1dp^2_1 - p^2_1dp^1_1 - \left((p^1_1)^2 + (p^2_1)^2\right)\phi^1_2. \end{array} \tag{3.37}$$

We complete these to a coframe dual to the vector fields in $\mathbf{\Gamma}$

$$\begin{array}{rcl} \beta_1 & = & -\frac{p^2_2}{p^1_1p^2_2-p^1_2p^2_1}\theta^1 + \frac{p^1_2}{p^1_1p^2_2-p^1_2p^2_1}\theta^2 \quad = \quad \frac{p^1_2}{p^1_1p^2_2-p^1_2p^2_1}\phi^2 - \frac{p^2_2}{p^1_1p^2_2-p^1_2p^2_1}\phi^1 + \omega^1, \\ \beta_2 & = & +\frac{p^2_1}{p^1_1p^2_2-p^1_2p^2_1}\theta^1 - \frac{p^1_1}{p^1_1p^2_2-p^1_2p^2_1}\theta^2 \quad = \quad \frac{p^2_1}{p^1_1p^2_2-p^1_2p^2_1}\phi^1 - \frac{p^1_1}{p^1_1p^2_2-p^1_2p^2_1}\phi^2 + \omega^2, \\ \tilde{\alpha}_1 & = & \frac{1}{(p^1_2)^2+(p^2_2)^2}(p^1_2\pi^1_2 + p^2_2\pi^2_2) \quad = \quad \frac{1}{(p^2_2)^2+(p^1_2)^2}(p^1_2dp^1_2 + p^2_2dp^2_2), \\ \tilde{\zeta}_1 & = & \frac{1}{(p^1_1)^2+(p^2_1)^2}(p^1_1\pi^1_1 + p^2_1\pi^2_1) \quad = \quad \frac{1}{(p^1_1)^2+(p^2_1)^2}(p^1_1dp^1_1 + p^2_1dp^2_1). \end{array} \tag{3.38}$$

We note that in order to work with this coframe we must further restrict U to the open set where $\det(p) \neq 0$. The corresponding condition on $\bar{U}$ is $s_0 \neq t_0$. In this coframe the EDS $\mathcal{E}$ in equation (3.12) is

$$\mathcal{E} = \left\{ \beta_1, \beta_2, \tilde{\alpha}_1 \wedge \tilde{\tau}, \tilde{\zeta}_1 \wedge \tilde{\sigma}, \tilde{\tilde{\eta}}_1 \wedge \tilde{\tau}, \tilde{\tilde{\xi}}_1 \wedge \tilde{\sigma} \right\}. \tag{3.39}$$

We can now give the main theorem of this section.

Theorem 3.2. *Let $\mathcal{E}$ be the EDS in equation (3.39) and $\mathfrak{q} : U \to \bar{U}$ the quotient. Then the quotient EDS $\bar{\mathcal{E}}$ is an $s = 0$ hyperbolic system [11] (modulo the characteristic*

$\partial_{\phi_2^1}$). *Moreoever, for a cross-section* $\mathfrak{s} : \bar{U} \to U$

$$\bar{\mathcal{E}} = \left\{ \mathfrak{s}^* \left(\tilde{\tilde{\eta}}_1 \wedge \tilde{\tau} \right), \mathfrak{s}^* \left(\tilde{\tilde{\xi}}_1 \wedge \tilde{\sigma} \right) \right\}. \tag{3.40}$$

Proof. The semi-basic forms in $\mathcal{E}$ are

$$\begin{aligned} \mathcal{A}^1_{sb} &= \mathcal{E} \cap \Omega^1(\mathbf{\Gamma}^\perp) = \{0\} \\ \mathcal{A}^2_{sb} &= \mathcal{E} \cap \Omega^2(\mathbf{\Gamma}^\perp) = \operatorname{span} \left\{ \tilde{\tilde{\eta}}_1 \wedge \tilde{\tau}, \tilde{\tilde{\xi}}_1 \wedge \tilde{\sigma} \right\}. \end{aligned} \tag{3.41}$$

Therefore the EDS $\mathcal{E}$ in equation (3.39) satisfies the hypotheses of theorem B.6, therefore the quotient system is algebraically generated by 1-forms and 2-forms. Pulling the semi-basic 2-forms from equation (3.41) back by the cross-section $\mathfrak{s}$ gives equation (3.40). Since the semi-basic 2-forms in equation (3.41) are decomposable and transverse, their pullback is decomposable and transverse. Therefore $\bar{\mathcal{E}}$ in equation (3.40) is an $s = 0$ hyperbolic system.

□

3.5 Coordinate Computations and the Final Coframe

In this section we write $\mathcal{E}$ and $\bar{\mathcal{E}}$ in coordinates and give the structure equations of the final coframe to be used for computations. Let $\mathfrak{s} : \bar{U} \to U$ be the cross-section

(3.42)

$$\mathfrak{s}(y, v, s_0, t_0) = \left[x^1 = 0, x^2 = 0, y, v, p^1_1 = \cos(s_0), p^2_1 = \sin(s_0), p^1_2 = \cos(t_0), p^2_2 = \sin(t_0) \right].$$

This induces slice coordinates $\gamma : \mathbb{R}^{1,1} \times \mathcal{F}(Q) \times \mathbb{R}^2 \times S^1 \times S^1 \to U$ on U given by

(3.43)

$$\gamma(x, y, v, q_0, r_0, s_0, t_0) = [x, y, v, p^1_1 = e^{q_0} \cos(s_0), p^1_2 = e^{r_0} \cos(t_0), p^2_1 = e^{q_0} \sin(s_0), p^2_2 = e^{r_0} \sin(t_0)].$$

In these coordinates on U, $Z^{(0)}$ given in equation (3.34) is then

$$Z^{(0)} = f\partial_{\omega^1} + g\partial_{\omega^2} - f'\partial_{q_0} - g'\partial_{r_0}, \tag{3.44}$$

and the corresponding integrable distribution is

$$\mathbf{\Gamma} = \{\partial_{x^1}, \partial_{x^2}, \partial_{q_0}, \partial_{r_0}\}. \tag{3.45}$$

Using the cross-section $\mathfrak{s}$ in equation (3.42) let

$$\begin{array}{rcccl} \tilde{\eta}_1 & = & \mathfrak{s}^*\tilde{\tilde{\eta}}_1 & = & dt_0 - \phi^1_2, \\ \tilde{\xi}_1 & = & \mathfrak{s}^*\tilde{\tilde{\xi}}_1 & = & ds_0 - \phi^1_2, \\ \sigma & = & \mathfrak{s}^*\tilde{\sigma} & = & -\frac{\sin(t_0)}{\sin(\delta)}\phi^1 + \frac{\cos(t_0)}{\sin(\delta)}\phi^2, \\ \tau & = & \mathfrak{s}^*\tilde{\tau} & = & \frac{\sin(s_0)}{\sin(\delta)}\phi^1 - \frac{\cos(s_0)}{\sin(\delta)}\phi^2, \end{array} \tag{3.46}$$

where $\delta = t_0 - s_0$. From equation (3.40) we have the quotient EDS is

$$\bar{\mathcal{E}} = \{\tilde{\eta}_1 \wedge \tau, \tilde{\xi}_1 \wedge \sigma\}. \tag{3.47}$$

We now pullback the forms from equation (3.46) by the quotient map $\mathfrak{q} : U \to \bar{U}$ to get $\mathcal{G}$-basic forms on U. By abuse of notation we write

$$\begin{array}{rcccl} \tilde{\eta}_1 & = & \mathfrak{q}^*\tilde{\eta}_1 & = & dt_0 - \phi^1_2, \\ \tilde{\xi}_1 & = & \mathfrak{q}^*\tilde{\xi}_1 & = & ds_0 - \phi^1_2, \\ \sigma & = & \mathfrak{q}^*\sigma & = & -\frac{\sin(t_0)}{\sin(\delta)}\phi^1 + \frac{\cos(t_0)}{\sin(\delta)}\phi^2, \\ \tau & = & \mathfrak{q}^*\tau & = & \frac{\sin(s_0)}{\sin(\delta)}\phi^1 - \frac{\cos(s_0)}{\sin(\delta)}\phi^2, \end{array} \tag{3.48}$$

These can be viewed as forms on either U or $\bar{U}$. The forms for the completed coframe from equation (3.38) are

$$
\begin{aligned}
\beta_1 &= -\tfrac{\sin(t_0)}{e^{q_0}\sin(\delta)}\phi^1 + \tfrac{\cos(t_0)}{e^{q_0}\sin(\delta)}\phi^2 + \omega^1, \\
\beta_2 &= \tfrac{\sin(s_0)}{e^{r_0}\sin(\delta)}\phi^1 - \tfrac{\cos(s_0)}{e^{r_0}\sin(\delta)}\phi^2 + \omega^2, \\
\tilde{\alpha}_1 &= dr_0, \\
\tilde{\zeta}_1 &= dq_0.
\end{aligned}
\tag{3.49}
$$

In the coframe on U

$$
\{\beta_1, \beta_2, \tilde{\alpha}_1, \tilde{\zeta}_1, \tilde{\eta}_1, \tilde{\xi}_1, \sigma, \tau, \phi^1_2\}
\tag{3.50}
$$

the contact system is $I = \{\beta_1, \beta_2\}$, the independence condition is $\sigma \wedge \tau \neq 0$, and

$$
\mathcal{E} = \{\beta_1, \beta_2, \tilde{\alpha}_1 \wedge \tau, \tilde{\zeta}_1 \wedge \sigma, \tilde{\eta}_1 \wedge \tau, \tilde{\xi}_1 \wedge \sigma\}.
\tag{3.51}
$$

The structure equations for the coframe in equation (3.50) are easily computed from equations (3.48), (3.49), and (3.4) to be

$$
\begin{aligned}
d\beta_1 &= -\tfrac{1}{e^{q_0}\sin(\delta)}\tilde{\eta}_1 \wedge \tau - \tfrac{1}{e^{q_0}}\tilde{\zeta}_1 \wedge \sigma - \tfrac{\cos(\delta)}{e^{q_0}\sin(\delta)}\tilde{\xi}_1 \wedge \sigma, \\
d\beta_2 &= \tfrac{1}{e^{r_0}\sin(\delta)}\tilde{\xi}_1 \wedge \sigma - \tfrac{1}{e^{r_0}}\tilde{\alpha}_1 \wedge \tau + \tfrac{\cos(\delta)}{e^{r_0}\sin(\delta)}\tilde{\eta}_1 \wedge \tau, \\
d\tilde{\alpha}_1 &= 0, \\
d\tilde{\zeta}_1 &= 0, \\
d\tilde{\eta}_1 &= -\kappa\sin(\delta)\sigma \wedge \tau, \\
d\tilde{\xi}_1 &= -\kappa\sin(\delta)\sigma \wedge \tau, \\
d\sigma &= \tfrac{\cos(\delta)}{\sin(\delta)}\tilde{\xi}_1 \wedge \sigma + \tfrac{1}{\sin(\delta)}\tilde{\eta}_1 \wedge \tau, \\
d\tau &= -\tfrac{1}{\sin(\delta)}\tilde{\xi}_1 \wedge \sigma - \tfrac{\cos(\delta)}{\sin(\delta)}\tilde{\eta}_1 \wedge \tau, \\
d\phi^1_2 &= \kappa\sin(\delta)\sigma \wedge \tau.
\end{aligned}
\tag{3.52}
$$

The involution ψ in equation (3.26) in these coordinates is

$$\psi(x^1, x^2, y, v, q_0, r_0, s_0, t_0) = (x^2, x^1, y, v, r_0, q_0, t_0, s_0). \tag{3.53}$$

Using the local coordinate on Q as in section 3.1 we have the forms

$$\begin{aligned} \phi^1 &= e^\rho dy^1, \\ \phi^2 &= e^\rho dy^2, \\ \phi^1_2 &= \tfrac{\partial \rho}{\partial y^2} dy^1 - \tfrac{\partial \rho}{\partial y^1} dy^2. \end{aligned} \tag{3.54}$$

From this we calculate the reduced EDS to be

$$\begin{aligned} \bar{\mathcal{E}} = \Bigg\{ &\left(dt_0 - \frac{\partial \rho}{\partial y^2} dy^1 + \frac{\partial \rho}{\partial y^1} dy^2 \right) \wedge (\sin(s_0) dy^1 - \cos(s_0) dy^2), \\ &\left(ds_0 - \frac{\partial \rho}{\partial y^2} dy^1 + \frac{\partial \rho}{\partial y^1} dy^2 \right) \wedge (-\sin(t_0) dy^1 + \cos(t_0) dy^2) \Bigg\}, \end{aligned} \tag{3.55}$$

with independence condition $dy^1 \wedge dy^2$. This defines a pair of differential equations

$$\cos(s_0) \left(\frac{\partial \rho}{\partial y^2} - \frac{\partial t_0}{\partial y^1} \right) - \sin(s_0) \left(\frac{\partial \rho}{\partial y^1} + \frac{\partial t_0}{\partial y^2} \right) = 0, \tag{3.56}$$

$$\cos(t_0) \left(\frac{\partial \rho}{\partial y^2} - \frac{\partial s_0}{\partial y^1} \right) - \sin(t_0) \left(\frac{\partial \rho}{\partial y^1} + \frac{\partial s_0}{\partial y^2} \right) = 0. \tag{3.57}$$

3.6 Darboux Integrability

As was noted in section 3.2, $\mathcal{E}$ in equation (3.51) is decomposable of type $[3, 3]$. In the coframe of equation (3.50) the associated singular Pfaffian systems from equation (3.14) are

$$\begin{aligned} \hat{V} &= \{\beta_1, \beta_2, \tilde{\alpha}_1, \tilde{\eta}_1, \tau\}, \\ \check{V} &= \{\beta_1, \beta_2, \tilde{\zeta}_1, \tilde{\xi}_1, \sigma\}. \end{aligned} \tag{3.58}$$

This is as expected from the form of $\mathcal{E}$ in equation (3.51). The function ψ given in coordinates in equation (3.53) is an involution of the pair $\{\hat{V}, \check{V}\}$.

Since $|\hat{V}| = |\check{V}| = 5$ and $|\mathcal{C}^{\perp}| = 8$ the condition for Darboux integrability in equation (2.7) immediately gives the necessary condition

$$|\hat{V}^{\infty}|, |\check{V}^{\infty}| \geq 3. \tag{3.59}$$

The goal of this thesis is to characterize metrics by the jet level at which $\mathcal{E}$ is Darboux integrable using the properties of $\bar{\mathcal{E}}$. We begin this process by noting that $\bar{\mathcal{E}}$ in equation (3.47) is decomposable of type $[2, 2]$, with associated singular Pfaffian systems

$$\hat{W} \;=\; \{\tilde{\eta}_1, \tau\} \qquad \check{W} \;=\; \{\tilde{\xi}_1, \sigma\}. \tag{3.60}$$

Lemma 3.3. *$\mathcal{E}$ is Darboux integrable if and only if $\hat{W}^{\infty}$ is non-trivial.*

Proof. To prove necessity, we assume $\mathcal{E}$ is Darboux integrable. From equation (2.7) and the coframe in equation 3.50 this implies that

$$\hat{V}^{\infty} + \check{V} = \Omega^1(\beta_1, \beta_2, \tilde{\alpha}_1, \tilde{\zeta}_1, \tilde{\eta}_1, \tilde{\xi}_1, \sigma, \tau). \tag{3.61}$$

By equation (3.52), $\hat{V}$ in equation (3.58) has derived system

$$\begin{aligned} \hat{V}' \;&=\; \{\tilde{\alpha}_1, \tilde{\eta}_1, \beta_2 - e^{-r_0}\tau\} \\ &=\; \{dq_0, \tilde{\eta}_1, \omega^2\}. \end{aligned} \tag{3.62}$$

We now see that we have the direct sum

$$\hat{V}' \oplus \check{V} = \Omega^1(\beta_1, \beta_2, \tilde{\alpha}_1, \tilde{\zeta}_1, \tilde{\eta}_1, \tilde{\xi}_1, \sigma, \tau). \tag{3.63}$$

Since $\hat{V}^\infty \subset \hat{V}'$ in order for equation (3.61) to be satisfied it must be that

$$\hat{V}^\infty = \hat{V}'. \tag{3.64}$$

The EDS $\hat{V}$ in equation (3.58) satisfies the hypotheses of theorem B.4, therefore the quotient system $\hat{V}/\mathcal{G}$ is Pfaffian. The semi-basic forms in $\hat{V}$ are

$$\hat{V} \cap \mathbf{\Gamma}^\perp = \{\tilde{\eta}_1, \tau\}. \tag{3.65}$$

From equation (3.60) we get $\hat{W} = \hat{V}/\mathcal{G}$, and from theorem 2.9 in [7] $\hat{W}^\infty = \hat{V}^\infty/\mathcal{G}$. Finally we use equation (3.64) to get $\hat{W}^\infty = \{\tilde{\eta}_1\}$.

To prove sufficiency we assume that $\hat{W}^\infty$ is non-trivial. From the structure equations we have that the derived system of $\hat{W}$ in equation (3.60) is

$$\hat{W}' = \{\tilde{\eta}_1\}. \tag{3.66}$$

This implies that there exists $a \in \mathcal{C}^\infty(U/\mathcal{G}, \mathbb{R})$ such that $a\tilde{\eta}_1$ is closed. Pulling this back by the quotient gives $\mathfrak{q}^*(a\tilde{\eta}_1) = (\mathfrak{q} \circ a)\tilde{\eta}_1 \in \hat{V}$, which is also closed and in $\hat{V}^\infty$, thus equation (3.61) holds. The pullback of equation (3.61) by the involution ψ is

$$\hat{V} + \check{V}^\infty = \Omega^1(\beta_1, \beta_2, \tilde{\alpha}_1, \tilde{\zeta}_1, \tilde{\eta}_1, \tilde{\xi}_1, \sigma, \tau). \tag{3.67}$$

Thus $\mathcal{E}$ satisfies condition (2.7).

To show that the condition of equation (2.8) holds we apply the involution ψ in equation (3.53) to equation (3.62) to find

$$\check{V}' = \{dr_0, \tilde{\xi}_1, \omega^1\}. \tag{3.68}$$

Using $\hat{V}^\infty \subset \hat{V}'$ and $\check{V}^\infty \subset \check{V}'$ we get

$$\hat{V}^\infty \cap \check{V}^\infty = 0. \tag{3.69}$$

Therefore $\mathcal{E}$ satisfies condition (2.8) and is Darboux integrable.

□

We use the condition on $\bar{\mathcal{E}}$ in lemma 3.3 to show that the only wave-maps which are Darboux integrable (without prolongation) are those for which (Q, h) has zero Gaussian curvature.

Theorem 3.4. *The EDS $\mathcal{E}$ is Darboux integrable if and only if $\kappa = 0$.*

Proof. By the structure equations (3.52), $d\tilde{\eta}_1 \wedge \tilde{\eta}_1 = \kappa \sin(\delta)\tilde{\eta}_1 \wedge \tau \wedge \sigma$. If $\kappa \neq 0$, then $\hat{W}$ does not have an integrable subsystem, but if $\kappa = 0$ then $\hat{W}^\infty = \{\tilde{\eta}_1\}$. Apply lemma 3.3 to complete the proof.

□

3.7 Noether's Theorem

At this point we digress to ask what conservation laws are predicted by Noether's theorem and how they are related to our EDS. By equation (2.83) the conformal symmetry in equation (3.34) produces the conserved density

$$Z^{(0)} \lrcorner \Lambda = f\left[\left((p_1^1)^2 + (p_1^2)^2\right)\omega^1 + p_1^1\theta^1 + p_1^2\theta^2\right] + g\left[\left((p_2^1)^2 + (p_2^2)^2\right)\omega^2 + p_2^1\theta^1 + p_2^2\theta^2\right]. \tag{3.70}$$

That is $d(Z^{(0)} \lrcorner \Lambda) \in \mathcal{E}_\Lambda$, since $\theta^1, \theta^2 \in \mathcal{E}_\Lambda$, this implies that for

$$\varphi_Z = f\left((p_1^1)^2 + (p_1^2)^2\right)\omega^1 + g\left((p_2^1)^2 + (p_2^2)^2\right)\omega^2, \tag{3.71}$$

$$d\varphi_Z = 2f(p_1^1 dp_1^1 + p_1^2 dp_1^2) \wedge \omega^1 + 2g(p_2^1 dp_2^1 + p_2^2 dp_2^2) \wedge \omega^2 \in \mathcal{E}_\Lambda. \tag{3.72}$$

In particular, this is true for $f = 1, g = 0$ and $f = 0, g = 1$, giving

$$\begin{aligned} d((p_1^1)^2 + (p_1^2)^2) \wedge \omega^1 &\in \mathcal{E}, \\ d((p_2^1)^2 + (p_2^2)^2) \wedge \omega^2 &\in \mathcal{E}. \end{aligned} \tag{3.73}$$

From equation (3.14)

$$\begin{aligned} &\omega^1 \in \check{V}, \\ &\omega^2 \in \hat{V}, \end{aligned} \tag{3.74}$$

thus we find that $(p_2^1)^2 + (p_2^2)^2$ and $(p_1^1)^2 + (p_1^2)^2$ are first integrals of $\hat{V}$ and $\check{V}$, respectively.

3.8 Bi-Symplectic Structures

We now examine some special properties of the $s = 0$ hyperbolic system $\bar{\mathcal{E}}$.

Theorem 3.5. *The quotient EDS $\bar{\mathcal{E}}$ is generated by a special bi-symplectic structure.*

Proof. A special bi-symplectic structure for $\bar{\mathcal{E}}$ is two 2-forms Φ, Ψ satisfying $\Phi \wedge \Psi = 0$, $d\Phi = d\Psi = 0$, and where $\bar{\mathcal{E}} = \{\Phi, \Psi\}$. We will use

$$\Phi = \frac{1 + \cos(\delta)}{\sin(\delta)}(\tilde{\eta}_1 \wedge \tau + \tilde{\xi}_1 \wedge \sigma) = d(\sigma - \tau) \tag{3.75}$$

$$\Psi = \frac{1 - \cos(\delta)}{\sin(\delta)}(\tilde{\eta}_1 \wedge \tau - \tilde{\xi}_1 \wedge \sigma) = d(\sigma + \tau). \tag{3.76}$$

From [11], section 1.6.2, the Poincaré-Cartan forms are

$$\begin{aligned} d\Omega_1 &= \Phi \wedge (\sigma + \tau) = \tfrac{1+\cos(\delta)}{\sin(\delta)}(\tilde{\xi}_1 \wedge \sigma \wedge \tau - \tilde{\eta}_1 \wedge \sigma \wedge \tau) \\ d\Omega_2 &= \Psi \wedge (\sigma - \tau) = -\tfrac{1-\cos(\delta)}{\sin(\delta)}(\tilde{\xi}_1 \wedge \sigma \wedge \tau - \tilde{\eta}_1 \wedge \sigma \wedge \tau), \end{aligned} \tag{3.77}$$

and the Lagrangians are

$$
\begin{aligned}
\Omega_1 &= (1+\cos(\delta))\sigma\wedge\tau \\
\Omega_2 &= (1-\cos(\delta))\sigma\wedge\tau.
\end{aligned} \tag{3.78}
$$

□

There is a $\mathcal{G}$-basic representative of the energy functional, $\bar{\Lambda} = \cos(\delta)\sigma\wedge\tau$, and $d\bar{\Lambda} = \frac{1}{\sin(\delta)}(\tilde{\xi}_1\wedge\sigma\wedge\tau - \tilde{\eta}_1\wedge\sigma\wedge\tau)$. We were unable to find a symplectic structure on $\mathcal{F}$ for which the Euler-Lagrange system of $\bar{\Lambda}$ was our quotient system $\bar{\mathcal{E}}$.

CHAPTER 4

THE PROLONGED EDS AND METRIC CLASSIFICATION

In this chapter we classify the metrics on Q for which the prolonged system $\mathcal{E}^{(1)}$ is Darboux integrable. We do this by first calculating the prolongation of both $\mathcal{E}$ and the infinitesimal generators of $\mathcal{G}$. As in chapter 3 we compute the quotient system $\overline{\mathcal{E}^{(1)}}$ by the prolonged conformal transformations. Again we give necessary and sufficient conditions for $\mathcal{E}^{(1)}$ to be Darboux integrable in terms of properties of $\overline{\mathcal{E}^{(1)}}$. We write these conditions in terms of the curvature of (Q, h). These conditions are then integrated to find the corresponding metrics for which the wave maps are Darboux integrable. Besides the flat case there are essentially two inequivalent metric.

4.1 1^{st} Prolongation and Quotient

In order to prolong $\mathcal{E}$ in equation (3.51) we use the coframe in equation (3.50)

$$\{\beta_1, \beta_2, \tilde{\alpha}_1, \tilde{\zeta}_1, \tilde{\eta}_1, \tilde{\xi}_1, \sigma, \tau, \phi^1_2\}, \tag{4.1}$$

and let

$$\{\partial_{x^1}, \partial_{x^2}, \partial_{r_0}, \partial_{q_0}, \partial_{t_0}, \partial_{s_0}, X, Y, \partial_{\phi^1_2}\}. \tag{4.2}$$

be the dual frame.

Theorem 4.1. *The prolonged manifold is*

$$U^{(1)} = U \times \mathbb{R}^4. \tag{4.3}$$

Proof. The integral 2-planes of $\mathcal{E}$ with independence condition $\sigma \wedge \tau \neq 0$ and no characteristic directions, $\phi^1_2 = 0$, are

$$E_{(q_1,r_1,s_1,t_1)} = \text{span}\{X + q_1\partial_{q_0} + s_1\partial_{s_0}, Y + r_1\partial_{r_0} + t_1\partial_{t_0}\}, \tag{4.4}$$

where $q_1, r_1, s_1, t_1 \in \mathbb{R}$, and thus the prolonged manifold is

$$U^{(1)} = G_2(\mathcal{E}) = \{(x, y, v, q_0, r_0, s_0, t_0, E_{(q_1,r_1,s_1,t_1)})\} = U \times \mathbb{R}^4. \tag{4.5}$$

□

To find the prolonged EDS we take the perpendicular space of the integral plane in equation (4.4), and pullback by the projection $\pi : U^{(1)} \to U$ to get

$$\mathcal{E}^{(1)} = \langle \pi^* E^{\perp}_{(q_1,r_1,s_1,t_1)} \rangle = \langle \beta_1, \beta_2, \alpha_1, \zeta_1, \eta_1, \xi_1 \rangle, \tag{4.6}$$

where $\beta_1, \beta_2 \in E^{\perp}_{(q_1,r_1,s_1,t_1)}$ by equations (4.1), (4.2), and (4.4). Again using equations (4.1), (4.2), and (4.4) we find that

$$\begin{aligned} \alpha_1 &= dr_0 - r_1\tau, \\ \zeta_1 &= dq_0 - q_1\sigma, \\ \eta_1 &= dt_0 - \phi^1_2 - t_1\tau, \\ \xi_1 &= ds_0 - \phi^1_2 - s_1\sigma, \end{aligned} \tag{4.7}$$

are also in $E^{\perp}_{(q_1,r_1,s_1,t_1)}$.

The structure equations for $\mathcal{E}^{(1)}$ are

$$
\begin{aligned}
d\beta_1 &= \sigma \wedge \left(\tfrac{1}{e^{q_0}}\zeta_1 + \tfrac{\cos(\delta)}{e^{q_0}\sin(\delta)}\xi_1\right) && + \tau \wedge \tfrac{1}{e^{q_0}\sin(\delta)}\eta_1, \\
d\beta_2 &= \sigma \wedge \tfrac{-1}{e^{r_0}\sin(\delta)}\xi_1 && + \tau \wedge \left(\tfrac{1}{e^{r_0}}\alpha_1 - \tfrac{\cos(\delta)}{e^{r_0}\sin(\delta)}\eta_1\right), \\
d\alpha_1 &= \sigma \wedge \tfrac{-r_1}{\sin(\delta)}\xi_1 && + \tau \wedge \left(\tilde{\alpha}_2 - \tfrac{r_1\cos(\delta)}{\sin(\delta)}\eta_1\right), \\
d\zeta_1 &= \sigma \wedge \left(\tilde{\zeta}_2 + \tfrac{q_1\cos(\delta)}{\sin(\delta)}\xi_1\right) && + \tau \wedge \tfrac{q_1}{\sin(\delta)}\eta_1, \\
d\eta_1 &= \sigma \wedge \tfrac{-t_1}{\sin(\delta)}\xi_1 && + \tau \wedge \tilde{\eta}_2, \\
d\xi_1 &= \sigma \wedge \tilde{\xi}_2 && + \tau \wedge \tfrac{s_1}{\sin(\delta)}\eta_1,
\end{aligned} \tag{4.8}
$$

where

$$
\begin{aligned}
\tilde{\alpha}_2 &= dr_1, \\
\tilde{\zeta}_2 &= dq_1, \\
\tilde{\eta}_2 &= dt_1 + \kappa\sin(\delta)\sigma - \tfrac{t_1\cos(\delta)}{\sin(\delta)}\eta_1, \\
\tilde{\xi}_2 &= ds_1 - \kappa\sin(\delta)\tau + \tfrac{s_1\cos(\delta)}{\sin(\delta)}\xi_1.
\end{aligned} \tag{4.9}
$$

The involution ψ in equation (3.53) can be extended to $U^{(1)}$ by pushing forward the integral planes in equation (4.4). This gives

$$
\begin{aligned}
\psi_* E_{(q_1,r_1,s_1,t_1)} &= \operatorname{span}\{\psi_*(X + q_1\partial_{q_0} + s_1\partial_{s_0}), \psi_*(Y + r_1\partial_{r_0} + t_1\partial_{t_0}\} \\
&= \operatorname{span}\{Y + q_1\partial_{r_0} + s_1\partial_{t_0}, X + r_1\partial_{q_0} + t_1\partial_{s_0}\} \\
&= E_{(r_1,q_1,t_1,s_1)}.
\end{aligned} \tag{4.10}
$$

Therefore ψ acts on $U^{(1)}$ by

$$
\psi(x^1, x^2, y, v, q_0, r_0, s_0, t_0, q_1, r_1, s_1, t_1) = (x^2, x^1, y, v, r_0, q_0, t_0, s_0, r_1, q_1, t_1, s_1). \tag{4.11}
$$

We prolong the action of $\mathcal{G}$ to $U^{(1)}$ by prolonging the infinitesimal generator in equation (3.44). To do this we let

$$\tilde{Z} = f(x^1)\partial_{\omega^1} + g(x^2)\partial_{\omega^2} - f(x^1)'\partial_{q_0} - g(x^2)'\partial_{r_0} + a_1\partial_{q_1} + a_2\partial_{r_1} + a_3\partial_{s_1} + a_4\partial_{t_1}, \tag{4.12}$$

and compute the Lie Derivative of $\mathcal{E}^{(1)}$ in the direction of $\tilde{Z}$.

$$\begin{aligned} \tilde{Z}(\alpha_1) &= -g''\beta_2 + (g''e^{-r_0} - a_2)\tau \\ \tilde{Z}(\zeta_1) &= -f''\beta_1 + (f''e^{-q_0} - a_1)\sigma \\ \tilde{Z}(\eta_1) &= -a_4\tau \\ \tilde{Z}(\xi_1) &= -a_3\sigma \end{aligned} \tag{4.13}$$

Forcing the results to be in $\mathcal{E}^{(1)}$ we get $a_1 = f''e^{-r_0}$, $a_2 = g''e^{-q_0}$, and $a_3 = a_4 = 0$. Therefore the prolonged infinitesimal generator is

$$Z^{(1)} = f\partial_{x^1} + g\partial_{x^2} - f'\partial_{q_0} - g'\partial_{r_0} + f''e^{-q_0}\partial_{q_1} + g''e^{-r_0}\partial_{r_1}. \tag{4.14}$$

The corresponding integrable distribution is

$$\mathbf{\Gamma}^{(1)} = \{\partial_{x^1}, \partial_{x^2}, \partial_{q_0}, \partial_{r_0}, \partial_{q_1}, \partial_{r_1}\}. \tag{4.15}$$

Theorem 4.2. *The quotient of $U^{(1)}$ is*

$$\overline{U^{(1)}} = \bar{U} \times \mathbb{R}^2. \tag{4.16}$$

Moreover, the quotient of $\mathcal{E}^{(1)}$ is an $s = 2$ hyperbolic system.

Proof. It is clear from the form of the distribution $\mathbf{\Gamma}$ in equation (4.15) that the quotient map $\mathfrak{q} : U \times \mathbb{R}^4 \to \bar{U} \times \mathbb{R}^2$ is

$$\mathfrak{q}(x, y, v, q_0, q_1, r_0, r_1, s_0, s_1, t_0, t_1) = (y, v, s_0, s_1, t_0, t_1). \tag{4.17}$$

To compute the quotient of $\mathcal{E}^{(1)}$ in equation (4.6) we note that the coframe

$$\{\beta_1, \beta_2, \alpha_1, \tilde{\alpha}_2, \zeta_1, \tilde{\zeta}_2, \eta_1, \tilde{\eta}_2, \xi_1, \tilde{\xi}_2, \sigma, \tau, \phi^1_2\} \tag{4.18}$$

and the structure equations (4.8) satisfy the hypotheses of proposition B.1. Therefore the quotient is Pfaffian, and since the semi-basic 1-forms in $\mathcal{E}^{(1)}$ are

$$\mathcal{E}^{(1)} \cap \mathbf{\Gamma}^{\perp} = \{\eta_1, \xi_1\}, \tag{4.19}$$

we can pull these back by the cross-section

$$\mathfrak{s}(y, v, s_0, s_1, t_0, t_1) = (0, 0, y, v, 0, 0, 0, 0, s_0, s_1, t_0, t_1) \tag{4.20}$$

to get the quotient system on $\bar{U} \times \mathbb{R}^2$

$$\overline{\mathcal{E}^{(1)}} = \langle \eta_1, \xi_1 \rangle. \tag{4.21}$$

From the structure equations (4.8) we see that this is an $s = 2$ hyperbolic system.

□

This quotient commutes with prolongation, as is shown in the next proposition.

Proposition 4.3. *Quotienting the prolonged manifold, $U^{(1)}$ by $\mathcal{G}$ gives the same result as prolonging the quotient manifold, $\bar{U}$, that is*

$$\overline{U^{(1)}} = \bar{U}^{(1)}. \tag{4.22}$$

The same result holds for the EDS:

$$\overline{\mathcal{E}^{(1)}} = \bar{\mathcal{E}}^{(1)}. \tag{4.23}$$

Proof. Let

$$\{\tilde{\eta}_1, \tilde{\xi}_1, \sigma, \tau, \phi_2^1\}. \tag{4.24}$$

be a coframe of $\bar{U}$ with dual frame

$$\{\partial_{t_0}, \partial_{s_0}, X, Y, \partial_{\phi_2^1}\} \tag{4.25}$$

Then the integral 2-planes of $\bar{\mathcal{E}}$ with independence condition $\sigma \wedge \tau \neq 0$ and no characteristic directions, $\phi_2^1 = 0$, are

$$E_{(s_1,t_1)} = \operatorname{span}\{X + s_1\partial_{s_0}, Y + t_1\partial t_0\} \tag{4.26}$$

and thus the prolonged manifold is

$$\bar{U}^{(1)} = G_2(\bar{\mathcal{E}}) = \{(y, v, s_0, t_0, E_{(s_1,t_1)})\} = \bar{U} \times \mathbb{R}^2. \tag{4.27}$$

Pulling $E^{\perp}_{(s_1,t_1)}$ back by the projection to $\bar{U}$ gives the Pfaffian system on $\bar{U}^{(1)}$,

$$\bar{\mathcal{E}}^{(1)} = \langle \eta_1, \xi_1 \rangle. \tag{4.28}$$

This is the same system as in equation (4.23).

□

4.2 Darboux Integrability

From the structure equations (4.8) we see that $\mathcal{E}^{(1)}$ is decomposable of type $[3,3]$, with associated singular Pfaffian systems

$$\begin{aligned} \hat{V} &= \{\beta_1, \beta_2, \alpha_1, \tilde{\alpha}_2, \zeta_1, \eta_1, \tilde{\eta}_2, \xi_1, \tau\}, \\ \check{V} &= \{\beta_1, \beta_2, \alpha_1, \zeta_1, \tilde{\zeta}_2, \eta_1, \xi_1, \tilde{\xi}_2, \sigma\}. \end{aligned} \tag{4.29}$$

From equations (4.7) and (4.9) we again have that ψ in equation (4.11) is an involution of the pair $\hat{V}, \check{V}$.

The structure equations (4.8) also show that the quotient system $\bar{\mathcal{E}}^{(1)}$ in equation (4.28) is decomposable of type $[2,2]$, with associated singular Pfaffian systems

$$\hat{W} = \{\eta_1, \tilde{\eta}_2, \xi_1, \tau\} \qquad \check{W} = \{\eta_1, \xi_1, \tilde{\xi}_2, \sigma\}. \tag{4.30}$$

Again we have similar results as in section 3.6.

Lemma 4.4. *The EDS $\mathcal{E}^{(1)}$ is Darboux integrable if and only if $\hat{W}^{\infty}$ is non-trivial.*

Proof. To prove necessity, we assume $\mathcal{E}^{(1)}$ is Darboux integrable. From equation (2.7) we then have

$$\hat{V}^{\infty} + \check{V} = \Omega^1(\beta_1, \beta_2, \alpha_1, \tilde{\alpha}_2, \zeta_1, \tilde{\zeta}_2, \eta_1, \tilde{\eta}_2, \xi_1, \tilde{\xi}_2, \sigma, \tau). \tag{4.31}$$

Using

$$d\kappa = \kappa_1\phi^1 + \kappa_2\phi^2 = \kappa_\sigma\sigma + \kappa_\tau\tau, \tag{4.32}$$

with

$$\begin{aligned} \kappa_\sigma &= -(\kappa_1\cos(s_0) + \kappa_2\sin(s_0)), \\ \kappa_\tau &= -(\kappa_1\cos(t_0) + \kappa_2\sin(t_0)) \end{aligned} \tag{4.33}$$

we compute,

$$\begin{aligned} d\tilde{\alpha}_2 &= 0, \\ d\tilde{\zeta}_2 &= 0, \\ d\tilde{\eta}_2 &= \sigma\wedge(-\kappa\cos(\delta)\eta_1) + \tau\wedge(-\kappa\eta_1) - (\sin(\delta)\kappa_\tau + \kappa t_1\cos(\delta))\sigma\wedge\tau, \\ d\tilde{\xi}_2 &= \sigma\wedge(-\kappa\xi_1) + \tau\wedge(-\kappa\cos(\delta)\xi_1) - (\sin(\delta)\kappa_\sigma - \kappa s_1\cos(\delta))\sigma\wedge\tau. \end{aligned} \tag{4.34}$$

This gives the derived systems

$$\begin{aligned} \hat{V}' &= \{\beta_1, \beta_2, \alpha_1, \tilde{\alpha}_2, \eta_1, \tilde{\eta}_2, \tau\}, \\ \hat{V}'' &= \{\alpha_1 + r_1\tau, \tilde{\alpha}_2, \eta_1 + t_1\tau, \tilde{\eta}_2 + \tfrac{t_1\cos(\delta)}{\sin(\delta)}\eta_1, e^{-r_0}\tau - \beta_2\} \\ &= \{dr_0, dr_1, dt_0 - \phi^1_2, dt_1 + \kappa\sin(\delta)\sigma, \omega^2\}. \end{aligned} \tag{4.35}$$

Since dr_0, dr_1, and ω^2 are closed, equation (4.31) implies that there exists $a_1, a_2 \in C^\infty(U^{(1)}, \mathbb{R})$ such that $a_2 \neq 0$ and

$$\{dr_0, dr_1, a_1(dt_0 - \phi^1_2) + a_2(dt_1 + \kappa\sin(\delta)\sigma), \omega^2\} \subset \hat{V}^\infty, \tag{4.36}$$

The EDS $\hat{V}$ in equation (4.29) satisfies the hypotheses of theorem B.4, therefore the quotient system $\hat{V}/\mathcal{G}$ is Pfaffian. The semi-basic forms in $\hat{V}$ are

$$\hat{V} \cap \mathbf{\Gamma}^{\perp} = \{\eta_1, \tilde{\eta}_2, \xi_1, \tau\}. \tag{4.37}$$

From equation (4.30) we get $\hat{W} = \hat{V}/\mathcal{G}$, and again from theorem 2.9 in [7] $\hat{W}^{\infty} = \hat{V}^{\infty}/\mathcal{G}$. Finally we use equation (4.36) to get

$$\{\mathfrak{s} \circ a_1(dt_0 - \phi^1_2) + \mathfrak{s} \circ a_2(dt_1 + \kappa \sin(\delta)\sigma)\} \subset \hat{W}^{\infty}, \tag{4.38}$$

where $\mathfrak{s}$ is the cross-section in equation (4.20).

To prove sufficiency we assume that $\hat{W}^{\infty}$ is non-trivial, but since

$$\begin{aligned} \hat{W}' &= \{\eta_1, \tilde{\eta}_2, \tau\} \\ \hat{W}'' &= \{\eta_1 + t_1\tau, \tilde{\eta}_2 + \tfrac{t_1 \cos(\delta)}{\sin(\delta)}\eta_1\} \\ &= \{dt_0 - \phi^1_2, dt_1 + \kappa \sin(\delta)\sigma\}, \end{aligned} \tag{4.39}$$

this implies that there exists $a_1, a_2 \in \mathcal{C}^{\infty}(\bar{U}^{(1)}, \mathbb{R})$ such that $a_1(dt_0 - \phi^1_2) + a_2(dt_1 + \kappa \sin(\delta)\sigma)$ is closed. This in turn implies $(\mathfrak{q} \circ a_1)(dt_0 - \phi^1_2) + (\mathfrak{q} \circ a_2)(dt_1 + \kappa \sin(\delta)\sigma)$ is also closed and therefore in $\hat{V}^{\infty}$. Therefore equation (4.31) holds. The pullback of equation (4.31) by the involution, ψ is

$$\hat{V} + \check{V}^{\infty} = \Omega^1(\beta_1, \beta_2, \alpha_1, \tilde{\alpha}_2, \zeta_1, \tilde{\zeta}_2, \eta_1, \tilde{\eta}_2, \xi_1, \tilde{\xi}_2, \sigma, \tau). \tag{4.40}$$

Thus $\mathcal{E}^{(1)}$ satisfies condition (2.7).

To show condition (2.8) is satisfied we pullback equation (4.35) by ψ to get

$$\check{V}'' = \{dq_0, dq_1, ds_0 - \phi^1_2, ds_1 - \kappa \sin(\delta)\tau, \omega^1\}. \tag{4.41}$$

Using $\hat{V}^{\infty} \subset \hat{V}''$ and $\check{V}^{\infty} \subset \check{V}''$ we get

$$\hat{V}^{\infty} \cap \check{V}^{\infty} = 0. \tag{4.42}$$

Therefore $\mathcal{E}$ satisfies condition (2.8) and is Darboux integrable.

$\square$

Given that

$$\begin{aligned} d\kappa_1 &= \kappa_{11}\phi^1 + \kappa_{12}\phi^2 - \kappa_2\phi^1_2 \\ d\kappa_2 &= \kappa_{12}\phi^1 + \kappa_{22}\phi^2 + \kappa_1\phi^1_2, \end{aligned} \tag{4.43}$$

we have the main theorem.

Theorem 4.5. *Assuming $\kappa \neq 0$, $\mathcal{E}^{(1)}$ is Darboux integrable if and only if*

$$\kappa_{11} = \frac{{\kappa_1}^2 - \kappa^3}{\kappa}, \qquad \kappa_{12} = \frac{\kappa_1\kappa_2}{\kappa}, \qquad \kappa_{22} = \frac{{\kappa_2}^2 - \kappa^3}{\kappa}. \tag{4.44}$$

Proof. Using lemma 4.4 we can show that $\hat{W}^{\infty}, \check{W}^{\infty}$ are non trivial if and only if (4.44) is satisfied.

When $\kappa \neq 0$,

$$\hat{W}''' = \left\{\varrho = -\kappa_\tau(dt_0 - \phi^1_2) + \kappa(dt_1 + \kappa\sin(\delta)\sigma)\right\} \tag{4.45}$$

To prove sufficiency, assume (4.44). This implies $d\varrho \wedge \varrho = 0$, therefore $\hat{W}^{\infty}$ is non-trivial.

For necessity, assume $\hat{W}^{\infty}$ non-trivial. Since $\hat{W}^{\infty} \subset \hat{W}'''$, it must be that $d\varrho \wedge \varrho = 0$.

Using

(4.46)

$$\begin{aligned} d\varrho \wedge \varrho|_{s_0=0,t_0=0,s_1=0,t_1=0} &= (\kappa^4 - \kappa{\kappa_1}^2 + \kappa^2\kappa_{11})\phi^1 \wedge \phi^2 \wedge \phi^1_2 \\ d\varrho \wedge \varrho|_{s_0=0,t_0=\frac{\pi}{4},s_1=0,t_1=0} &= (\kappa^2\kappa_{12} - \kappa\kappa_1\kappa_2 + \tfrac{1}{2}(\kappa^2\kappa_{22} - \kappa{\kappa_2}^2 - \kappa{\kappa_1}^2 + \kappa^2\kappa_{11} + 2\kappa^4))\phi^1 \wedge \phi^2 \wedge \phi^1_2 \\ d\varrho \wedge \varrho|_{s_0=0,t_0=\frac{\pi}{2},s_1=0,t_1=0} &= (\kappa^4 + \kappa^2\kappa_{22} - \kappa{\kappa_2}^2)\phi^1 \wedge \phi^2 \wedge \phi^1_2, \end{aligned}$$

and forcing these to zero gives (4.44). □

From equation (4.36) there are always three intermediate integrals, but since $dr_0 - e^{r_0}r_1\omega^2 \in \check{V}$ another intermediate integral is required for Darboux integrability by condition (2.7). Theorem 4.5 gives conditions for the existence of an extra intermediate integral, and shows that, under these conditions, one can be found that is conformally invariant.

4.3 The Metrics

We now wish to compute the metrics that satisfy equations (4.44). To do this we search for closed forms spanning $\{\phi^1, \phi^2\}$. Using these forms we can write the metric $h = (\phi^1)^2 + (\phi^2)^2$ in coordinates. Natural forms to choose would be

(4.47)
$$\begin{aligned} d\kappa &= \kappa_1\phi^1 + \kappa_2\phi^2, \\ d({\kappa_1}^2 + {\kappa_2}^2) &= ((\kappa_1\kappa_{11} + \kappa_2\kappa_{12})\phi^1 + (\kappa_1\kappa_{12} + \kappa_2\kappa_{22})\phi^2). \end{aligned}$$

Substituting equation (4.44) these become

(4.48)
$$\begin{aligned} d\kappa &= \kappa_1\phi^1 + \kappa_2\phi^2, \\ d({\kappa_1}^2 + {\kappa_2}^2) &= 2\left(\frac{{\kappa_1}^2 + {\kappa_2}^2 - \kappa^3}{\kappa}\right) d\kappa. \end{aligned}$$

This implies that ${\kappa_1}^2 + {\kappa_2}^2 = H(\kappa)$ where H satisfies the differential equation

$$2\frac{H(\kappa) - \kappa^3}{\kappa} = H'(\kappa). \tag{4.49}$$

The solutions to equation (4.49) are

$$H(\kappa) = -2\kappa^2(\kappa - C) \tag{4.50}$$

for some constant $C \in \mathbb{R}$.

Since the forms in equation (4.47) did not span $\{\phi^1, \phi^2\}$ we must choose a different frame. The next most obvious coframe to work in is (see [10])

$$\begin{array}{rcl} \mu_1 & = & d\kappa & = & \kappa_1\phi^1 + \kappa^2\phi^2, \\ \mu_2 & = & \text{skew } d\kappa & = & -\kappa_2\phi^1 + \kappa_1\phi^2. \end{array} \tag{4.51}$$

In this coframe we have structure equations

$$\begin{array}{rcl} d\mu_1 & = & 0, \\ d\mu_2 & = & \frac{\kappa_{11}+\kappa_{22}}{{\kappa_1}^2+{\kappa_2}^2}\mu_1 \wedge \mu_2. \end{array} \tag{4.52}$$

Substituting (4.44) and (4.50) gives,

$$d\mu_2 = \frac{2\kappa - C}{\kappa(\kappa - C)} d\kappa \wedge \mu_2. \tag{4.53}$$

From this we find the integrating factor

$$e^{-\int \frac{2\kappa - C}{\kappa(\kappa - C)} d\kappa} = \frac{1}{\kappa(\kappa - C)}, \tag{4.54}$$

giving us the closed form $\frac{1}{\kappa(\kappa-C)}\mu_2$. Since $\frac{1}{\kappa(\kappa-C)}\mu_2$ is closed about each point there is an open set such that

$$\mu_2 = \frac{\kappa(\kappa - C)}{C} dz, \tag{4.55}$$

where we have divided by C for later convenience. Using equation (4.51) the metric in this coframe is,

$$h = (\phi^1)^2 + (\phi^2)^2 = \frac{{\mu_1}^2 + {\mu_2}^2}{{\kappa_1}^2 + {\kappa_2}^2} \tag{4.56}$$

Substituting equation (4.50), (4.51), and (4.55) we find h in coordinates (κ, z)

$$h = \frac{1}{-2\kappa^2(\kappa - C)} \left(d\kappa^2 + \left(\frac{\kappa(\kappa - C)}{C} dz \right)^2 \right). \tag{4.57}$$

Simplifying we get

$$h = -\frac{\kappa - C}{2C^2} \left(\left(\frac{C}{\kappa(\kappa - C)} d\kappa \right)^2 + dz^2 \right). \tag{4.58}$$

The analysis now splits into two cases.

Case 1: $\kappa(\kappa - C) > 0$

We let

$$w = \log\left(\frac{\kappa}{\kappa - C} \right), \tag{4.59}$$

giving

$$dw = -\frac{C}{\kappa(\kappa - C)} d\kappa. \tag{4.60}$$

Thus the metrics are

$$h = \frac{1}{2C}\frac{1}{1-e^w}(dw^2+dz^2). \tag{4.61}$$

Case 2: $\kappa(\kappa - C) < 0$

We let

$$w = \log\left(-\frac{\kappa}{\kappa - C}\right), \tag{4.62}$$

giving

$$dw = -\frac{C}{\kappa(\kappa - C)}d\kappa. \tag{4.63}$$

Thus the metric are

$$h = \frac{1}{2C}\frac{1}{1+e^w}(dw^2+dz^2). \tag{4.64}$$

Since the EDS $\mathcal{E}$ is invariant under constant scaling of the Lagrangian we get that there are two metrics with inequivalent Darboux integrable wave maps on the 2-jets

$$\frac{1}{1\pm e^w}(dw^2+dz^2). \tag{4.65}$$

For these metrics $\rho(w,z) = -\frac{1}{2}\log(1\pm e^w)$, and equations (3.1) and (3.2) become

$$w_{12} = \frac{w_1 w_2 - z_1 z_2}{2(1\pm e^{-w})}, \tag{4.66}$$

$$z_{12} = \frac{w_1 z_2 + z_1 w_2}{2(1\pm e^{-w})}. \tag{4.67}$$

The general solution to these equations is given in [2]. The reduced equations (3.56) and (3.57) become

$$-\cos(s_0)\frac{\partial t_0}{\partial w} + \sin(s_0)\left(\frac{1}{2(1 \pm e^{-w})} - \frac{\partial t_0}{\partial z}\right) = 0, \tag{4.68}$$

$$-\cos(t_0)\frac{\partial s_0}{\partial w} + \sin(t_0)\left(\frac{1}{2(1 \pm e^{-w})} - \frac{\partial s_0}{\partial z}\right) = 0. \tag{4.69}$$

CHAPTER 5

EQUIVALENCE OF DARBOUX INTEGRABILITY ON THE QUOTIENT

In this chapter we present the main result of the thesis. We show that $\mathcal{E}^{(k)}$ is Darboux integrable if and only if $\bar{\mathcal{E}}^{(k+1)}$ is Darboux integrable. This is accomplished by using a coframe on the quotient which is similar to the Laplace adapted coframe in [12]. We use the structure equations of this coframe to show that vanishing of the generalized Laplace invariant H_{k+1} is necessary and sufficient for Darboux integrability of the quotient system $\bar{\mathcal{E}}^{(k+1)}$. This is equivalent to the vanishing of the invariant F_{k+2}, which necessary and sufficient for Darboux integrability of $\mathcal{E}^{(k)}$.

5.1 k^{th} Prolongation

In this section we give a formula for the k^{th} prolongation of $\mathcal{E}$ using the coframe in equation (4.18).

Theorem 5.1. *For $k > 1$, there exists a coframe*

(5.1)

$$\{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \ldots, \zeta_k, \tilde{\zeta}_{k+1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_k, \tilde{\xi}_{k+1}, \sigma, \tau, \phi^1_2\}$$

with dual frame

(5.2)

$$\{B_1, B_2, I_1, \ldots, I_k, \tilde{I}_{k+1}, J_1, \ldots, J_k, \tilde{J}_{k+1}, V_1, \ldots, V_k, \tilde{V}_{k+1}, W_1, \ldots, W_k, \tilde{W}_{k+1}, X, Y, \partial_{\phi^1_2}\}$$

on some open subset of $U \times \mathbb{R}^{4k}$ such that

$$\mathcal{E}^{(k)} = \langle \beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \zeta_1, \ldots, \zeta_k, \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k \rangle \tag{5.3}$$

is a Pfaffian system with structure equations,

$$
\begin{aligned}
d\alpha_i &= \sigma \wedge \tfrac{-r_i}{\sin(\delta)}\xi_1 && + \tau \wedge \left(\alpha_{i+1} - \tfrac{r_i \cos(\delta)}{\sin(\delta)}\eta_1\right),\\
d\zeta_i &= \sigma \wedge \left(\zeta_{i+1} + \tfrac{q_i \cos(\delta)}{\sin(\delta)}\xi_1\right) && + \tau \wedge \tfrac{q_i}{\sin(\delta)}\eta_1,\\
d\alpha_k &= \sigma \wedge \tfrac{-r_k}{\sin(\delta)}\xi_1 && + \tau \wedge \left(\tilde{\alpha}_{k+1} - \tfrac{r_k \cos(\delta)}{\sin(\delta)}\eta_1\right),\\
d\zeta_k &= \sigma \wedge \left(\tilde{\zeta}_{k+1} + \tfrac{q_k \cos(\delta)}{\sin(\delta)}\xi_1\right) && + \tau \wedge \tfrac{q_k}{\sin(\delta)}\eta_1,
\end{aligned}
\tag{5.4}
$$

where $1 \leq i < k$, *and*

(5.5)

$$
\begin{aligned}
d\eta_1 &= \sigma \wedge H_0\xi_1 && + \tau \wedge (\eta_2 - A_1\eta_1) && \\
d\xi_1 &= \sigma \wedge (\xi_2 - E_1\xi_1) && + \tau \wedge K_0\eta_1 && \\
d\eta_i &= \sigma \wedge H_{i-1}\eta_{i-1} && + \tau \wedge (\eta_{i+1} - A_i\eta_i) && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_{i+1}, \xi_{i+1}),\\
d\xi_i &= \sigma \wedge (\xi_{i+1} - E_i\xi_i) && + \tau \wedge K_{i-1}\xi_{i-1} && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_{i+1}, \xi_{i+1}),\\
d\eta_k &= \sigma \wedge H_{k-1}\eta_{k-1} && + \tau \wedge \tilde{\eta}_{k+1} && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_k, \xi_k, \tilde{\eta}_{k+1}, \tilde{\xi}_{k+1}),\\
d\xi_k &= \sigma \wedge \tilde{\xi}_{k+1} && + \tau \wedge K_{k-1}\xi_{k-1} && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_k, \xi_k, \tilde{\eta}_{k+1}, \tilde{\xi}_{k+1}),
\end{aligned}
$$

where $1 < i < k$, *and*

$$
\begin{array}{lll}
H_0 = \frac{-t_1}{\sin(\delta)} & K_0 = \frac{s_1}{\sin(\delta)} & \\
A_0 = 0 & E_0 = 0 & \\
A_i = A_{i-1} - Y(\ln H_{i-1}) & E_i = E_{i-1} - X(\ln K_{i-1}) & 1 \leq i \leq p\\
H_1 = H_0K_0 + X(A_1) & K_1 = H_0K_0 + Y(E_1) & \\
H_i = H_{i-1} + X(A_i) & K_i = K_{i-1} + Y(E_i) & 1 < i < p\\
H_i = 0 & K_i = 0 & p \leq i < k\\
A_i = 0 & E_i = 0 & p < i < k,
\end{array}
\tag{5.6}
$$

and p *is the lowest integer such that* $H_p = 0$.

Furthermore, for $1 \leq i < k$,

$$
(5.7) \qquad \begin{aligned}
d\eta_i &= 0 \mod \Omega^1(\xi_1, \eta_1, \ldots, \eta_{i+1}) \\
d\xi_i &= 0 \mod \Omega^1(\eta_1, \xi_1, \ldots, \xi_{i+1}) \\
d\eta_k &= 0 \mod \Omega^1(\xi_1, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}) \\
d\xi_k &= 0 \mod \Omega^1(\eta_1, \xi_1, \ldots, \xi_k, \tilde{\xi}_{k+1})
\end{aligned}
$$

Proof. The proof is by induction starting with $k = 2$.

5.1.1 Coframe for $k = 2$

This proof is accomplished in three steps.

Step 1: Prolongation

We begin with the coframe in equation (4.18)

$$
(5.8) \qquad \{\beta_1, \beta_2, \alpha_1, \tilde{\alpha}_2, \zeta_1, \tilde{\zeta}_2, \eta_1, \tilde{\eta}_2, \xi_1, \tilde{\xi}_2, \sigma, \tau, \phi^1_2\},
$$

and let

$$
(5.9) \qquad \{B_1, B_2, I_1, \tilde{I}_2, J_1, \tilde{J}_2, V_1, \tilde{V}_2, W_1, \tilde{W}_2, X, Y, \partial_{\phi^1_2}\}
$$

be the dual frame on $U^{(1)}$. Using the structure equations (4.8), the integral 2-planes of $\mathcal{E}^{(1)}$ with independence condition $\sigma \wedge \tau \neq 0$ and no characteristic directions, $\phi^1_2 = 0$, are

$$
(5.10) \qquad E_{(q_2, r_2, s_2, t_2)} = \mathrm{span}\{X + q_2\tilde{J}_2 + s_2\tilde{W}_2, Y + r_2\tilde{I}_2 + t_2\tilde{V}_2\},
$$

where $q_2, r_2, s_2, t_2 \in \mathbb{R}$. Therefore the prolonged manifold is

$$(5.11)\quad U^{(2)} = G_2(\mathcal{E}^{(1)}) = \{(x, y, v, q_0, r_0, s_0, t_0, q_1, r_1, s_1, t_1, E_{(q_2,r_2,s_2,t_2)})\} = U^{(1)} \times \mathbb{R}^4.$$

To find the prolonged EDS we take the perpendicular space of the integral plane in equation (5.10), and pullback by the projection $\pi : U^{(2)} \to U^{(1)}$ to get

$$\mathcal{E}^{(2)} = \langle \pi^*(E^{\perp}_{(q_2,r_2,s_2,t_2)}) \rangle = \langle \beta_1, \beta_2, \alpha_1, \alpha_2, \zeta_1, \zeta_2, \eta_1, \eta_2, \xi_1, \xi_2 \rangle, \tag{5.12}$$

where $\beta_1, \beta_2, \alpha_1, \zeta_1, \eta_1, \xi_1 \in E^{\perp}_{(q_1,r_1,s_1,t_1)}$ by equations (5.8), (5.9), and (5.10). Again using equations (5.8), (5.9), and (5.10) we find that

$$\begin{aligned} \alpha_2 &= dr_1 - r_2\tau, \\ \zeta_2 &= dq_1 - q_2\sigma, \\ \eta_2' &= \tilde{\eta}_2 - t_2\tau, \\ \xi_2' &= \tilde{\xi}_2 - s_2\sigma, \end{aligned} \tag{5.13}$$

are also in $E^{\perp}_{(q_2,r_2,s_2,t_2)}$.

By equations (5.13), (5.8), and (5.9) using the vectors

$$\begin{aligned} \tilde{X} &= X + q_2\tilde{J}_2 + s_2\tilde{W}_2, \\ \tilde{Y} &= Y + r_2\tilde{I}_2 + t_2\tilde{V}_2 \end{aligned} \tag{5.14}$$

makes the coframe of $U^{(2)}$

$$\{\beta_1, \beta_2, \alpha_1, \alpha_2, dr_2, \zeta_1, \zeta_2, dq_2, \eta_1, \eta_2, dt_2, \xi_1, \xi_2, ds_2, \sigma, \tau, \phi_2^1\} \tag{5.15}$$

dual to

$$\{B_1, B_2, I_1, \tilde{I}_2, \partial_{r_2}, J_1, \tilde{J}_2, \partial_{q_2}, \tilde{V}_1, \tilde{V}_2, \partial_{t_2}, \tilde{W}_1, \tilde{W}_2, \partial_{s_2}\tilde{X}, \tilde{Y}, \partial_{\phi_2^1}\}. \tag{5.16}$$

Step 2: Coframe Adaptation

Using equations (5.14) and (5.6) we calculate the invariants

$$\begin{aligned} A_1 &= -\tilde{Y}(H_0)/H_0 &&= -\tfrac{t_2}{t_1} + \tfrac{t_1\cos(\delta)}{\sin(\delta)}, \\ E_1 &= -\tilde{X}(K_0)/K_0 &&= -\tfrac{s_2}{s_1} - \tfrac{s_1\cos(\delta)}{\sin(\delta)}. \end{aligned} \tag{5.17}$$

We now adapt the coframe in equation (5.15) by setting

$$\begin{aligned} \eta_2 &= \eta_2' + A_1\eta_1, \\ \xi_2 &= \xi_2' + E_1\xi_1, \\ \tilde{\alpha}_3 &= dr_2, \\ \tilde{\zeta}_3 &= dq_2, \\ \tilde{\eta}_3 &= \tilde{Y}(\eta_2) = dt_2 - t_1\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1 + \kappa\sin(\delta)\sigma, \\ \tilde{\xi}_3 &= \tilde{X}(\xi_2) = ds_2 + s_1\tfrac{\cos(\delta)}{\sin(\delta)}\xi_1 - \kappa\sin(\delta)\tau, \end{aligned} \tag{5.18}$$

and make the corresponding adaptations to the frame in equation (5.16) gives

$$
\begin{array}{lcl}
\tilde{X}' & = & \tilde{X} - \kappa \sin(\delta)\partial_{t_2}, \\
\tilde{Y}' & = & \tilde{Y} + \kappa \sin(\delta)\partial_{s_2}, \\
\tilde{V}_1 & = & V_1 - A_1\tilde{V}_2 + t_1\frac{\cos(\delta)}{\sin(\delta)}\partial_{t_2}, \\
\tilde{W}_1 & = & W_1 - E_1\tilde{W}_2 - s_1\frac{\cos(\delta)}{\sin(\delta)}\partial_{s_2}, \\
\tilde{I}_3 & = & \partial_{r_2}, \\
\tilde{J}_3 & = & \partial_{q_2}, \\
\tilde{V}_3 & = & \partial_{t_2}, \\
\tilde{W}_3 & = & \partial_{s_2}.
\end{array} \tag{5.19}
$$

Now the coframe of $U^{(2)}$

$$\{\beta_1, \beta_2, \alpha_1, \alpha_2, \tilde{\alpha}_3, \zeta_1, \zeta_2, \tilde{\zeta}_3, \eta_1, \eta_2, \tilde{\eta}_3, \xi_1, \xi_2, \tilde{\xi}_3, \sigma, \tau, \phi^1_2\} \tag{5.20}$$

is dual to the frame

$$\{B_1, B_2, I_1, \tilde{I}_2, \tilde{I}_3, J_1, \tilde{J}_2, \tilde{J}_3, \tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{X}', \tilde{Y}', \partial_{\phi^1_2}\}. \tag{5.21}$$

We check that this adaptation does not affect the calculation in equation (5.17) by computing

$$\frac{\partial H_0}{\partial s_2} = 0, \tag{5.22}$$

$$\frac{\partial K_0}{\partial t_2} = 0. \tag{5.23}$$

By equations (5.19) and (5.17) we have

$$\begin{aligned} A_1 &= -\tilde{Y}'(H_0)/H_0, \\ E_1 &= -\tilde{X}'(K_0)/K_0. \end{aligned} \tag{5.24}$$

Using equations (5.6) and (5.19) we calculate the invariants

$$\begin{aligned} H_1 &= H_0K_0 + \tilde{X}'(A_1) &= \tfrac{\sin(\delta)(\kappa_\tau t_1 - t_2\kappa)}{{t_1}^2}, \\ K_1 &= H_0K_0 + \tilde{Y}'(E_1) &= \tfrac{\sin(\delta)(s_2\kappa - \kappa_\sigma s_1)}{{s_1}^2}. \end{aligned} \tag{5.25}$$

Step 3: Verification of the Structure Equations

We now wish to verify the structure equations (5.4). We begin by calculating the bracket relation $[\tilde{X}', \tilde{Y}']$. Using the structure equations (3.52) we find

$$\begin{aligned} \beta_1([\tilde{X}', \tilde{Y}']) &= 0, \\ \beta_2([\tilde{X}', \tilde{Y}']) &= 0, \\ \sigma([\tilde{X}', \tilde{Y}']) &= 0, \\ \tau([\tilde{X}', \tilde{Y}']) &= 0, \\ \phi_2^1([\tilde{X}', \tilde{Y}']) &= -\kappa\sin(\delta). \end{aligned} \tag{5.26}$$

Similarly, the structure equations (4.8) show

$$\begin{aligned} \alpha_1([\tilde{X}', \tilde{Y}']) &= 0, \\ \zeta_1([\tilde{X}', \tilde{Y}']) &= 0, \\ \eta_1([\tilde{X}', \tilde{Y}']) &= 0, \\ \xi_1([\tilde{X}', \tilde{Y}']) &= 0. \end{aligned} \tag{5.27}$$

Finally, equation (5.18) shows

$$
\begin{aligned}
\alpha_2([\tilde{X}', \tilde{Y}']) &= 0, \\
\zeta_2([\tilde{X}', \tilde{Y}']) &= 0, \\
\eta_2([\tilde{X}', \tilde{Y}']) &= 0, \\
\xi_2([\tilde{X}', \tilde{Y}']) &= 0, \\
\tilde{\alpha}_3([\tilde{X}', \tilde{Y}']) &= 0, \\
\tilde{\zeta}_3([\tilde{X}', \tilde{Y}']) &= 0,
\end{aligned}
\tag{5.28}
$$

giving the bracket relation

$$
[\tilde{X}', \tilde{Y}'] = -\kappa \sin(\delta) \partial_{\phi_2^1} \mod \tilde{V}_3, \tilde{W}_3. \tag{5.29}
$$

This shows that $\tilde{X}'$ and $\tilde{Y}'$ commute on forms of subscript less than 2. We use this to calculate

$$
\begin{aligned}
\tilde{X}'(\alpha_1) &= \tfrac{-r_1}{\sin(\delta)} \xi_1, \\
\tilde{Y}'(\alpha_1) &= \alpha_2, \\
\tilde{X}'(\zeta_1) &= \zeta_2, \\
\tilde{Y}'(\zeta_1) &= \tfrac{q_1}{\sin(\delta)} \eta_1, \\
\tilde{X}'(\eta_1) &= H_0 \xi_1, \\
\tilde{Y}'(\eta_1) &= \eta_2 - A_1 \eta_1, \\
\tilde{X}'(\xi_1) &= \xi_2 - E_1 \xi_1, \\
\tilde{Y}'(\xi_1) &= K_0 \eta_1, \\
\tilde{X}'(\alpha_2) &= \tfrac{-r_2}{\sin(\delta)} \xi_1, \\
\tilde{Y}'(\alpha_2) &= \tilde{\alpha}_3, \\
\tilde{X}'(\zeta_2) &= \tilde{\zeta}_3, \\
\tilde{Y}'(\zeta_2) &= \tfrac{q_2}{\sin(\delta)} \eta_1,
\end{aligned}
\tag{5.30}
$$

$$
\begin{aligned}
\tilde{X}'(\eta_2) &= \tilde{X}'(\tilde{Y}'(\eta_1) + A_1\eta_1), \\
&= \tilde{Y}'(\tilde{X}'(\eta_1)) + \tilde{X}'(A_1)\eta_1 + A_1\tilde{X}'(\eta_1), \\
&= \tilde{Y}'(H_0\xi_1) + \tilde{X}'(A_1)\eta_1 + A_1H_0\xi_1, \\
&= \tilde{Y}'(H_0)\xi_1 + H_0\tilde{Y}'(\xi_1) + \tilde{X}'(A_1)\eta_1 + A_1H_0\xi_1, \\
&= H_0K_0\eta_1 + \tilde{X}'(A_1)\eta_1, \\
&= H_1\eta_1, \\
\tilde{Y}'(\eta_2) &= \tilde{\eta}_3, \\
\tilde{X}'(\xi_2) &= \tilde{\xi}_3, \\
\tilde{Y}'(\xi_2) &= \tilde{Y}'(\tilde{X}'(\xi_1) + E_1\xi_1), \\
&= \tilde{X}'(\tilde{Y}'(\xi_1)) + \tilde{Y}'(E_1)\xi_1 + E_1\tilde{Y}'(\xi_1), \\
&= \tilde{X}'(K_0\eta_1) + \tilde{Y}'(E_1)\xi_1 + E_1K_0\eta_1, \\
&= \tilde{X}'(K_0)\eta_1 + K_0\tilde{X}'(\eta_1) + \tilde{Y}'(E_1)\xi_1 + E_1K_0\eta_1, \\
&= K_0H_0\xi_1 + \tilde{Y}'(E_1)\xi_1, \\
&= K_1\xi_1.
\end{aligned}
\tag{5.31}
$$

Therefore the structure equations (5.4) hold for $k = 2$.

5.1.2 Inductive Step

This proof requires an extra adaptation and is completed in four steps.

Step 1: Prolongation

Assume that (5.4) holds for k and let

(5.32)

$$\{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \ldots, \zeta_k, \tilde{\zeta}_{k+1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_k, \tilde{\xi}_{k+1}, \sigma, \tau, \phi^1_2\}$$

be the coframe in equation (5.1) and

(5.33)

$$\{X, Y, \partial_{\phi^1_2}, B_1, B_2, I_1, \ldots, I_k, \tilde{I}_{k+1}, J_1, \ldots, J_k, \tilde{J}_{k+1}, V_1, \ldots, V_k, \tilde{V}_{k+1}, W_1, \ldots, W_k, \tilde{W}_{k+1}\}$$

the dual frame in equation (5.2). Using the structure equations (5.4), the integral 2-planes of $\mathcal{E}^{(k)}$ with independence condition $\sigma \wedge \tau \neq 0$ and no characteristic directions, $\phi^1_2 = 0$, are

$$E_{(q_{k+1},r_{k+1},s_{k+1},t_{k+1})} = \operatorname{span}\{X + q_{k+1}\tilde{J}_{k+1} + s_{k+1}\tilde{W}_{k+1}, Y + r_{k+1}\tilde{I}_{k+1} + t_{k+1}\tilde{V}_{k+1}\}, \tag{5.34}$$

where $q_{k+1}, r_{k+1}, s_{k+1}, t_{k+1} \in \mathbb{R}$. Therefore the prolonged manifold is

$$\begin{aligned} U^{(k+1)} &= G_2(\mathcal{E}^{(k)}) \\ &= \{(x, y, v, q_0, r_0, s_0, t_0, \ldots, q_k, r_k, s_k, t_k, E_{(q_{k+1},r_{k+1},s_{k+1},t_{k+1})})\} \\ &= U^{(k)} \times \mathbb{R}^4. \end{aligned} \tag{5.35}$$

To find the prolonged EDS we take the perpendicular space of the integral plane in equation (5.34), and pullback by the projection $\pi : U^{(k+1)} \to U^{(k)}$ to get

$$\begin{aligned} \mathcal{E}^{(k+1)} &= \langle \pi^*(E^{\perp}_{(q_{k+1},r_{k+1},s_{k+1},t_{k+1})})\rangle \\ &= \langle \beta_1, \beta_2, \alpha_1, \ldots, \alpha_{k+1}, \zeta_1, \ldots, \zeta_{k+1}, \eta_1, \ldots, \eta_{k+1}, \xi_1, \ldots, \xi_{k+1}\rangle, \end{aligned} \tag{5.36}$$

where $\beta_1, \beta_2, \alpha_1, \zeta_1, \eta_1, \xi_1, \ldots, \alpha_k, \zeta_k, \eta_k, \xi_k \in E^{\perp}_{(q_{k+1},r_{k+1},s_{k+1},t_{k+1})}$ by equations (5.32), (5.33), and (5.34). Again using equations (5.32), (5.33), and (5.34) we find that

$$\begin{aligned} \alpha_{k+1} &= dr_k - r_{k+1}\tau, \\ \zeta_{k+1} &= dq_k - q_{k+1}\sigma, \\ \eta'_{k+1} &= \tilde{\eta}_{k+1} - t_{k+1}\tau, \\ \xi'_{k+1} &= \tilde{\xi}_{k+1} - s_{k+1}\sigma, \end{aligned} \tag{5.37}$$

are also in $E^{\perp}_{(q_{k+1},r_{k+1},s_{k+1},t_{k+1})}$.

By equations (5.37), (5.32), and (5.33) using the vectors

$$\begin{aligned}\tilde{X} &= X + q_{k+1}\tilde{J}_{k+1} + s_{k+1}\tilde{W}_{k+1} &&= X + q_{k+1}\partial_{q_k} + s_{k+1}\partial_{s_k} \\ \tilde{Y} &= Y + r_{k+1}\tilde{I}_{k+1} + t_{k+1}\tilde{V}_{k+1} &&= Y + r_{k+1}\partial_{r_k} + t_{k+1}\partial_{t_k}\end{aligned} \tag{5.38}$$

makes the coframe of $U^{(k+1)}$

$$\begin{aligned}&\{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_{k+1}, dr_{k+1}, \zeta_1, \ldots, \zeta_{k+1}, dq_{k+1} \\ &\qquad \eta_1, \ldots, \eta_k, \eta'_{k+1}, dt_{k+1}, \xi_1, \ldots, \xi_k, \xi'_{k+1}, ds_{k+1}, \sigma, \tau, \phi^1_2\}\end{aligned} \tag{5.39}$$

dual to

$$\begin{aligned}&\{B_1, B_2, I_1, \ldots, I_k, \tilde{I}_{k+1}, \partial_{r_{k+1}}, J_1, \ldots, J_k, \tilde{J}_{k+1}, \partial_{q_{k+1}}, \\ &\qquad V_1, \ldots, V_k, \tilde{V}_{k+1}, \partial_{t_{k+1}}, W_1, \ldots, W_k, \tilde{W}_{k+1}, \partial_{s_{k+1}}, \tilde{X}, \tilde{Y}, \partial_{\phi^1_2}\}.\end{aligned} \tag{5.40}$$

To check that this change of frame does not affect the calculation of the invariants in equation (5.6) we note that H_{k-2} is defined on $U^{(k-1)}$ and hence

$$\tilde{I}_{k+1}(H_{k-2}) = \tilde{V}_{k+1}(H_{k-2}) = \tilde{W}_{k+1}(H_{k-2}) = 0. \tag{5.41}$$

Likewise

$$\tilde{J}_{k+1}(A_{k-2}) = \tilde{W}_{k+1}(A_{k-2}) = 0. \tag{5.42}$$

From the structure equations (5.4) we have the bracket relation

$$[Y, \tilde{W}_{k+1}] = 0 \text{ mod } \tilde{V}_{k+1}, \tilde{W}_{k+1}. \tag{5.43}$$

Using equations (5.38), (5.41), (5.42), and (5.43) we have

$$
\begin{aligned}
\tilde{W}_{k+1}(A_{k-1}) &= \tilde{W}_{k+1}(A_{k-2}) - \tilde{W}_{k+1}(Y(\log(H_{k-2}))) \\
&= -Y(\tilde{W}_{k+1}(\log(H_{k-2}))) + [Y, \tilde{W}_{k+1}](\log(H_{k-2})) \\
&= 0.
\end{aligned}
\tag{5.44}
$$

Substituting this into equation (5.6) we get

$$
\begin{aligned}
A_{k-1} &= A_{k-2} - \tilde{Y}(H_{k-2})/H_{k-2} \\
H_{k-1} &= H_{k-2} + \tilde{X}(A_{k-1}).
\end{aligned}
\tag{5.45}
$$

A Similar calculation shows

$$
\begin{aligned}
E_{k-1} &= E_{k-2} - \tilde{X}(K_{k-2})/K_{k-2} \\
K_{k-1} &= K_{k-2} + \tilde{Y}(E_{k-1}).
\end{aligned}
\tag{5.46}
$$

Step 2: First Adaptation

We now adapt the coframe in equation (5.39) by setting

$$
\begin{aligned}
\eta''_{k+1} &= \tilde{Y}(\eta_k) = \tilde{\eta}_{k+1} - t_{k+1}(\tau - a_1^1\eta_1 - \cdots - a_k^1\eta_k - a_1^2\xi_1 - \cdots - a_k^2\xi_k), \\
\xi''_{k+1} &= \tilde{X}(\xi_k) = \tilde{\xi}_{k+1} - s_{k+1}(\sigma - b_1^1\xi_1 - \cdots - b_k^1\xi_k - b_1^2\eta_1 - \cdots - b_k^2\eta_k),
\end{aligned}
\tag{5.47}
$$

and making the corresponding changes to the frame in equation (5.40)

$$
\begin{aligned}
\tilde{V}_i' &= V_i - t_{k+1}(b_i^2\tilde{W}_{k+1} + a_i^1\tilde{V}_{k+1}) \quad && i \leq k, \\
\tilde{W}_i' &= W_i - s_{k+1}(a_i^2\tilde{V}_{k+1} + b_i^1\tilde{W}_{k+1}) \quad && i \leq k,
\end{aligned}
\tag{5.48}
$$

where

$$
\begin{aligned}
a_i^1 &= d\eta_k(\tilde{V}_{k+1}, V_i), \\
a_i^2 &= d\eta_k(\tilde{V}_{k+1}, W_i), \\
b_i^1 &= d\xi_k(\tilde{W}_{k+1}, W_i), \\
b_i^2 &= d\xi_k(\tilde{W}_{k+1}, V_i).
\end{aligned} \tag{5.49}
$$

We now have that the coframe on $U^{(k+1)}$

$$
\begin{aligned}
&\{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_{k+1}, dr_{k+1}, \zeta_1, \ldots, \zeta_{k+1}, dq_{k+1}, \\
&\quad \eta_1, \ldots, \eta_k, \eta''_{k+1}, dt_{k+1}, \xi_1, \ldots, \xi_k, \xi''_{k+1}, ds_{k+1}, \sigma, \tau, \phi_2^1\}
\end{aligned} \tag{5.50}
$$

is dual to the frame

$$
\begin{aligned}
&\{B_1, B_2, I_1, \ldots, I_k, \tilde{I}_{k+1}, \partial_{r_{k+1}}, J_1, \ldots, J_k, \tilde{J}_{k+1}, \partial_{q_{k+1}}, \\
&\quad \tilde{V}'_1, \ldots, \tilde{V}'_k, \tilde{V}_{k+1}, \partial_{t_{k+1}}, \tilde{W}'_1, \ldots, \tilde{W}'_k, \tilde{W}_{k+1}, \partial_{s_{k+1}}, \tilde{X}, \tilde{Y}, \partial_{\phi_2^1}\}.
\end{aligned} \tag{5.51}
$$

Step 3: Second Adaptation

Using equations (5.38) and (5.6) we calculate the invariants

$$
\begin{aligned}
A_k &= A_{k-1} - \tilde{Y}(H_{k-1})/H_{k-1}, \\
E_k &= E_{k-1} - \tilde{X}(K_{k-1})/K_{k-1}
\end{aligned} \tag{5.52}
$$

We now adapt the coframe in equation (5.50) by setting

$$
\begin{aligned}
\eta_{k+1} &= \eta''_{k+1} + A_k \eta_k, \\
\xi_{k+1} &= \xi''_{k+1} + E_k \xi_k, \\
\tilde{\alpha}_{k+2} &= dr_{k+1}, \\
\tilde{\zeta}_{k+2} &= dq_{k+1}, \\
\tilde{\eta}_{k+2} &= \tilde{Y}(\eta_{k+1}) &&= dt_{k+1} + a\sigma \bmod \mathcal{E}^{(k+1)}, \\
\tilde{\xi}_{k+2} &= \tilde{X}(\xi_{k+1}) &&= ds_{k+1} + b\tau \bmod \mathcal{E}^{(k+1)},
\end{aligned}
\tag{5.53}
$$

and let the frame,

$$
\{B_1, B_2, I_1, \ldots, I_k, \tilde{I}_{k+1}, \tilde{I}_{k+2}, J_1, \ldots, J_k, \tilde{J}_{k+1}, \tilde{J}_{k+2},
\tilde{V}_1, \ldots, \tilde{V}_{k+2}, \tilde{W}_1, \ldots, \tilde{W}_{k+2}, \tilde{X}', \tilde{Y}', \partial_{\phi_2^1}\}
\tag{5.54}
$$

be dual to the coframe on $U^{(k+1)}$

$$
\{\sigma, \tau, \beta_1, \beta_2, \alpha_1, \ldots, \alpha_{k+1}, \tilde{\alpha}_{k+2}, \zeta_1, \ldots, \zeta_{k+1}, \tilde{\zeta}_{k+2}, \eta_1, \ldots, \eta_{k+1}, \tilde{\eta}_{k+2}, \xi_1, \ldots, \xi_{k+1}, \tilde{\xi}_{k+2}\}.
\tag{5.55}
$$

We note that this implies that

$$
\begin{aligned}
\tilde{X}' &= \tilde{X} + a\partial_{t_{k+1}}, \\
\tilde{Y}' &= \tilde{Y} + b\partial_{s_{k+1}}.
\end{aligned}
\tag{5.56}
$$

We again check that this adaptation does not affect the calculation of our invariant in equation (5.52) by computing

$$
\frac{\partial H_{k-1}}{\partial s_{k+1}} = 0,
\tag{5.57}
$$

$$\frac{\partial K_{k-1}}{\partial t_{k+1}} = 0. \tag{5.58}$$

By equations (5.56) and (5.52) we have

$$\begin{aligned} A_k &= A_{k-1} - \tilde{Y}'(H_{k-1})/H_{k-1}, \\ E_k &= E_{k-1} - \tilde{X}'(K_{k-1})/K_{k-1}. \end{aligned} \tag{5.59}$$

Using equations (5.6) and (5.56) we calculate the invariants

$$\begin{aligned} H_k &= H_{k-1} + \tilde{X}'(A_k), \\ K_k &= K_{k-1} + \tilde{Y}'(E_k). \end{aligned} \tag{5.60}$$

Step 4: Verification of the Structure Equations

Using the structure equations (5.4) and equation (5.53) we get the bracket relation

$$[\tilde{X}', \tilde{Y}'] = -\kappa \sin(\delta)\partial_{\phi_2^1} \mod \tilde{V}_{k+2}, \tilde{W}_{k+2}. \tag{5.61}$$

This shows that $\tilde{X}'$ and $\tilde{Y}'$ commute on forms of subscript less than or equal to k. We use this to calculate

(5.62)

$$
\begin{aligned}
\tilde{X}'(\alpha_k) &= \tfrac{-r_k}{\sin(\delta)}\xi_1, \\
\tilde{Y}'(\alpha_k) &= \alpha_{k+1}, \\
\tilde{X}'(\zeta_k) &= \zeta_{k+1}, \\
\tilde{Y}'(\zeta_k) &= \tfrac{q_k}{\sin(\delta)}\eta_1, \\
\tilde{X}'(\eta_k) &= H_{k-1}\eta_{k-1}, \\
\tilde{Y}'(\eta_k) &= \eta_{k+1} - A_k\eta_k, \\
\tilde{X}'(\xi_k) &= \xi_{k+1} - E_k\xi_k, \\
\tilde{Y}'(\xi_k) &= K_{k-1}\xi_{k-1}, \\
\tilde{X}'(\alpha_{k+1}) &= \tfrac{-r_{k+1}}{\sin(\delta)}\xi_1, \\
\tilde{Y}'(\alpha_{k+1}) &= \tilde{\alpha}_{k+2}, \\
\tilde{X}'(\zeta_{k+1}) &= \tilde{\zeta}_{k+2}, \\
\tilde{Y}'(\zeta_{k+1}) &= \tfrac{q_{k+1}}{\sin(\delta)}\eta_1, \\
\tilde{X}'(\eta_{k+1}) &= \tilde{X}'(\tilde{Y}'(\eta_k) + A_k\eta_k), \\
&= \tilde{Y}'(\tilde{X}'(\eta_k)) + \tilde{X}'(A_k)\eta_k + A_k\tilde{X}'(\eta_k), \\
&= \tilde{Y}'(H_{k-1}\eta_{k-1}) + \tilde{X}'(A_k)\eta_k + A_kH_{k-1}\xi_{k-1}, \\
&= \tilde{Y}'(H_{k-1})\eta_{k-1} + H_{k-1}\tilde{Y}'(\eta_{k-1}) + \tilde{X}'(A_k)\eta_k + A_kH_{k-1}\eta_{k-1}, \\
&= \tilde{Y}'(H_{k-1})\eta_{k-1} + H_{k-1}(\eta_k - A_{k-1}\eta_{k-1}) + \tilde{X}'(A_k)\eta_k + A_kH_{k-1}\eta_{k-1}, \\
&= (\tilde{Y}'(H_{k-1}) - H_{k-1}A_{k-1} + A_kH_{k-1})\eta_{k-1} + (H_{k-1} + \tilde{X}'(A_k))\eta_k, \\
&= H_k\eta_k, \\
\tilde{Y}'(\eta_{k+1}) &= \tilde{\eta}_{k+2},
\end{aligned}
$$

(5.63)

$$\begin{aligned}
\tilde{X}'(\xi_{k+1}) &= \tilde{\xi}_{k+2}, \\
\tilde{Y}'(\xi_{k+1}) &= \tilde{Y}'(\tilde{X}'(\xi_k) + E_k\xi_k), \\
&= \tilde{X}'(\tilde{Y}'(\xi_k)) + \tilde{Y}'(E_k)\xi_k + E_k\tilde{Y}'(\xi_k), \\
&= \tilde{X}'(K_{k-1}\xi_{k-1}) + \tilde{Y}'(E_k)\xi_k + E_kK_{k-1}\xi_{k-1}, \\
&= \tilde{X}'(K_{k-1})\xi_{k-1} + K_{k-1}\tilde{X}'(\xi_{k-1}) + \tilde{Y}'(E_k)\xi_k + E_kK_{k-1}\xi_{k-1}, \\
&= \tilde{X}'(K_{k-1})\xi_{k-1} + K_{k-1}(\xi_k - E_{k-1}\xi_{k-1}) + \tilde{Y}'(E_k)\xi_k + E_kK_{k-1}\xi_{k-1}, \\
&= (\tilde{X}'(K_{k-1}) - K_{k-1}E_{k-1} + E_kK_{k-1})\xi_{k-1} + (K_{k-1} + \tilde{Y}'(E_k))\xi_k, \\
&= K_k\xi_k.
\end{aligned}$$

This completes the inductive argument, showing that the structure equation (5.4) hold.

The proof of equation (5.7) is also by induction. From the structure equations, (5.4), it is true for $i = 1$. We assume it is true for $i = l < k$, and using equation (5.4), calculate

$$\begin{aligned}
d\eta_{l+1} &= d(Y(\eta_l) + A_l\eta_l) = Y(d\eta_l) + dA_l \wedge \eta_l + A_l d\eta_l \\
d\xi_{l+1} &= d(X(\xi_l) + E_l\xi_l) = X(d\xi_l) + dE_l \wedge \xi_l + E_l d\xi_l.
\end{aligned} \tag{5.64}$$

By induction, equation (5.7) holds.

□

It remains to prove the assumption implicit in the statement of equation (5.6), namely that $H_i = 0$ if and only if $K_i = 0$. This is proved using the involution ψ given in the following proposition.

Proposition 5.2. *The involution ψ extended to $U^{(k)}$ is*

(5.65)

$$\psi(x^1, x^2, y, v, q_0, r_0, s_0, t_0, \ldots, q_k, r_k, s_k, t_k) = (x^2, x^1, y, v, r_0, q_0, t_0, s_0, \ldots, r_k, q_k, t_k, s_k).$$

Proof. Equation (5.65) is true for $k = 0, 1$. If we assume it is true on $U^{(k)}$, then since $\tilde{\alpha}_{k+1} = \partial_{r_k}$, $\tilde{\zeta}_{k+1} = \partial_{q_k}$, $\tilde{\eta}_{k+1} = \partial_{t_k}$, and $\tilde{\xi}_{k+1} = \partial_{s_k}$, it is a simple matter to push-forward the integral 2-planes of $\mathcal{E}^{(k)}$. These are defined in equation (5.34), and pushing forward by ψ_* gives

(5.66)
$$\begin{aligned}\psi_* E_{(q_{k+1}, r_{k+1}, s_{k+1}, t_{k+1})} &= \operatorname{span}\{X + q_{k+1}\tilde{I}_{k+1} + s_{k+1}\tilde{V}_{k+1}, Y + r_{k+1}\tilde{J}_{k+1} + t_{k+1}\tilde{W}_{k+1}\} \\ &= E_{(r_{k+1}, q_{k+1}, t_{k+1}, s_{k+1})}.\end{aligned}$$

By induction equation (5.65) is true.

For $i < k$ we pullback $X(\eta_{i+1})$ by ψ to get

$$\psi^*(X(\eta_{i+1})) = Y(\xi_{i+1}) \tag{5.67}$$

or

$$\psi^*(H_i)\xi_i = K_i\xi_i. \tag{5.68}$$

Injectivity of ψ gives $H_i = 0$ if and only if $K_i = 0$. □

5.2 Quotient EDS

We now take the quotient of the EDS $\mathcal{E}$ by the action of $\mathcal{G}$ prolonged to $U^{(k)}$. The first step is to compute the infinitesimal generators of $\mathcal{G}$.

Theorem 5.3. *The infinitesimal generators of $\mathcal{G}$ on $U^{(k)}$ are*

$$Z^{(k)} = f\partial_{x^1} + g\partial_{x^2} + a_0\partial_{q_0} + b_0\partial_{r_0} + \cdots + a_k\partial_{q_k} + b_k\partial_{r_k}$$

where $f = f(x^1)$, $g = g(x^2)$ *and*

$$a_0 = -f' \qquad b_0 = -g' \tag{5.69}$$
$$a_i = X(a_{i-1}) \quad b_i = Y(b_{i-1}) \quad 1 \leq i \leq k.$$

Furthermore

$$\begin{aligned} a_i &= (-1)^{i+1} f^{(i+1)} e^{-iq_0} \quad \mod f'', \ldots, f^{(i)}, \\ b_i &= (-1)^{i+1} g^{(i+1)} e^{-ir_0} \quad \mod g'', \ldots, g^{(i)} \end{aligned} \tag{5.70}$$

for $0 \leq i \leq k$.

Proof. Since $f = f(x^1)$ and $g = g(x^2)$ we have

$$\begin{aligned} d(-f') &= -f''\omega^1 = -f''(\beta_1 - e^{-q_0}\sigma), \\ d(-g') &= -g''\omega^2 = -g''(\beta_2 - e^{-r_0}\tau), \end{aligned} \tag{5.71}$$

we have

$$\begin{aligned} X(-f') &= f''e^{-q_0}, \\ Y(-g') &= g''e^{-r_0}. \end{aligned} \tag{5.72}$$

Using equation (4.14) proves the theorem for $k = 1$.

To prolong the group action from level k to level $k+1$ we compute the Lie Derivatives of $\mathcal{E}^{(k+1)}$ in the direction of $\tilde{Z} = Z^{(k)} + c_1\partial_{q_{k+1}} + c_2\partial_{r_{k+1}} + c_3\partial_{s_{k+1}} + c_4\partial_{t_{k+1}}$

giving,

$$
\begin{aligned}
\tilde{Z}(\alpha_{k+1}) &= (Y(b_k) - c_2)\tau \mod \mathcal{E}^{(k+1)},\\
\tilde{Z}(\zeta_{k+1}) &= (X(a_k) - c_1)\sigma \mod \mathcal{E}^{(k+1)},\\
\tilde{Z}(\eta_{k+1}) &= -c_4\tau,\\
\tilde{Z}(\xi_{k+1}) &= -c_3\sigma.
\end{aligned} \tag{5.73}
$$

Forcing the results to be in $\mathcal{E}^{(k+1)}$ we get $c_1 = Y(a_k)$, $c_2 = X(b_k)$, and $c_3 = c_4 = 0$. This completes the inductive step, proving equation (5.69).

By equations (5.69) and (5.72), equation (5.70) is true for $i = 0, 1$. Inductively we assume

$$
a_l = (-1)^{l+1} f^{(l+1)} e^{-lq_0} + \sum_{i=2}^{l} c_i f^{(i)} \tag{5.74}
$$

for $2 \leq l < k$ and $c_i \in C^\infty(U^{(k)}, \mathbb{R})$. We compute $X(a_l)$,

$$
X(a_l) = (-1)^{l+1}(X(f^{(l+1)})e^{-lq_0} + f^{(l+1)}X(e^{-lq_0})) + \sum_{i=2}^{l} \left(X(c_i)f^{(i)} + c_i X(f^{(i)})\right). \tag{5.75}
$$

Substituting $X(f^{(i)}) = -f^{(i+1)}e^{-q_0}$, this becomes

$$
\begin{aligned}
X(a_l) = (-1)^{l+2}X(f^{(l+2)})e^{-(l+1)q_0} &+ (-1)^{l+1} f^{(l+1)} X(e^{-lq_0}))\\
&+ \sum_{i=2}^{l} \left(X(c_i)f^{(i)} - c_i e^{-q_0} f^{(i+1)})\right).
\end{aligned} \tag{5.76}
$$

This gives

$$
a_{l+1} = (-1)^{l+2} X(f^{(l+2)}) e^{-(l+1)q_0} \mod f'', \ldots, f^{(l+1)}. \tag{5.77}
$$

A similar argument holds for b_l, proving equation (5.70).

$\square$

To find the quotient EDS we must find the forms semi-basic to the action of $\mathcal{G}$, and pull them back by a cross-section of $U^{(k)}$. To do this we will find the integrable distribution corresponding to $Z^{(k)}$ in a corollary to theorem 5.3.

Corollary 5.4. *The integrable distribution corresponding to $Z^{(k)}$ is*

$$\mathbf{\Gamma}^{(k)} = \{\partial_{\omega^1}, \partial_{\omega^2}, \partial_{q_0}, \ldots, \partial_{q_k}, \partial_{r_0}, \ldots, \partial_{r_k}\}. \tag{5.78}$$

Proof. Let $\mathbf{\Gamma}^{(k)}$ be the distribution corresponding to $Z^{(k)}$. Working at the point,

$$(x_0^1, x_0^2, y_0, v_0, Q_0, R_0, S_0, T_0, \ldots, Q_k, R_k, S_k, T_k)$$

we let

$$f_i(x^1) = (x_0^1 - x^1)^i; \quad g_i(x^2) = (x_0^2 - x^2)^i \tag{5.79}$$

for $0 \leq i \leq k+1$. Using equation (5.70) we have that for $f = f_0$, $g = 0$

$$Z^{(k)}_{(x_0^1, x_0^2, y_0, v_0, Q_0, R_0, S_0, T_0, \ldots, Q_k, R_k, S_k, T_k)} = \partial_{\omega^1}, \tag{5.80}$$

for $1 \leq l \leq k+1$ and $f = f_l$, $g = 0$

$$Z^{(k)}_{(x_0^1, x_0^2, y_0, v_0, Q_0, R_0, S_0, T_0, \ldots, Q_k, R_k, S_k, T_k)} = l! e^{-(l-1)Q_0} \partial_{q_{l-1}}, \tag{5.81}$$

for $f=0$, $g=g_0$

$$Z^{(k)}_{(x_0^1,x_0^2,y_0,v_0,Q_0,R_0,S_0,T_0,\ldots,Q_k,R_k,S_k,T_k)} = \partial_{\omega^2}, \tag{5.82}$$

for $1 \leq l \leq k+1$ and $f=0$, $g=g_l$

$$Z^{(k)}_{(x_0^1,x_0^2,y_0,v_0,Q_0,R_0,S_0,T_0,\ldots,Q_k,R_k,S_k,T_k)} = l!e^{-(l-1)R_0}\partial_{r_{l-1}}. \tag{5.83}$$

This gives

$$\operatorname{span}\{\partial_{x^1},\partial_{x^2},\partial_{q_0},\ldots,\partial_{q_k},\partial_{r_0},\ldots,\partial_{r_k}\} \subset \Gamma^{(k)}_{(x_0^1,x_0^2,y_0,v_0,Q_0,R_0,S_0,T_0,\ldots,Q_k,R_k,S_k,T_k)}, \tag{5.84}$$

and since for any f and g,

$$Z^{(k)}_{(x_0^1,x_0^2,y_0,v_0,Q_0,R_0,S_0,T_0,\ldots,Q_k,R_k,S_k,T_k)} \in \operatorname{span}\{\partial_{x^1},\partial_{x^2},\partial_{q_0},\ldots,\partial_{q_k},\partial_{r_0},\ldots,\partial_{r_k}\}, \tag{5.85}$$

we get equality.

□

Theorem 5.5. *The quotient of $U^{(k)}$ by $\mathcal{G}$ is*

$$\overline{U^{(k)}} = \bar{U} \times \mathbb{R}^{2k}. \tag{5.86}$$

Moreover, the quotient system $\overline{\mathcal{E}^{(k)}}$ is an $s=2k$ hyperbolic system.

Proof. It is clear from the form of the integrable distribution corresponding to $\mathcal{G}$ that the quotient map is $\mathfrak{q}: U^{(k)} = \bar{U} \times \mathbb{R}^{2k}$ given by

$$\mathfrak{q}(x,y,v,q_0,r_0,s_0,t_0,\ldots,q_k,r_k,s_k,t_k) = (y,v,s_0,t_0,\ldots,s_k,t_k). \tag{5.87}$$

Using the structure equations (5.4) we see that the coframe in theorem 5.1 satisfies the hypotheses of theorem B.5. Therefore the quotient is Pfaffian. The semi-basic forms in $\mathcal{E}^{(k)}$ are generated by

$$\mathcal{A}^1_{sb} = \{\eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k\}. \tag{5.88}$$

These are all $\mathcal{G}$-basic, and by our usual abuse of notation we write

$$\bar{\mathcal{E}}^{(k)} = \langle \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k \rangle. \tag{5.89}$$

The structure equations for $\bar{\mathcal{E}}^{(k)}$ are

(5.90)

$$\begin{aligned}
d\eta_1 &= \sigma \wedge H_0\xi_1 && + \tau \wedge (\eta_2 - A_1\eta_1)\,, && \\
d\xi_1 &= \sigma \wedge (\xi_2 - E_1\xi_1) && + \tau \wedge K_0\eta_1, && \\
d\eta_i &= \sigma \wedge H_{i-1}\eta_{i-1} && + \tau \wedge (\eta_{i+1} - A_i\eta_i) && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_{i+1}, \xi_{i+1}), \\
d\xi_i &= \sigma \wedge (\xi_{i+1} - E_i\xi_i) && + \tau \wedge K_{i-1}\xi_{i-1} && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_{i+1}, \xi_{i+1}), \\
d\eta_k &= \sigma \wedge H_{k-1}\eta_{k-1} && + \tau \wedge \tilde{\eta}_{k+1} && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_k, \xi_k, \tilde{\eta}_{k+1}, \tilde{\xi}_{k+1}), \\
d\xi_k &= \sigma \wedge \tilde{\xi}_{k+1} && + \tau \wedge K_{k-1}\xi_{k-1} && \mod \Omega^2(\eta_1, \xi_1, \ldots, \eta_k, \xi_k, \tilde{\eta}_{k+1}, \tilde{\xi}_{k+1}),
\end{aligned}$$

where $1 < i < k$. From this we see that $\bar{\mathcal{E}}^{(k)}$ is an $s = 2k$ hyperbolic system.

□

In this frame, vanishing of a certain invariant guarantees the existence of a closed form in each of the associated differential systems of $\mathcal{E}^{(k)}$. This will in turn ensure that $\hat{V}$ and $\check{V}$ have large enough integrable subsets to ensure Darboux integrability of $\mathcal{E}^{(k)}$.

5.3 Darboux Integrability of $\mathcal{E}^{(k)}$

As in the previous chapters we will show the equivalence of Darboux integrability of $\mathcal{E}^{(k)}$ to the existence of a closed form on the quotient. We us this in the next section will be to relate Darboux integrability of $\mathcal{E}^{(k)}$ and its quotient. First we use the structure equations (5.4) to see that $\mathcal{E}^{(k)}$ in equation (5.3) is decomposable of type $[3, 3]$, and the associated singular Pfaffian systems are

$$\begin{aligned} \hat{V} &= \{\beta_1, \beta_2, \alpha_1, \dots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \dots, \zeta_k, \eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \dots, \xi_k, \tau\}, \\ \check{V} &= \{\beta_1, \beta_2, \alpha_1, \dots, \alpha_k, \zeta_1, \dots, \zeta_k, \tilde{\zeta}_{k+1}, \eta_1, \dots, \eta_k, \xi_1, \dots, \xi_k, \tilde{\xi}_{k+1}, \sigma\}. \end{aligned} \tag{5.91}$$

As before the involution ψ given in proposition 5.2 is an involution of the pair $\hat{V}, \check{V}$.

By the structure equations (5.90), the quotient system $\bar{\mathcal{E}}^{(k)}$ in equation (5.89) is decomposable of type $[2, 2]$, with associated singular Pfaffian systems

$$\hat{W} = \{\eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \dots, \xi_k, \tau\} \qquad \check{W} = \{\eta_1, \dots, \eta_k, \xi_1, \dots, \xi_k, \tilde{\xi}_{k+1}, \sigma\}. \tag{5.92}$$

Before proving the main lemma we establish the low order derived systems of $\hat{V}$.

Lemma 5.6. *For $0 \leq i \leq k$, the i^{th} derived system of $\hat{V}$ in equation (5.91) is*

$$\hat{V}^i = \{\beta_1, \beta_2, \alpha_1, \dots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \dots, \zeta_{k-i}, \eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \dots, \xi_{k-i}, \tau\}. \tag{5.93}$$

Furthermore, for $1 \leq i \leq k$

$$e^{-r_0}\tau - \beta_2, \alpha_i + r_i\tau, \tilde{\alpha}_{k+1} \in \hat{V}^\infty \subset \Omega^1(\beta_2, \alpha_1, \dots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \dots, \zeta_{k-i}, \eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \tau). \tag{5.94}$$

Proof. We will prove the lemma by induction. Equation (5.93) is clearly true for $i = 0$. Assuming that it is true for some $1 \leq l < k$, we have

$$\hat{V}^l = \{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \ldots, \zeta_{k-l}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_{k-l}, \tau\}, \tag{5.95}$$

and we want to show that

$$(\hat{V}^l)' = \{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \ldots, \zeta_{k-l-1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_{k-l-1}, \tau\}. \tag{5.96}$$

First, from the structure equations, (5.4), (3.52), and (5.7) we see that

$$\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \zeta_1, \ldots, \zeta_{k-l-1}, \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_{k-l-1}, \tau \in (\hat{V}^l)'. \tag{5.97}$$

Second, from equation (5.53) we have that $\tilde{\alpha}_{k+1} = dr_k$, thus $d\tilde{\alpha}_{k+1} = 0$ and $\tilde{\alpha}_{k+1} \in (\hat{V}^l)'$. Third, using the structure equations (5.4) we compute $\tilde{\eta}_{k+1} = Y(\eta_k)$. From this and equation (5.7) we have that

$$d\tilde{\eta}_{k+1} = dY(\eta_k) = Y(d\eta_k) = 0 \mod \tau, \xi_1, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \tag{5.98}$$

showing that $\tilde{\eta}_{k+1} \in (\hat{V}^l)'$.

It remains to show that if

$$\rho = a\zeta_{k-l} + b\xi_{k-l} \in (\hat{V}^l)', \tag{5.99}$$

then $\rho = 0$. Since $\sigma \notin \hat{V}^l$ it is necessary for $d\rho \in \hat{V}^l$ that

$$X(\rho) = 0 \mod \hat{V}^l. \tag{5.100}$$

Therefore we compute $X(\rho)$

$$X(\rho) = X(a)\zeta_{k-l} + aX(\zeta_{k-l}) + X(b)\xi_{k-l} + bX(\xi_{k-l}). \tag{5.101}$$

Which, using the structure equations, (5.4), is

$$X(\rho) = a\zeta_{k-l+1} + b\xi_{k-l+1} \mod \hat{V}^l. \tag{5.102}$$

We conclude $a = b = 0$, proving equation (5.93).

In order to prove equation (5.94) we observe

$$\begin{aligned} e^{-r_0}\tau - \beta_2 &= \omega^2 \\ \alpha_i + r_i\tau &= dr_{i-1} \quad 1 \le i \le k \\ \tilde{\alpha}_{k+1} &= dr_k. \end{aligned} \tag{5.103}$$

These forms are in $\hat{V}$ and are closed, therefore they are in $\hat{V}^\infty$. Next we let $\rho \in \hat{V}^{k+1}$, that is

$$\rho = b_0\tau + b_1\beta_1 + b_2\beta_2 + \sum_{i=1}^{k}(a_i\alpha_i + c_i\eta_i) + a_{k+1}\tilde{\alpha}_{k+1} + c_{k+1}\tilde{\eta}_{k+1} \in \hat{V}^k, \tag{5.104}$$

and $d\rho \in \hat{V}^k$. Again we use the necessary condition (5.100), and compute $X(\rho)$

$$\begin{aligned} X(\rho) = {} & X(b_0)\tau + X(b_1)\beta_1 + X(b_2)\beta_2 + b_0X(\tau) + b_1X(\beta_1) + b_2X(\beta_2) \\ & + \textstyle\sum_{i=1}^k (X(a_i)\alpha_i + X(c_i)\eta_i) + \sum_{i=1}^k (a_iX(\alpha_i) + c_iX(\eta_i)) + X(a_{k+1})\tilde{\alpha}_{k+1} \\ & + X(c_{k+1})\tilde{\eta}_{k+1} + a_{k+1}X(\tilde{\alpha}_{k+1}) + c_{k+1}X(\tilde{\eta}_{k+1}). \end{aligned} \tag{5.105}$$

Using the structure equations, (5.4), equation (5.98) gives

$$X(\rho) = b_1 e^{-q_0}\zeta_1 \mod \hat{V}^k, \xi_1. \tag{5.106}$$

This shows that $b_1 = 0$, since $\hat{V}^\infty \subset \hat{V}^{k+1}$ we have completed the proof of equation (5.94).

□

Similar arguments give a simple corollary.

Corollary 5.7. *For $0 \leq i \leq k$, the i^{th} derived system of $\hat{W}$ in equation (5.92) is*

$$\hat{W}^i = \{\eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_{k-i}, \tau\}. \tag{5.107}$$

As in the previous sections we will now show Darboux integrability of $\mathcal{E}^{(k)}$ is equivalent to the existence of a form in $\hat{W}^\infty$.

Lemma 5.8. *$\mathcal{E}^{(k)}$ is Darboux integrable if and only if there exists a form, $\rho \in \hat{W}^\infty$ such that $\tilde{V}_{k+1} \lrcorner\, \rho \neq 0$, where $\tilde{V}_{k+1}$ is dual to $\tilde{\eta}_{k+1}$.*

Proof. Assuming $\mathcal{E}^{(k)}$ is Darboux integrable condition (2.7) is

$$\hat{V}^\infty + \check{V} = \Omega^1(\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \ldots, \zeta_k, \tilde{\zeta}_{k+1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_k, \tilde{\xi}_{k+1}, \sigma, \tau). \tag{5.108}$$

This implies there exists a form, $\rho \in \hat{V}^\infty$ such that $\tilde{V}_{k+1} \lrcorner\, \rho \neq 0$. By equation (5.94) in lemma 5.6

$$\rho \in \Omega^1(\beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \tau). \tag{5.109}$$

Using equation (5.94) we can assume

$$\rho \in \Omega^1(\eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \tau) \tag{5.110}$$

Since ρ is semi-basic $\mathfrak{s}^*\rho \in \hat{W}^\infty$ and $\mathfrak{s}^*\rho \neq 0$, where $\mathfrak{s}$ is the cross-section of $U^{(k)}$, $x^1 = x^2 = q_i = r_i = 0$ for $i = 0, \dots, k$. This proves necessity.

To prove sufficiency, we assume that there exists a form $\bar{\rho} \in \hat{W}^\infty$ such that $\tilde{V}_{k+1} \lrcorner\, \bar{\rho} \neq 0$. Pulling this back by the quotient gives a form $\rho = \mathfrak{q}^*\bar{\rho} \in \hat{V}^\infty$ such that $\tilde{V}_{k+1} \lrcorner\, \rho \neq 0$. Using this, the forms in equation (5.94), and the definition of $\check{V}$ in equation (5.91), we conclude that equation (5.108) holds. applying the involution ψ^* in equation (5.65) to equation (5.108) gives

(5.111)

$$\hat{V} + \check{V}^\infty = \Omega^1(\beta_1, \beta_2, \alpha_1, \dots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \dots, \zeta_k, \tilde{\zeta}_{k+1}, \eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \dots, \xi_k, \tilde{\xi}_{k+1}, \sigma, \tau).$$

Therefore condition (2.7) in definition 2.4 is satisfied.

Also, applying the involution ψ^* in equation (5.65) to equation (5.93) gives

$$\check{V}^k = \{\zeta_1, \dots, \zeta_k, \xi_1, \dots, \xi_k, \sigma\}. \tag{5.112}$$

Using $\hat{V}^\infty \subset \hat{V}^k$ and $\check{V}^\infty \subset \check{V}^k$ we get

$$\hat{V}^\infty \cap \check{V}^\infty = 0. \tag{5.113}$$

Therefore $\mathcal{E}^{(k)}$ satisfies condition (2.8) and is Darboux integrable.

□

This lemma allows us to study Darboux integrability of $\mathcal{E}^{(k)}$ by the existence of an intermediate integral for $\bar{\mathcal{E}}^{(k)}$.

Corollary 5.9. *The EDS $\mathcal{E}^{(k)}$ is Darboux integrable if and only if it has a conformally invariant intermediate integral.*

Proof. The pullback of the intermediate integral for $\bar{\mathcal{E}}^{(k)}$ to U is a conformally invariant intermediate integral for $\mathcal{E}^{(k)}$.

□

We will work in the co-frame whose existence was proved in theorem 5.1,

$$\{\xi_1, \ldots, \xi_k, \tilde{\xi}_{k+1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \sigma, \tau, \phi_2^1\}, \tag{5.114}$$

with dual frame

$$\{W_1, \ldots, W_k, \tilde{W}_{k+1}, V_1, \ldots, V_k, \tilde{V}_{k+1}, X, Y, \partial_{\phi_2^1}\}. \tag{5.115}$$

We define the η and ξ order of a function as follows.

Definition 5.10. *For $a \in \mathcal{C}^\infty(\bar{U}^{(k)}, \mathbb{R})$,*

$$\begin{aligned} \mathfrak{o}_\eta(a) &= \mathit{min}\{j \leq k | \tfrac{\partial a}{\partial t_i} = 0 \text{ for all } i > j\} \\ \mathfrak{o}_\xi(a) &= \mathit{min}\{j \leq k | \tfrac{\partial a}{\partial s_i} = 0 \text{ for all } i > j\} \end{aligned} \tag{5.116}$$

The effect of X and Y on these orders is determined in the following lemma.

Lemma 5.11. *Let $a \in \mathcal{C}^\infty(\bar{U}^{(k)}, \mathbb{R})$ with order*

$$\begin{aligned} \mathfrak{o}_\eta(a) &= t \leq k \\ \mathfrak{o}_\xi(a) &= s \leq k, \end{aligned} \tag{5.117}$$

then

$$
\begin{aligned}
\mathfrak{o}_\eta(X(a)) &\leq t \\
\mathfrak{o}_\eta(Y(a)) &\leq t+1 \\
\mathfrak{o}_\xi(X(a)) &\leq s+1 \\
\mathfrak{o}_\xi(Y(a)) &\leq s.
\end{aligned} \tag{5.118}
$$

Proof. We begin by pulling a back by the projection from $\bar{U}^{(k+2)}$ to $\bar{U}^{(k)}$. From the structure equations (5.90) we get the bracket relations on $\bar{U}^{(k+2)}$,

$$
\begin{aligned}
[X, V_i] &= -H_i V_{i+1} && \mathrm{mod}\, \tilde{V}_{k+1}, \tilde{W}_{k+1} \\
[Y, V_i] &= -V_{i-1} + A_i V_i && \mathrm{mod}\, \tilde{V}_{k+1}, \tilde{W}_{k+1} \\
[X, W_i] &= -W_{i-1} + E_i W_i && \mathrm{mod}\, \tilde{V}_{k+1}, \tilde{W}_{k+1} \\
[Y, W_i] &= -K_i W_{i+1} && \mathrm{mod}\, \tilde{V}_{k+1}, \tilde{W}_{k+1}
\end{aligned} \tag{5.119}
$$

for $1 < i \leq k+1$. By equation (5.53) we have $V_i = \partial_{t_{i-1}} \mod \partial_{t_i}, \ldots \partial_{t_{k+2}}$, thus for $k \geq T > t+1$, $k \geq S > s+1$ we have

$$
\begin{aligned}
V_T(a) &= 0 \\
W_S(a) &= 0.
\end{aligned} \tag{5.120}
$$

Using this and equation (5.119) gives

$$
\begin{aligned}
V_T(X(a)) &= -H_T V_{T+1}(a) + X(V_T(a)) &&= 0 \\
V_{T+1}(Y(a)) &= -V_T(a) + A_{T+1} V_{T+1}(a) + Y(V_{T+1}(a)) &&= 0 \\
W_{S+1}(X(a)) &= -W_S(a) + E_{S+1} W_{S+1}(a) + X(W_{S+1}(a)) &&= 0 \\
W_S(Y(a)) &= -K_S W_{S+1}(a) + Y(W_S(a)) &&= 0.
\end{aligned} \tag{5.121}
$$

This completes the proof.

□

To find necessary and sufficient conditions that $\hat{W}$ be non-trivial we pull back by the projection from $\bar{U}^{(k+2)}$ to $\bar{U}^{(k)}$. The pullback of $\hat{W}^k$ is a subsystem of $\Omega^1(\eta_1, \ldots, \eta_{k+1}, \tau)$. Any form in the pullback of $\hat{W}^\infty$ must be X-invariant, and when $H_0, \ldots, H_k \neq 0$, the X-invariant forms in $\Omega^1(\eta_1, \ldots, \eta_{k+1}, \tau)$ must have a τ term.

Proposition 5.12. *Let $H_0, \ldots, H_k \neq 0$ and $\rho \in \Omega^1(\eta_1, \ldots, \eta_{k+1}, \tau)$ on $U^{(k+2)}$, such that $X(\rho) = 0$, then $Y \lrcorner \rho \neq 0$.*

Proof. The proof shall be by contradiction, thus we assume that $Y \lrcorner \rho = 0$. We can now write ρ as

$$\rho = \sum_{i=1}^{k+1} b_i \eta_i. \tag{5.122}$$

We calculate $X(\rho)$ to be

$$X(\rho) = \sum_{i=1}^{k+1} (X(b_i)\eta_i + b_i X(\eta_i)). \tag{5.123}$$

By the structure equations, (5.90), this is

$$X(\tilde{\rho}) = X(b_1)\eta_1 + b_1 H_0 \xi_1 + \sum_{i=2}^{k+1} (X(b_i)\eta_i + b_i H_{i-1}\eta_{i-1}). \tag{5.124}$$

The interior product of W_1 with ρ then gives

$$W_1 \lrcorner X(\rho) = H_0 b_1 = 0. \tag{5.125}$$

Since $H_0 \neq 0$, if $X(\rho) = 0$ then $b_1 = 0$. In this manner, the interior product of $V_1, \ldots, W_k$ with $X(\rho)$ will give $b_2 = \cdots = b_{k+1} = 0$. This proves the claim, and we have $Y \lrcorner \rho \neq 0$.

□

If we assume the existence of a form ρ in the pullback of $\hat{W}^\infty$ and set $\Sigma = \tau - \rho/(Y \lrcorner \rho)$ then $\Sigma \in \Omega^1(\eta_1, \ldots, \eta_{k+1})$ on $\bar{U}^{(k+2)}$ such that

$$d(\tau - \Sigma) \wedge (\tau - \Sigma) = 0, \tag{5.126}$$

and thus

$$X(\tau - \Sigma) = 0. \tag{5.127}$$

We now show that the existence of such a Σ is equivalent to the vanishing of an invariant, F_{k+2}, where

$$F_1 = -\frac{1}{\sin(\delta)} \qquad \text{and} \qquad F_i = -X\left(\frac{F_{i-1}}{H_{i-2}}\right) \qquad 2 \leq i \leq k+2 \tag{5.128}$$

on $\bar{U}^{(k+2)}$.

Theorem 5.13. *Let $H_0, \ldots, H_k \neq 0$ and $\Sigma \in \Omega^1(\eta_1, \ldots, \eta_{k+1})$ on $\bar{U}^{(k+2)}$, then*

$$X(\Sigma) = \frac{1}{\sin(\delta)}\xi_1 \tag{5.129}$$

if and only if

$$F_{k+2} = 0. \tag{5.130}$$

Proof. Let

$$\Sigma = \sum_{i=1}^{k} g_i \xi_i + \sum_{i=1}^{k+1} f_i \eta_i. \tag{5.131}$$

Using the structure equations (5.90) on $\bar{U}^{(k+2)}$ gives

$$\begin{aligned} X(\Sigma) &= \sum_{i=1}^{k+1} X(f_i)\eta_i + f_1 H_0 \xi_1 + \sum_{i=2}^{k+1} f_i H_{i-1}\eta_{i-1} \\ &= \sum_{i=1}^{k} X(f_i)\eta_i + X(f_{k+1})\eta_{k+1} + f_1 H_0 \xi_1 + \sum_{i=1}^{k} f_{i+1} H_i \eta_i \\ &= f_1 H_0 \xi_1 + \sum_{i=1}^{k} (X(f_i) + f_{i+1} H_i)\eta_i + X(f_{k+1})\eta_{k+1}. \end{aligned}$$

Setting this equal to $\frac{1}{\sin(\delta)}\xi_1$, as in equation (5.129), gives necessary conditions

$$\begin{aligned} f_{i+1} H_i &= -X(f_i) \quad 1 \leq i \leq k \\ X(f_{k+1}) &= 0 \\ f_1 H_0 &= \tfrac{1}{\sin(\delta)}. \end{aligned} \tag{5.132}$$

These are clearly sufficient as well, and setting

$$F_i = -f_i H_{i-1}, \tag{5.133}$$

they become

$$\begin{aligned} F_{i+1} &= -X\left(\tfrac{F_i}{H_{i-1}}\right) \qquad 1 \leq i \leq k \\ -F_{k+2} &= X\left(\tfrac{F_{k+1}}{H_k}\right) = 0 \\ F_1 &= -\tfrac{1}{\sin(\delta)}. \end{aligned} \tag{5.134}$$

This gives the definition (5.128) and the condition (5.130), completing the proof.

□

However, we have not addressed the case where the lowest integer p such that $H_p = 0$ is less than or equal to k. We will address this in the following theorem, but first prove a technical lemma on the order and relation of the invariants.

Lemma 5.14. *For $i \geq 1$,*

$$\begin{aligned} \mathfrak{o}_\eta(H_i) &\leq i+1 \\ \mathfrak{o}_\xi(H_i) &\leq i-1 \\ \mathfrak{o}_\eta(A_{i+1}) &\leq i+2 \\ \mathfrak{o}_\xi(A_{i+1}) &\leq i-1 \\ \mathfrak{o}_\eta(F_i) &\leq i-1 \\ \mathfrak{o}_\xi(F_i) &\leq i-1 \end{aligned} \tag{5.135}$$

on $\bar{U}^{(k+2)}$. Furthermore,

$$V_{i+2}(H_i) = F_{i+1} \qquad 1 \leq i \leq k \tag{5.136}$$

Proof. Equation (5.135) is true for $i = 1$, and equation (5.118) completes the inductive step, proving equation (5.135). The proof of equation (5.136) is also by induction, and is true for $i = 1$. We then assume it is true for $l - 1$, that is

$$V_{l+1}(H_{l-1}) = F_l. \tag{5.137}$$

From equation (5.6) we get

$$H_l = H_{l-1} + X(A_{l-1}) - X(Y(\log H_{l-1})). \tag{5.138}$$

Thus

$$V_{l+2}(H_l) = V_{l+2}(H_{l-1}) + V_{l+2}(X(A_{l-1})) - V_{l+2}(X(Y(\log H_{l-1}))). \tag{5.139}$$

By equation (5.135), $V_{l+2}(H_{l-1}) + V_{l+2}(X(A_{l-1})) = 0$, giving

$$\begin{aligned} V_{l+2}(H_l) &= -V_{l+2}(X(Y(\log H_{l-1}))) \\ &= [X, V_{l+2}](Y(\log H_{l-1})) - X(V_{l+2}(Y(\log H_{l-1}))). \end{aligned} \tag{5.140}$$

From the bracket relations, equation (5.119),

$$\begin{aligned} V_{l+2}(H_l) &= -X(V_{l+2}(Y(\log H_{l-1}))) \\ &= X([Y, V_{l+2}](\log H_{l-1}) - Y(V_{l+2}(\log H_{l-1}))). \end{aligned} \tag{5.141}$$

Again using (5.119), and the orders from equation (5.135), we have

$$\begin{aligned} V_{l+2}(H_l) &= -X(V_{l+2}(Y(\log H_{l-1}))) \\ &= X((-V_{l+1} + A_{l+2}V_{l+2})(\log H_{l-1})) \\ &= -X(V_{l+1}(\log H_{l-1})). \end{aligned} \tag{5.142}$$

Last we use the inductive assumption from equation (5.138), giving

$$V_{l+2}(H_l) = -X\left(\frac{F_l}{H_{l-1}}\right) = F_{l+1}. \tag{5.143}$$

This completes the inductive step, proving equation (5.136).

□

From equation (5.136) see that $H_k = 0$ implies that $F_{k+1} = 0$, we will show that the converse is also true.

Theorem 5.15. *The invariant $H_k = 0$ if and only if $F_{k+1} = 0$.*

Proof. We need only prove sufficiency, thus we let $F_{k+1} = 0$. From theorem 5.13 we get

$$\Sigma = -\sum_{i=1}^{k} \frac{F_i}{H_{i-1}} \eta_i \tag{5.144}$$

such that $X(\tau - \Sigma) = 0$. We will see that $Y(\tau - \Sigma) \in \Omega^1(\eta_1, \ldots, \eta_{k+1}, \tau)$ with $X(Y(\tau - \Sigma))$ and $Y \lrcorner Y(\tau - \Sigma) = 0$. Start by computing $Y(\tau - \Sigma)$,

$$Y(\tau - \Sigma) = Y(\tau) - Y(\Sigma). \tag{5.145}$$

From the structure equations (3.52) we get

$$Y(\tau) = \frac{\cos(\delta)}{\sin(\delta)} \eta_1, \tag{5.146}$$

and from equation (5.144) and the structure equations (5.90),

$$Y(\Sigma) \in \Omega^1(\eta_1, \ldots, \eta_{k+1}). \tag{5.147}$$

This gives

$$Y(\tau - \Sigma) \in \Omega^1(\eta_1, \ldots, \eta_{k+1}), \tag{5.148}$$

therefore $Y \lrcorner Y(\tau - \Sigma) = 0$. Using equation (5.61) we see that

$$X(Y(\tau - \Sigma)) = Y(X(\tau - \Sigma)) = 0. \tag{5.149}$$

Now if $H_0, \ldots, H_k \neq 0$ then $Y(\tau - \Sigma)$ contradicts proposition 5.12. Therefore there exists an integer $0 \leq p \leq k$ such that $H_p = 0$. Finally the definition (5.6) gives us that $H_i = 0$ for all $i \geq p$.

$\square$

Since the Darboux integrability of $\mathcal{E}^{(k-1)}$ implies that $\mathcal{E}^{(k)}$ is Darboux integrable, the interesting case is when $F_1, \ldots, F_{k+1} \neq 0$, and $F_{k+2} = 0$. This justifies the assumption that $H_0, \ldots, H_k \neq 0$. The next proposition will study these conditions.

Proposition 5.16. *Let* $H_0, \ldots, H_k \neq 0$, $\Sigma \in \Omega^1(\eta_1, \ldots, \eta_{k+1})$ *on* $\bar{U}^{(k+2)}$, *then*

$$d(\tau - \Sigma) \wedge (\tau - \Sigma) = 0 \tag{5.150}$$

if and only if the condition (5.129) holds.

Proof. By the structure equations, (5.90) on $\bar{U}^{(k+2)}$, we have

$$d\Sigma = \sigma \wedge \nu + \tau \wedge \gamma + \Omega, \tag{5.151}$$

where

$$\nu = X(\Sigma), \qquad \gamma = Y(\Sigma), \tag{5.152}$$

and

$$\begin{aligned} \nu &\in \Omega^1(\xi_1, \eta_1, \ldots, \eta_{k+1}), \\ \gamma &\in \Omega^1(\eta_1, \ldots, \eta_{k+2}), \\ \Omega &\in \Omega^2(\xi_1, \ldots, \xi_{k+2}, \tilde{\xi}_{k+3}, \eta_1, \ldots, \eta_{k+2}, \tilde{\eta}_{k+3}). \end{aligned} \tag{5.153}$$

Using equation (3.52) gives

$$
\begin{aligned}
d(\tau - \Sigma) &= -\tfrac{1}{\sin(\delta)}\xi_1 \wedge \sigma - \tfrac{\cos(\delta)}{\sin(\delta)}\eta_1 \wedge \tau - \sigma \wedge \nu - \tau \wedge \gamma - \Omega \\
&= \sigma \wedge (\tfrac{1}{\sin(\delta)}\xi_1 - \nu) + \tau \wedge (\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma) - \Omega \\
&= \sigma \wedge (\tfrac{1}{\sin(\delta)}\xi_1 - \nu) + \Sigma \wedge (\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma) - \Omega \bmod \tau - \Sigma.
\end{aligned} \tag{5.154}
$$

Thus the necessary and sufficient conditions are equation (5.129) and

$$
\Omega = \Sigma \wedge \left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right). \tag{5.155}
$$

We will now show that equation (5.129) implies equation (5.155). Assuming equation (5.129) gives

$$
d\Sigma = \frac{1}{\sin(\delta)}\sigma \wedge \xi_1 + \tau \wedge \gamma + \Omega. \tag{5.156}
$$

By proposition 5.16,

$$
\Sigma = -\sum_{i=1}^{k+1} \frac{F_i}{H_{i-1}}\eta_i, \tag{5.157}
$$

and by equation (5.135), this refines equation (5.153) so that

$$
\Omega \in \Omega^2(\xi_1, \ldots, \xi_{k+1}, \eta_1, \ldots, \eta_{k+1}) \tag{5.158}
$$

Using equations (3.52), (5.90), (5.156) and $d(\delta) = \eta_1 - \xi_1 - s_1\sigma + t_1\tau$ gives

(5.159)

$$
\begin{aligned}
d^2\Sigma &= d\left(\tfrac{1}{\sin(\delta)}\right) \wedge \sigma \wedge \xi_1 + \tfrac{1}{\sin(\delta)}d\sigma \wedge \xi_1 - \tfrac{1}{\sin(\delta)}\sigma \wedge d\xi_1 + d\tau \wedge \gamma - \tau \wedge d\gamma + d\Omega, \\
0 &= -\tfrac{\cos(\delta)}{\sin^2(\delta)}(\eta_1 + t_1\tau) \wedge \sigma \wedge \xi_1 + \tfrac{1}{\sin^2(\delta)}\eta_1 \wedge \tau \wedge \xi_1 - \tfrac{s_1}{\sin^2(\delta)}\sigma \wedge \tau \wedge \eta_1 \\
&\quad + (\tfrac{1}{\sin(\delta)}\sigma \wedge \xi_1 + \tfrac{\cos(\delta)}{\sin(\delta)}\tau \wedge \eta_1) \wedge \gamma - \tau \wedge d\gamma + d\Omega,
\end{aligned}
$$

solving for $d\Omega$ gives

$$
\begin{aligned}
d\Omega &= \tfrac{\cos(\delta)}{\sin^2(\delta)}(\eta_1 + t_1\tau)\wedge\sigma\wedge\xi_1 - \tfrac{1}{\sin^2(\delta)}\eta_1\wedge\tau\wedge\xi_1 + \tfrac{s_1}{\sin^2(\delta)}\sigma\wedge\tau\wedge\eta_1 \\
&\quad -(\tfrac{1}{\sin(\delta)}\sigma\wedge\xi_1 + \tfrac{\cos(\delta)}{\sin(\delta)}\tau\wedge\eta_1)\wedge\gamma + \tau\wedge d\gamma \\
&= \sigma\wedge\left(\tfrac{\cos(\delta)}{\sin^2(\delta)}\xi_1\wedge\eta_1 - \tfrac{t_1\cos(\delta)}{\sin^2(\delta)}\tau\wedge\xi_1 + \tfrac{s_1}{\sin^2(\delta)}\tau\wedge\eta_1 - \tfrac{1}{\sin(\delta)}\xi_1\wedge\gamma\right) \\
&\quad +\tau\wedge\left(\tfrac{1}{\sin^2(\delta)}\eta_1\wedge\xi_1 - \tfrac{\cos(\delta)}{\sin(\delta)}\eta_1\wedge\gamma + d\gamma\right).
\end{aligned} \tag{5.160}
$$

Using $X \lrcorner\, \Omega = 0$ and $X \lrcorner\, \gamma$ from equation (5.158) we compute

$$X(\Omega) = \frac{\cos(\delta)}{\sin^2(\delta)}\xi_1\wedge\eta_1 - \frac{t_1\cos(\delta)}{\sin^2(\delta)}\tau\wedge\xi_1 + \frac{s_1}{\sin^2(\delta)}\tau\wedge\eta_1 - \frac{1}{\sin(\delta)}\xi_1\wedge\gamma - \tau\wedge X(\gamma). \tag{5.161}$$

By equation (5.152),

$$X(\gamma) = X(Y(\Sigma)) \tag{5.162}$$

Using equation (5.61) and $\tilde{V}_{k+3}(\Sigma) = \tilde{W}_{k+3}(\Sigma) = 0$ gives

$$
\begin{aligned}
X(\gamma) &= Y(X(\Sigma)) + [X, Y](\Sigma) \\
&= Y(X(\Sigma)).
\end{aligned} \tag{5.163}
$$

Substituting equation (5.129) gives

$$X(\gamma) = Y\left(\frac{1}{\sin(\delta)}\xi_1\right). \tag{5.164}$$

Finally, using the structure equations, (5.90), we have

$$X(\gamma) = -\frac{t_1\cos(\delta)}{\sin^2(\delta)}\xi_1 + \frac{s_1}{\sin^2(\delta)}\eta_1. \tag{5.165}$$

Substituting into equation (5.160) gives

$$X(\Omega) = \frac{\cos(\delta)}{\sin^2(\delta)}\xi_1 \wedge \eta_1 - \frac{1}{\sin(\delta)}\xi_1 \wedge \gamma. \tag{5.166}$$

We also have

(5.167)

$$\begin{aligned} X\left(\Sigma \wedge \left(\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right)\right) &= X(\Sigma) \wedge \left(\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right) + \Sigma \wedge \left(X\left(\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1\right) - X(\gamma)\right) \\ &= \tfrac{\cos(\delta)}{\sin^2(\delta)}\xi_1 \wedge \eta_1 - \tfrac{1}{\sin(\delta)}\xi_1 \wedge \gamma + \Sigma \wedge \left(X\left(\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1\right) - X(\gamma)\right) \\ &= X(\Omega) + \Sigma \wedge \left(X\left(\tfrac{\cos(\delta)}{\sin(\delta)}\eta_1\right) - X(\gamma)\right) \end{aligned}$$

The structure equations (5.90) also give

$$X\left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1\right) = \frac{s_1}{\sin^2(\delta)}\eta_1 - \frac{t_1\cos(\delta)}{\sin^2(\delta)}\xi_1 = X(\gamma). \tag{5.168}$$

Now we have

$$X\left(\Omega - \Sigma \wedge \left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right)\right) = 0. \tag{5.169}$$

By equations (5.153), (5.158), and the definition of Σ

$$\Omega - \Sigma \wedge \left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right) \in \Omega^2(\xi_1, \ldots, \xi_{k+1}, \eta_1, \ldots, \eta_k, \eta_{k+2}). \tag{5.170}$$

Using lemma 5.17 completes the proof.

□

Lemma 5.17. *Let $H_0, \ldots, H_k \neq 0$ and $\Omega \in \Omega^2(\xi_1, \ldots, \xi_{k+1}, \eta_1, \ldots, \eta_{k+2})$ on $\bar{U}^{(k+2)}$. If $X(\Omega) = 0$ then $\Omega = 0$.*

Proof. We begin by writing

$$\Omega = \sum_{i=1}^{k+1} (\mu_i \wedge \xi_i + \mu_{k+i+1} \wedge \eta_i) \tag{5.171}$$

where

$$\begin{array}{lcll} \mu_i & \in & \Omega^1(\xi_1, \dots, \xi_{i-1}, \eta_1, \dots, \eta_{k+2}) & 1 \leq i \leq k+1, \\ \mu_{k+i+1} & \in & \Omega^1(\eta_{i+1}, \dots, \eta_{k+2}) & 1 \leq i \leq k+1. \end{array} \tag{5.172}$$

Using this and the structure equations, (5.90), we have

$$\begin{array}{lcll} X(\mu_i) & \in & \Omega^1(\xi_1, \dots, \xi_i, \eta_1, \dots, \eta_{k+2}) & 1 \leq i \leq k+1, \\ X(\mu_{k+i+1}) & \in & \Omega^1(\eta_i, \dots, \eta_{k+2}) & 1 < i \leq k+1. \end{array} \tag{5.173}$$

From equation (5.171) we get,

$$X(\Omega) = \sum_{i=1}^{k+1} (X(\mu_i) \wedge \xi_i + \mu_i \wedge X(\xi_i) + X(\mu_{k+i+1}) \wedge \eta_i + \mu_{k+i+1} \wedge X(\eta_i)) \tag{5.174}$$

Substituting the structure equations for $\bar{U}^{(k+2)}$, (5.90), this becomes

$$\begin{aligned} X(\Omega) = \sum_{i=1}^{k+1} (X(\mu_i) \wedge \xi_i + \mu_i \wedge (\xi_{i+1} - E_i \xi_i) + X(\mu_{k+i+1}) \wedge \eta_i) \\ + H_0 \mu_{k+2} \wedge \xi_1 + \sum_{i=2}^{k+1} H_{i-1} \mu_{k+i+1} \wedge \eta_{i-1}. \end{aligned} \tag{5.175}$$

Since W_{k+2} is dual to ξ_{k+2}, by equations (5.172) and (5.173) we get $W_{k+2} \lrcorner\, \mu_i = 0$ and $W_{k+2} \lrcorner\, X(\mu_i) = 0$ for $1 \leq i \leq 2k+2$. Knowing this, we compute the interior product of W_{k+2} with $X(\Omega)$. From equation (5.175) to get

$$W_{k+2} \lrcorner\, X(\Omega) = \mu_{k+1}. \tag{5.176}$$

If $X(\Omega) = 0$ then $\mu_{k+1} = 0$. Continuing in this manner, the interior product of $W_{k+1}, \ldots, W_2$ with $X(\Omega)$ gives $\mu_{k+1} = \cdots = \mu_1 = 0$. Equation (5.171) now becomes

$$\Omega = \sum_{i=1}^{k+1} \mu_{k+i+1} \wedge \eta_i \tag{5.177}$$

and equation (5.175) becomes

$$X(\Omega) = \sum_{i=1}^{k+1} X(\mu_{k+i+1}) \wedge \eta_i + H_0 \mu_{k+2} \wedge \xi_1 + \sum_{i=2}^{k+1} H_{i-1} \mu_{k+i+1} \wedge \eta_{i-1}. \tag{5.178}$$

Once again equations (5.172), (5.173) give that $W_1 \lrcorner\, \mu_i = 0$ and $W_1 \lrcorner\, X(\mu_i) = 0$ for $k+1 \leq i \leq 2k+2$. Computing the interior product of W_1 with $X(\Omega)$ from equation (5.178) gives

$$W_1 \lrcorner\, X(\Omega) = -H_0 \mu_{k+2}. \tag{5.179}$$

Since $H_0 \neq 0$, if $X(\Omega) = 0$ then $\mu_{k+2} = 0$. Continuing inductively, the interior products of $V_1, \ldots, V_k$ with $X(\Omega)$ gives $\mu_{k+3} = \cdots = \mu_{2k+2} = 0$. Therefore, if $X(\Omega) = 0$ then $\Omega = 0$.

□

We note that when p is the lowest integer such that $H_p = 0$ since $F_{p+1} = 0$ proposition 5.16 can be applied to the case $k = p - 1$. This will be used in the next theorem which proves the assertion that $F_{k+2} = 0$ is equivalent to Darboux integrability of $\mathcal{E}^{(k)}$.

Theorem 5.18. *The non-negative integer k is the smallest integer such that $\mathcal{E}^{(k)}$ is Darboux integrable if and only if k is the smallest integer such that $F_{k+2} = 0$.*

Proof. To prove sufficiency, assume

$$F_1, \ldots, F_{k+1} \neq 0 \tag{5.180}$$

and

$$F_{k+2} = 0. \tag{5.181}$$

By theorem 5.15, equation (5.180) implies

$$H_0, \ldots, H_k \neq 0. \tag{5.182}$$

Using Σ guaranteed by and proposition 5.16, equation (5.157), let

$$\begin{aligned} \tilde{\rho} &= H_k(\tau - \Sigma) \\ &= H_k\tau + H_k \sum_{i=1}^{k} \frac{F_i}{H_{i-1}} \eta_i + F_{k+1}\eta_{k+1}. \end{aligned} \tag{5.183}$$

If $\iota : U^{(k)} \to U^{(k+2)}$, is the inclusion, $s_{k+1} = s_{k+2} = t_{k+1} = t_{k+2} = 0$. Then $\rho = \iota^*\tilde{\rho}$ is a form in $\hat{W}^\infty$, and $\tilde{V}_{k+1} \lrcorner\, \rho = F_{k+1}$. By lemma 5.8 we have that $\mathcal{E}^{(k)}$ is Darboux integrable. To prove that k is the lowest integer such that $\mathcal{E}^{(k)}$ is Darboux integrable we shall assume to the contrary that there exists $l < k$, the lowest integer such that $\mathcal{E}^{(l)}$ is Darboux integrable. We now wish to show that $F_{l+2} = 0$, but first we must show that $H_0, \ldots, H_l \neq 0$. We shall prove this by contradiction as well, assuming that there exists $p \leq l$, the lowest integer such that $H_p = 0$. By equation (5.138) this implies that $F_{p+1} = 0$, and as above $\mathcal{E}^{(p-1)}$ is Darboux integrable. This violates equation (5.180) and the assumption that l is the lowest integer such that $\mathcal{E}^{(l)}$ is Darboux integrable, giving $H_0, \ldots, H_l \neq 0$. By lemma 5.8 there exists a non-zero closed form ρ in the l^{th} derived system of $\hat{W}$

$$\hat{W}^l = \{\eta_1, \ldots, \eta_l, \tilde{\eta}_{l+1}, \tau\}$$

on $\bar{U}^{(l)}$. Pulling ρ back by the projection from $\bar{U}^{(l+2)}$ to $\bar{U}^{(l)}$ gives a non-zero closed form

$$\tilde{\rho} \in \Omega^1(\eta_1, \ldots, \eta_{l+1}, \tau). \tag{5.184}$$

By proposition 5.12 we have $Y \lrcorner\, \rho \neq 0$, thus we let

$$\Sigma = \tau - \tilde{\rho}/(Y \lrcorner\, \tilde{\rho}). \tag{5.185}$$

By equation (5.184),

$$\Sigma \in \Omega^1(\eta_1, \ldots, \eta_{l+1}) \tag{5.186}$$

$$\tau - \Sigma = \tilde{\rho}/(Y \lrcorner\, \tilde{\rho}). \tag{5.187}$$

Since $\tilde{\rho}$ is closed,

$$d(\tau - \Sigma) \wedge (\tau - \Sigma) = d\left(\tilde{\rho}/(Y \lrcorner\, \tilde{\rho})\right) \wedge \left(\tilde{\rho}/(Y \lrcorner\, \tilde{\rho})\right) = 0. \tag{5.188}$$

Proposition 5.16 now implies that $F_{l+2} = 0$. This proves necessity as well, completing the proof. $\square$

In the thesis [3], the 2×2 matrices of Laplace invariants, $\mathbf{H}_i$ and $\mathbf{K}_i$, are found for a general system of two PDE's in the plane. It is found that a necessary condition for Darboux integrability of the system is

$$\det \mathbf{H}_p = 0 \qquad \text{and} \qquad \det \mathbf{K}_q = 0,$$

for some $p, q \geq 0$. For F-Gordon systems, including the system given by equations (3.1) and (3.2), it is found that the conditions

$$\mathbf{H}_p = 0 \qquad \text{and} \qquad \mathbf{K}_q = 0,$$

for some $p, q \geq 0$, are sufficient for Darboux integrability of the system. In contrast, by adapting the coframe to the conformal symmetry of the equations, we are able to find conditions for Darboux integrability which are both necessary and sufficient.

5.4 Darboux Integrability of $\bar{\mathcal{E}}^{(k)}$

In this section we show the equivalence of Darboux integrability of $\mathcal{E}^{(k)}$ and Darboux integrability of the prolongation of its quotient $\bar{\mathcal{E}}^{(k+1)}$.

Theorem 5.19. *The non-negative integer k is the smallest integer such that $\mathcal{E}^{(k)}$ is Darboux integrable if and only if k is the smallest integer such that $\bar{\mathcal{E}}^{(k+1)}$ is Darboux integrable.*

Before proving this theorem we establish a condition that $\bar{\mathcal{E}}^{(k)}$ be Darboux integrable.

Theorem 5.20. *The non-negative integer k is the smallest integer such that $\bar{\mathcal{E}}^{(k)}$ is Darboux integrable if and only if k is the smallest integer such that $H_k = 0$.*

Proof. We begin by showing $\bar{\mathcal{E}}^{(k)}$ is Darboux integrable implies that $H_k = 0$. If $\bar{\mathcal{E}}^{(k)}$ is Darboux integrable then condition (2.7) gives

$$\hat{W}^\infty + \check{W} = \Omega^1(\xi_1, \dots, \xi_k, \tilde{\xi}_{k+1}, \eta_1, \dots, \eta_k, \tilde{\eta}_{k+1}, \sigma, \tau). \tag{5.189}$$

By the definition of $\check{W}$ (5.92) there must exist $\rho_1, \rho_2 \in \hat{W}^\infty$ such that $Y \lrcorner \rho_1 \neq 0$ and $\tilde{V}_{k+1} \lrcorner \rho_2 \neq 0$. Without loss of generality we can assume that $Y \lrcorner \rho_2 = 0$. Since $\rho_2 \in \hat{W}^\infty$ it must be that $X(\rho_2) = 0$. If $H_0, \dots, H_k \neq 0$ then pulling ρ_2 back by the projection from $\bar{U}^{k+2}$ to $\bar{U}^{(k)}$ gives a form contradicting proposition 5.12. Thus there exists an integer $0 \leq p \leq k$ such that $H_p = 0$ implying that $H_k = 0$ also.

To prove sufficiency let $H_0, \dots, H_{k-1} \neq 0$ and $H_k = 0$. This implies $F_{k+1} = 0$, and by proposition 5.16 there exists a form on $\bar{U}^{(k+2)}$

$$\Sigma = -\sum_{i=1}^{k} \frac{F_i}{H_{i-1}} \eta_i, \tag{5.190}$$

such that

$$d(\tau - \Sigma) = (\tau - \Sigma) \wedge \left(\frac{\cos(\delta)}{\sin(\delta)} \eta_1 - \gamma \right), \tag{5.191}$$

where

$$\begin{aligned} \gamma &= Y(\Sigma) \\ &= -\sum_{i=1}^{k} Y\left(\frac{F_i}{H_{i-1}}\right) \eta_i - \sum_{i=1}^{k} \frac{F_i}{H_{i-1}} Y(\eta_i). \end{aligned} \tag{5.192}$$

Using the structure equations in (5.90) this becomes

(5.193)

$$
\begin{aligned}
\gamma &= -\sum_{i=1}^{k} Y\left(\frac{F_i}{H_{i-1}}\right)\eta_i - \sum_{i=1}^{k}\frac{F_i}{H_{i-1}}(\eta_{i+1} - A_i\eta_i)\\
&= -\sum_{i=1}^{k} Y\left(\frac{F_i}{H_{i-1}}\right)\eta_i - \sum_{i=1}^{k}\frac{F_i}{H_{i-1}}\eta_{i+1} + \sum_{i=1}^{k}\frac{A_iF_i}{H_{i-1}}\eta_i\\
&= -Y\left(\frac{F_1}{H_0}\right)\eta_1 - \sum_{i=2}^{k} Y\left(\frac{F_i}{H_{i-1}}\right)\eta_i - \sum_{i=2}^{k}\frac{F_{i-1}}{H_{i-2}}\eta_i - \frac{F_k}{H_{k-1}}\eta_{k+1} + \frac{A_1F_1}{H_0}\eta_1 + \sum_{i=2}^{k}\frac{A_iF_i}{H_{i-1}}\eta_i\\
&= \left(\frac{A_1F_1}{H_0} - Y\left(\frac{F_1}{H_0}\right)\right)\eta_1 + \sum_{i=2}^{k}\left(\frac{A_iF_i}{H_{i-1}} - Y\left(\frac{F_i}{H_{i-1}}\right) - \frac{F_{i-1}}{H_{i-2}}\right)\eta_i - \frac{F_k}{H_{k-1}}\eta_{k+1}.
\end{aligned}
$$

Taking the exterior derivative of equation (5.191) gives

$$
0 = d(\tau - \Sigma)\wedge d\left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right) - (\tau - \Sigma)\wedge d\left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right). \tag{5.194}
$$

Substituting equation (5.191) we get

$$
0 = -(\tau - \Sigma)\wedge d\left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right). \tag{5.195}
$$

Therefore

$$
d\left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right) = 0 \mod \tau - \Sigma. \tag{5.196}
$$

If $\iota : U^{(k)} \to U^{(k+2)}$ is the inclusion $s_{k+1} = s_{k+2} = t_{k+1} = t_{k+2} = 0$, then $\rho_1 = \iota^*(\tau - \Sigma)$ and $\rho_2 = \iota^*\left(\frac{\cos(\delta)}{\sin(\delta)}\eta_1 - \gamma\right)$ are forms in $\hat{W}^\infty$, and $Y \lrcorner\, \rho_1 = 1$, $\tilde{V}_{k+1} \lrcorner\, \rho_2 = \frac{F_k}{H_{k-1}}$. Using this and the definition of $\check{W}$ in equation (5.92) we get that equation (5.189) is satisfied. Viewing $\bar{U}^{(k)}$ as the cross-section $x^1 = x^2 = q_0 = \cdots = q_k = r_0 = \cdots = r_k = 0$, it is ψ-invariant. Therefore the involution ψ can be restricted to an involution of $\bar{U}^{(k)}$, $\tilde{\psi}$.

Pulling equation (5.189) back by $\tilde{\psi}$ gives

$$\hat{W} + \check{W}^{\infty} = \Omega^1(\xi_1, \ldots, \xi_k, \tilde{\xi}_{k+1}, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \sigma, \tau). \tag{5.197}$$

Also the pullback of the k^{th} derived system of $\hat{W}$ given in equation (5.107) by $\tilde{\psi}$ is

$$\check{W}^k = \{\xi_1, \ldots, \xi_k, \sigma\}. \tag{5.198}$$

Using $\hat{W}^{\infty} \subset \hat{W}^k$ and $\check{W}^{\infty} \subset \check{W}^k$ we get

$$\hat{W}^{\infty} \cap \check{W}^{\infty} = 0. \tag{5.199}$$

Therefore $\bar{\mathcal{E}}^{(k)}$ satisfies condtion (2.8) and is Darboux integrable. In fact k is the smallest integer such that $\bar{\mathcal{E}}^{(k)}$ is Darboux integrable. If $\bar{\mathcal{E}}^{(l)}$ is Darboux integrable for $l < k$ then $H_l = 0$, contradicting our assumption.

For necessity we assume that k is the smallest integer such that $\bar{\mathcal{E}}^{(k)}$ is Darboux integrable. We know that this implies $H_k = 0$. From the sufficiency proof, if there exists l such that $0 \leq l < k$, and l is the smallest integer such that $H_l = 0$ then $\bar{\mathcal{E}}^{(l)}$ is Darboux integrable. This contradicts our assumption, and therefore k is the smallest integer such that $H_k = 0$.

□

We can now prove theorem 5.19.

Proof. By theorem 5.18 k is the smallest integer such that $\mathcal{E}^{(k)}$ is Darboux integrable is equivalent to k being the smallest integer such that $F_{k+2} = 0$. By theorem 5.15 this is equivalent to k being the smallest integer such that $H_{k+1} = 0$. Finally, by theorem

5.20 this is equivalent to k being the smallest integer such that $\bar{\mathcal{E}}^{(k+1)}$ is Darboux integrable.

$\square$

CHAPTER 6

CALCULATION OF THE GENERALIZED LAPLACE INVARIANTS

Using the machinery developed in chapter 5, we reproduce the results of chapters 3 and 4. We also show that at the second prolongation there are no new metrics with Darboux integrable wave maps.

By the construction of the coframe in theorem 5.1 we have

$$\begin{array}{rclllll} d(\delta) & = & -s_1\sigma & +t_1\tau & +\eta_1-\xi_1, \\ ds_i & = & s_{i+1}\sigma & +Y(s_i)\tau & \mod \eta_1,\xi_1,\ldots,\eta_{i+1},\xi_{i+1}, \\ dt_i & = & X(t_i)\sigma & +t_{i+1}\tau & \mod \eta_1,\xi_1,\ldots,\eta_{i+1},\xi_{i+1}. \end{array} \tag{6.1}$$

Using this we compute

$$\begin{array}{rcl} X(\delta) & = & -s_1, \\ Y(\delta) & = & t_1, \\ X(s_i) & = & s_{i+1}, \\ Y(s_i) & = & Y(X^i(\delta)), \\ & = & X^i(Y(\delta)), \\ & = & X^i(t_1), \\ X(t_i) & = & -X(Y^i(\delta)), \\ & = & -Y^i(X(\delta)), \\ & = & Y^i(s_1), \\ Y(t_i) & = & t_{i+1}. \end{array} \tag{6.2}$$

From equations (5.6) and (5.128) we have the invariants on $U^{(1)}$

$$A_0 = 0, \tag{6.3}$$

$$H_0 = -\frac{t_1}{\sin(\delta)}, \tag{6.4}$$

$$F_1 = -\frac{1}{\sin(\delta)}. \tag{6.5}$$

6.1 0^{th} Prolongation and the Invariant F_2

Using the definitions in equations (5.6) and (5.128) we compute the invariants on $U^{(2)}$. This gives

$$\begin{aligned} A_1 &= -Y(H_0)/H_0 \\ &= -\frac{t_2}{t_1} + \frac{t_1\cos(\delta)}{\sin(\delta)}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} H_1 &= H_0 K_0 + X(A_1) \\ &= \frac{\sin(\delta)(\kappa_\tau t_1 - t_2\kappa)}{{t_1}^2}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} F_2 &= -X\left(\frac{F_1}{H_0}\right), \\ &= -\frac{\kappa\sin(\delta)}{{t_1}^2}. \end{aligned} \tag{6.8}$$

This gives the result of theorem 3.4. The EDS $\mathcal{E}$ is Darboux integrable if and only if $\kappa = 0$.

6.2 1^{st} Prolongation and the Invariant F_3

Again using the definitions in equations (5.6) and (5.128) we compute the invariants on $U^{(3)}$.

$$A_2 = A_1 - Y(H_1)/H_1$$

(6.9)
$$= \frac{{t_1}^2\kappa_{\tau\tau} + {t_2}^2\kappa - t_2t_1\kappa_\tau + t_2t_1\kappa^3\sin(\delta) - t_3t_1\kappa}{t_1(t_2\kappa - t_1\kappa_\tau)},$$

$$H_2 = H_1 + X(A_2)$$

(6.10)
$$= t_1(t_3\kappa_\sigma\kappa_\tau - t_3\kappa^3\cos(\delta) - t_3\kappa\kappa_{\sigma\tau} + t_2\kappa\kappa_{\sigma\tau\tau} - t_2\kappa_\sigma\kappa_{\tau\tau} + 3t_2\kappa^2\kappa_\tau\cos(\delta) + t_1\kappa^2\cos(\delta)\kappa_{\tau\tau}$$
$$- 3t_1\kappa{\kappa_\tau}^2\cos(\delta) + t_1\kappa_{\sigma\tau}\kappa_{\tau\tau} - {t_1}^2\kappa^2\kappa_\tau\sin(\delta) - t_1\kappa_\tau\kappa_{\sigma\tau\tau})/(t_2\kappa - t_1\kappa_\tau)^2,$$

$$F_3 = -X\left(\frac{F_2}{H_1}\right)$$

(6.11)
$$= \frac{t_1(\kappa_\sigma\kappa_\tau - \kappa^3\cos(\delta) - \kappa\kappa_{\sigma\tau})}{(t_2\kappa - t_1\kappa_\tau)^2},$$

where

$$\kappa_{\sigma\tau} = \kappa_{11}\cos(s_0)\cos(t_0) + \kappa_{12}\sin(s_0 + t_0) + \kappa_{22}\sin(s_0)\sin(t_0), \tag{6.12}$$

$$\kappa_{\tau\tau} = \kappa_{11}\cos^2(t_0) + \kappa_{12}\sin(2t_0) + \kappa_{22}\sin^2(t_0) + t_1(\kappa_1\sin(t_0) - \kappa_2\cos(t_0)), \tag{6.13}$$

(6.14)

$$\kappa_{\sigma\tau\tau} = -\left(\kappa_{111}\cos(t_0)^2\cos(s_0) + \kappa_{112}\left(\cos(t0)\cos(s_0+t_0) + \frac{1}{2}\cos(s_0)\sin(2t_0)\right)\right.$$
$$\left. + \kappa_{122}\left(\sin(t_0)\cos(s_0+t_0) + \frac{1}{2}\sin(s_0)\sin(2t_0)\right) + \kappa_{222}\sin^2(t_0)\sin(s_0)\right)$$
$$- t_1(\kappa_{11}\cos(s_0)\sin(t_0) - \kappa_{12}\cos(s_0+t_0) - \kappa_{22}\sin(s_0)\cos(t_0))$$
$$+ \kappa\left(\frac{1}{2}\kappa_1\sin(s_0)\sin(2t_0) + \kappa_2\cos(t_0)\sin(\delta)\right).$$

From equation (6.11) the condition that $\mathcal{E}^{(1)}$ is Darboux integrable is

$$\kappa_\sigma\kappa_\tau - \kappa^3\cos(\delta) - \kappa\kappa_{\sigma\tau} = 0. \tag{6.15}$$

We have

$$(\kappa_\sigma\kappa_\tau - \kappa^3\cos(\delta) - \kappa\kappa_{\sigma\tau})|_{s0=0,t_0=0} = -\kappa^3 + {\kappa_1}^2 - \kappa\kappa_{11} \tag{6.16}$$
$$(\kappa_\sigma\kappa_\tau - \kappa^3\cos(\delta) - \kappa\kappa_{\sigma\tau})|_{s0=0,t_0=\frac{\pi}{2}} = \kappa_1\kappa_2 - \kappa\kappa_{12} \tag{6.17}$$
$$(\kappa_\sigma\kappa_\tau - \kappa^3\cos(\delta) - \kappa\kappa_{\sigma\tau})|_{s0=\frac{\pi}{2},t_0=\frac{\pi}{2}} = -\kappa^3 + {\kappa_2}^2 - \kappa\kappa_{22} \tag{6.18}$$

Since substituting (4.44) into F_3 gives 0 we have the result of theorem 4.5. The EDS $\mathcal{E}^{(1)}$ is Darboux integrable if and only if the conditions (4.44) hold.

6.3 2^{nd} Prolongation and the Invariant F_4

On $U^{(4)}$ we use the definition (5.128) to compute the invariant

$$F_4 = -X\left(\frac{F_3}{H_2}\right) \tag{6.19}$$

$F_4 = (((-\kappa_{\sigma\tau}\kappa^4\cos(\delta) - \kappa^6 + \kappa_\sigma\kappa_\tau\cos(\delta)\kappa^3)s_1 - 3\kappa_\sigma^2\kappa_\tau\kappa^2\sin(\delta) + \kappa_{\sigma\sigma}\kappa_\tau\kappa^3\sin(\delta) +$

$3\kappa_{\sigma\tau}\kappa_\sigma\kappa^3\sin(\delta)-\kappa_{\sigma\sigma\tau}\kappa^4\sin(\delta))t_1+(-\kappa^4\sin(-\delta)\kappa_{\sigma\tau\tau}-\kappa^3\sin(\delta)\kappa_{\tau\tau}\kappa_\sigma-3\kappa^3\kappa_{\sigma\tau}\kappa_\tau\sin(\delta)+$
$3\kappa^2\kappa_\sigma\kappa_\tau^2\sin(\delta))s_1-\kappa_{\sigma\sigma\tau}\kappa\kappa_{\tau\tau}\kappa_\sigma-\kappa_\sigma\kappa_{\sigma\tau}\kappa_\tau\kappa^2\cos(\delta)+\kappa_{\sigma\sigma\tau\tau}\kappa_\sigma\kappa_\tau\kappa+3\kappa_{\sigma\tau\tau}\cos(\delta)\kappa^3\kappa_\sigma+$
$\kappa_{\sigma\sigma}\cos(\delta)\kappa^3\kappa_{\tau\tau}-3\kappa_{\sigma\sigma}\kappa_\tau^2\kappa^2\cos(\delta)+5\kappa_\sigma^2\kappa_\tau^2\kappa\cos(\delta)-3\kappa^2\cos(\delta)\kappa_{\tau\tau}\kappa_\sigma^2+\kappa_{\sigma\sigma}\kappa_{\sigma\tau}\kappa\kappa_{\tau\tau}+$
$3\kappa_{\sigma\sigma\tau}\kappa_\tau\kappa^3\cos(\delta)+\kappa_{\sigma\sigma\tau}\kappa_{\sigma\tau\tau}\kappa^2-\kappa_{\sigma\sigma}\kappa_\tau\kappa_{\sigma\tau\tau}\kappa-\kappa_{\sigma\sigma\tau\tau}\cos(\delta)\kappa^4+4\kappa_\sigma\kappa^4\cos(\delta)^2\kappa_\tau-\kappa_{\sigma\sigma\tau\tau}\kappa_{\sigma\tau}\kappa^2$
$+\kappa^4\kappa_\sigma\kappa_\tau-4\kappa_{\sigma\tau}^2\kappa^3\cos(\delta)-4\kappa_{\sigma\tau}\cos(\delta)^2\kappa^5-\kappa^5\kappa_{\sigma\tau}-\kappa^7\cos(\delta))t_2+((\kappa_{\sigma\tau}\kappa^3\kappa_\tau\cos(\delta)-$
$\kappa_\sigma\kappa_\tau^2\kappa^2\cos(\delta)+\kappa^5\kappa_\tau)s_1-\kappa_{\sigma\sigma}\kappa_\tau^2\kappa^2\sin(\delta)+3\kappa_\sigma^2\kappa_\tau^2\kappa\sin(\delta)+\kappa_{\sigma\sigma\tau}\kappa_\tau\kappa^3\sin(\delta)$
$-3\kappa_\sigma\kappa_{\sigma\tau}\kappa_\tau\kappa^2\sin(\delta))t_1^2+((3\kappa^2\kappa_{\sigma\tau}\kappa_\tau^2\sin(\delta)+\kappa^2\kappa_\sigma\kappa_\tau\kappa_{\tau\tau}\sin(\delta)-3\kappa\kappa_\sigma\kappa_\tau^3\sin(\delta)$
$-\kappa^3\sin(\delta)\kappa_\tau\kappa_{\sigma\tau\tau})s_1+3\kappa_{\sigma\sigma}\kappa_\tau^3\kappa\cos(\delta)-\kappa_{\sigma\sigma}\kappa_\tau\kappa^2\kappa_{\tau\tau}\cos(\delta)+4\kappa_{\sigma\tau}^2\kappa^2\kappa_\tau\cos(\delta)-\kappa_{\sigma\sigma}\kappa_\tau\kappa_{\tau\tau}\kappa_{\sigma\tau}+$
$\kappa^6\cos(\delta)\kappa_\tau-\kappa_{\sigma\sigma\tau\tau}\kappa_\sigma\kappa_\tau^2+\kappa^4\kappa_{\sigma\tau}\kappa_\tau-\kappa^3\kappa_\sigma\kappa_\tau^2+\kappa_{\sigma\sigma}\kappa_\tau^2\kappa_{\sigma\tau\tau}-3\kappa_{\sigma\tau\tau}\kappa_\sigma\kappa_\tau\kappa^2\cos(\delta)-$
$5\kappa_\sigma^2\kappa_\tau^3\cos(\delta)-3\kappa_{\sigma\sigma\tau}\kappa_\tau^2\kappa^2\cos(\delta)-\kappa_{\sigma\sigma\tau}\kappa\kappa_\tau\kappa_{\sigma\tau\tau}+4\kappa_{\sigma\tau}\cos^2(\delta)\kappa^4\kappa_\tau+\kappa_{\sigma\sigma\tau}\kappa_\sigma\kappa_\tau\kappa_{\tau\tau}+$
$3\kappa_\sigma^2\kappa_\tau\kappa\kappa_{\tau\tau}\cos(\delta)+\kappa_{\sigma\tau}\kappa_\sigma\kappa_\tau^2\kappa\cos(\delta)+\kappa_{\sigma\sigma\tau\tau}\kappa_{\sigma\tau}\kappa\kappa_\tau+\kappa_{\sigma\sigma\tau\tau}\cos(\delta)\kappa^3\kappa_\tau-4\kappa_\sigma\kappa^3\cos(\delta)^2\kappa_\tau^2)t_1$
$/((-\kappa_\sigma\kappa_\tau+\kappa_{\sigma\tau}\kappa+\kappa^3\cos(\delta))t_3+(-3\kappa^2\cos(\delta)\kappa_\tau+\kappa_{\tau\tau}\kappa_\sigma+\kappa^3t_1\sin(\delta)-\kappa_{\sigma\tau\tau}\kappa)t_2-$
$t_1^2\kappa_\tau\kappa^2\sin(\delta)+(\kappa_\tau\kappa_{\sigma\tau\tau}+3\kappa_\tau^2\kappa\cos(\delta)-\kappa^2\kappa_{\tau\tau}\cos(\delta)-\kappa_{\tau\tau}\kappa_{\sigma\tau})t_1)^2,$

where

$$\kappa_{\sigma\sigma}=\kappa_{11}\cos^2(s_0)+\kappa_{12}\sin(2s_0)+\kappa_{22}\sin^2(s_0)+s_1(\kappa_1\sin(s_0)-\kappa_2\cos(s_0)), \tag{6.20}$$

$\kappa_{\sigma\sigma\tau}=(-\kappa_{11}\cos(t_0)\sin(s_0)-\kappa_{12}\sin(t_0)\sin(s_0)+\cos(s_0)\kappa_{12}\cos(t_0)+$
$\kappa_{22}\sin(t_0)\cos(s_0))s_1+(-\sin(s_0-t_0)\kappa_1\sin(s_0)+\sin(s_0-t_0)\kappa_2\cos(s_0)-$
$2\cos(s_0)\sin(s_0)\cos(t_0)\kappa_2+\sin(s_0)^2\cos(t_0)\kappa_1)\kappa-\cos(s_0)^2\kappa_{111}\cos(t_0)-$
$2\cos(s_0)\sin(s_0)\cos(t_0)\kappa_{112}-\cos(s_0)^2\kappa_{112}\sin(t_0)-2\cos(s_0)\sin(s_0)\kappa_{122}\sin(t_0)-$
$\sin(s_0)^2\cos(t_0)\kappa_{122}-\sin(s_0)^2\kappa_{222}\sin(t_0),$
$\kappa_{\sigma\sigma\tau\tau}=3\kappa_2^2\sin(s_0)\cos(t_0)\cos(s_0)\sin(t_0)-2\kappa_{1222}\sin(t_0)\cos(t_0)\cos(s_0)^2-$
$2\kappa_{1222}\cos(t_0)^2\sin(s_0)\cos(s_0)+((\kappa_1\cos(s_0)\sin(t_0)\cos(t_0)+\kappa_2\sin(s_0)\sin(t_0)\cos(t_0)-$
$\kappa_2\cos(s_0)\cos(t_0)^2)\kappa+2\kappa_{112}\sin(s_0)\sin(t_0)\cos(t_0)+\kappa_{122}\sin(s_0)-$
$\kappa_{112}\cos(s_0)\cos(t_0)^2-\kappa_{122}\sin(s_0)\cos(t_0)^2-2\kappa_{122}\cos(s_0)\sin(t_0)\cos(t_0)+$

$\kappa_{111}\sin(s_0)\cos(t_0)^2 - \kappa_{222}\cos(s_0) + \kappa_{222}\cos(s_0)\cos(t_0)^2)s_1 - \kappa_{2222}\cos(t_0)^2 +$

$((\kappa_{11}\sin(t_0)\sin(s_0) - \sin(s_0)\kappa_{12}\cos(t_0) - \cos(s_0)\kappa_{12}\sin(t_0) + \kappa_{22}\cos(t_0)\cos(s_0))s_1 +$

$(\kappa_2\sin(t_0)\sin(s_0)\cos(s_0) - \kappa_2\cos(t_0)\cos(s_0)^2 + \kappa_1\cos(t_0)\sin(s_0)\cos(s_0))\kappa -$

$\kappa_{112}\cos(t_0)\cos(s_0)^2 - 2\kappa_{122}\cos(t_0)\sin(s_0)\cos(s_0) - \kappa_{122}\sin(t_0)\cos(s_0)^2 +$

$\kappa_{122}\sin(t_0) - \cos(t_0)\kappa_{222} + \kappa_{222}\cos(t_0)\cos(s_0)^2 + \kappa_{111}\sin(t_0)\cos(s_0)^2 +$

$2\kappa_{112}\sin(t_0)\sin(s_0)\cos(s_0))t_1 + \kappa_{1122}\cos(s_0)^2 + \kappa_{1122}\cos(t_0)^2 + \kappa_{2222} -$

$2\kappa_2^2\cos(t_0)^2\cos(s_0)^2 - \kappa_1\kappa_2\sin(s_0)\cos(s_0) - 2\kappa_{1122}\cos(t_0)^2\cos(s_0)^2 -$

$\kappa_1\kappa_2\sin(t_0)\cos(t_0) + 2\kappa_{1222}\sin(s_0)\cos(s_0) + 2\kappa_{1222}\sin(t_0)\cos(t_0) +$

$\kappa_{1111}\cos(t_0)^2\cos(s_0)^2 + \kappa_{2222}\cos(t_0)^2\cos(s_0)^2 + 2\kappa_1\kappa_2\cos(t_0)^2\sin(s_0)\cos(s_0) +$

$2\kappa_1\kappa_2\sin(t_0)\cos(t_0)\cos(s_0)^2 - \kappa_{2222}\cos(s_0)^2 + 2\kappa_{1112}\sin(t_0)\cos(t_0)\cos(s_0)^2 +$

$2\kappa_{1112}\cos(t_0)^2\sin(s_0)\cos(s_0) - \kappa_1^2\sin(s_0)\cos(t_0)\cos(s_0)\sin(t_0) +$

$4\kappa_{1122}\sin(s_0)\cos(t_0)\cos(s_0)\sin(t_0) + (-2\cos(s_0)\kappa_{12}\sin(s_0) -$

$4\kappa_{11}\sin(s_0)\cos(t_0)\cos(s_0)\sin(t_0) + 2\kappa_{22}\cos(s_0)^2 + 6\kappa_{12}\sin(t_0)\cos(t_0)\cos(s_0)^2 +$

$6\kappa_{12}\cos(t_0)^2\sin(s_0)\cos(s_0) + 2\kappa_{22}\cos(t_0)^2 - 2\cos(t_0)\kappa_{12}\sin(t_0) -$

$4\kappa_{22}\cos(t_0)^2\cos(s_0)^2 + 4\kappa_{22}\sin(s_0)\cos(t_0)\cos(s_0)\sin(t_0))\kappa + \kappa_2^2\cos(s_0)^2 + \kappa_2^2\cos(t_0)^2.$

The condition that $\mathcal{E}^{(2)}$ is Darboux integrable is $F_4 = 0$. We begin the analysis by defining,

$$
\begin{aligned}
e_{11} &= \kappa\kappa_{11} - {\kappa_1}^2 + \kappa^3 \\
e_{12} &= \kappa\kappa_{12} - \kappa_1\kappa_2 \\
e_{22} &= \kappa\kappa_{22} - {\kappa_2}^2 + \kappa^3
\end{aligned}
\tag{6.21}
$$

and

(6.22)

$$
\begin{aligned}
\tilde{F}_4 = F_4((&-\kappa_\sigma\kappa_\tau + \kappa_{\sigma\tau}\kappa + \kappa^3\cos(\delta))t_3 + (-3\kappa^2\cos(\delta)\kappa_\tau + \kappa_{\tau\tau}\kappa_\sigma + \kappa^3 t_1\sin(\delta) - \kappa_{\sigma\tau\tau}\kappa)t_2 \\
&- t_1^2\kappa_\tau\kappa^2\sin(\delta) + (\kappa_\tau\kappa_{\sigma\tau\tau} + 3\kappa_\tau^2\kappa\cos(\delta) - \kappa^2\kappa_{\tau\tau}\cos(\delta) - \kappa_{\tau\tau}\kappa_{\sigma\tau})t_1)^2.
\end{aligned}
$$

By equation (6.19) $\tilde{F}_4 = 0$ if and only if $F_4 = 0$. We calculate the the coefficient of $t_2 t_1 s_1$ in $\tilde{F}_4$ to be

$$[\tilde{F}_4]_{t_2 t_1 s_1} = 8({e_{12}}^2 - e_{11} e_{22}). \tag{6.23}$$

We will now consider two cases.

Case 1: $e_{22} \neq 0$.

In this case we we solve equation (6.23) for e_{11} and get

$$e_{11} = \frac{{e_{12}}^2}{e_{22}}. \tag{6.24}$$

After substituting this equation and all of its derivatives, and defining

(6.25)

$$\begin{aligned}
e_{221} &= \partial_{\phi^1} e_{22} &&= \kappa_{122}\kappa^2 - 2\kappa_2 e_{12} + \kappa_1(e_{22} - {\kappa_2}^2 + \kappa^3) \\
e_{222} &= \partial_{\phi^2} e_{22} &&= \kappa_{222}\kappa^2 - \kappa_2(e_{22} + {\kappa_2}^2 - 4\kappa^3) \\
e_{2221} &= \partial_{\phi^1} e_{222} &&= \kappa_{1222}\kappa^3 - \kappa_2 e_{122} + 2\kappa_1 e_{222} - e_{12}(e_{22} + 3{\kappa_2}^2 - \kappa^3) \\
& && \quad + \kappa_1\kappa_2(e_{22} - {\kappa_2}^2 + 4\kappa^3) \\
e_{2222} &= \partial_{\phi^2} e_{222} &&= \kappa_{2222}\kappa^3 + \kappa_2 e_{222} - e_{22}(e_{22} + 2{\kappa_2}^2 - 5\kappa^3) - {\kappa_2}^2({\kappa_2}^2 - 11\kappa^3) - 4\kappa^6,
\end{aligned}$$

the coefficient of $t_1 s_1$ and $t_2 \sin(s_0)\sin(t_0)$ in $\tilde{F}_4$ are

$$[\tilde{F}_4]_{t_1 s_1} = 8\frac{\kappa_2(e_{12}e_{222} - 2\kappa_2 e_{22} e_{12} - e_{22}e_{221} + 2\kappa_1 {e_{22}}^2)}{\kappa^2} = 0 \tag{6.26}$$

and

$$[\tilde{F}_4]_{t_2 \sin(s_0)\sin(t_0)} = 8\frac{\kappa_2 e_{222} e_{22} - e_{2222} e_{22} - \kappa^3 {e_{22}}^2 + {e_{222}}^2}{\kappa^2} = 0. \tag{6.27}$$

Assuming $\kappa_2 = 0$ gives that

$$[\tilde{F}_4]_{t_2 \sin(s_0)\sin(t_0)} = -8\kappa^7 = 0, \tag{6.28}$$

which implies that $e_{22} = 0$ contradicting our assumption; thus we get

$$e_{221} = \frac{e_{12}e_{222} - 2\kappa_2 e_{22}e_{12} + 2\kappa_1 {e_{22}}^2}{e_{22}} \tag{6.29}$$

and

$$e_{2222} = \frac{\kappa_2 e_{222}e_{22} - \kappa^3 {e_{22}}^2 + {e_{222}}^2}{e_{22}}. \tag{6.30}$$

Substituting these gives that the coefficient of $t_2\cos(s_0)\cos(t_0)$ and $t_2\sin(s_0)\cos^2(t_0)\sin(t_0)$ in $\tilde{F}_4$ are

$$[\tilde{F}_4]_{t_2\cos(s_0)\cos(t_0)} = -96\frac{e_{12}(e_{2221}{e_{22}}^2 - e_{222}(e_{222}e_{12} - 2\kappa_2 e_{12}e_{22} + 3\kappa_1 {e_{22}}^2) + \kappa^3 e_{12}{e_{22}}^2)}{\kappa^2 {e_{22}}^2} \tag{6.31}$$

and

$$\begin{aligned}[\tilde{F}_4]&_{t_2\sin(s_0)\cos^2(t_0)\sin(t_0)} \\ &= -8\frac{4e_{2221}e_{12}{e_{22}}^2 - e_{222}(-8\kappa_2 {e_{12}}^2 e_{22} + 4e_{222}{e_{12}}^2 + 12\kappa_1 e_{12}{e_{22}}^2) + \kappa^3 {e_{22}}^2(5{e_{12}}^2 + {e_{22}}^2)}{\kappa^2 {e_{22}}^2}.\end{aligned} \tag{6.32}$$

These are both zero if and only if $e_{12} = e_{22} = 0$. This is a contradiction, thus we move on to the next case.

Case 2: $e_{22} = 0$.

Equation (6.23) forces $e_{12} = 0$, giving that the coefficient of $t_2 \cos^3(s_0)\cos(t_0)$ of $\tilde{F}_4$ is

$$[\tilde{F}_4]_{t_2 \cos^3(s_0)\cos(t_0)} = -8\kappa {e_{11}}^2. \tag{6.33}$$

Thus $e_{11} = 0$ as well, showing that all metrics with Darboux integrable wave-maps on the 3-jets are Darboux integrable at a lower jet level.

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APPENDICES

APPENDIX A
THE ORTHONORMAL FRAME BUNDLES

Let P, Q be pseudo-Riemannian manifolds of dimension n, s, with metrics g, h, of signature (n_1, n_2), (s_1, s_2). The energy Lagrangian will be integrated over P, in order to do this, P must be an oriented manifold. Therefore we assume that P is oriented with the canonical volume form ν for g. Choose $\tilde{g} \in M_n(\mathbb{R})$, symmetric of signature (n_1, n_2), and $\tilde{h} \in M_s(\mathbb{R})$, symmetric of signature (s_1, s_2). These will be the normal forms of g, h in our formulation.

A.1 The Frame Bundle for P

The oriented orthonormal frame bundle for P is

(A.1)

$$\mathcal{F}(P) = \{(x, u) | x \in P, u : \mathbb{R}^n \to T_x P, g(u(e_i), u(e_j)) = \tilde{g}_{ij}, \nu(u(e_1), \ldots, u(e_n)) = 1\}$$

where u is a linear isomorphism, and e_i is the standard basis for $\mathbb{R}^n$. The map u provides a basis of tangent vectors, $u(e_i)$. With this in mind, the restrictions on u are that g be in its normal form, and that it preserves the orientation, ν.

The special orthogonal group,

$$SO(n_1, n_2) = \{a \in GL(n) | \tilde{g}_{ij} a^i_k a^j_l = \tilde{g}_{kl}, \det(a) = 1\}, \tag{A.2}$$

acts on $\mathcal{F}(P)$ on the right by,

$$a * (x, u) = (x, ua). \tag{A.3}$$

To show that this is well defined, we compute

$$g(ua(e_i), ua(e_j)) = g(u(a^i_k e_i), u(a^j_l e_j)). \tag{A.4}$$

Using linearity of u, and bilinearity of g, this becomes

$$g(ua(e_i), ua(e_j)) = g(u(e_i), u(e_j))a^i_k a^j_l. \tag{A.5}$$

By the definition of $\mathcal{F}(P)$, (A.1), this is

$$g(ua(e_i), ua(e_j)) = \tilde{g}_{ij} a^i_k a^j_l. \tag{A.6}$$

By the definition of $SO(n_1, n_2)$, (A.2), this is

$$g(ua(e_i), ua(e_j)) = \tilde{g}_{kl}. \tag{A.7}$$

Similarly,

$$\nu(ua(e_1), \ldots, ua(e_n)) = \det(a)\nu(u(e_1), \ldots, u(e_n)) = \nu(u(e_1), \ldots, u(e_n)). \tag{A.8}$$

Therefore $(x, ua) \in \mathcal{F}(P)$.

There is a global coframe of $\mathcal{F}(P)$, [9], composed of the $\mathbb{R}^n$-valued canonical form

$$\tilde{\omega} = u^{-1} \circ \left(\pi_P^{\mathcal{F}(P)}\right)_*, \tag{A.9}$$

and the $\mathfrak{so}(n_1, n_2)$-valued Levi-Civita connection form $\tilde{\boldsymbol{\varpi}}$. The structure equations for this frame are

$$
\begin{aligned}
d\tilde{\omega} &= -\tilde{\boldsymbol{\varpi}} \wedge \tilde{\omega} \\
d\tilde{\boldsymbol{\varpi}} &= -\tilde{\boldsymbol{\varpi}} \wedge \tilde{\boldsymbol{\varpi}} + \tilde{\boldsymbol{\Omega}}
\end{aligned}
\tag{A.10}
$$

where $\tilde{\boldsymbol{\Omega}}$ is the $\mathfrak{so}(n_1, n_2)$-valued curvature 2-form for $\tilde{\boldsymbol{\varpi}}$.

A.2 The Frame Bundle for Q

The orthonormal frame bundle for Q is

$$\mathcal{F}(Q) = \{(y, v) | y \in Q, v : \mathbb{R}^s \to T_yQ, h(v(e_\alpha), v(e_\beta)) = \tilde{h}_{\alpha\beta}\} \tag{A.11}$$

where v is a linear isomorphism, and e_α is the standard basis for $\mathbb{R}^s$. Again, the map, v provides a basis of tangent vectors, $u(e_i)$, where h is in its chosen normal form.

The orthogonal group,

$$O(s_1, s_2) = \{b \in GL(s) | \tilde{h}_{\alpha\beta} b^\alpha_\gamma b^\beta_\mu = \tilde{h}_{\gamma\mu}\}. \tag{A.12}$$

acts on $\mathcal{F}(Q)$ on the right by,

$$b * (y, v) = (y, vb). \tag{A.13}$$

To show that this is well defined, we compute

$$h(vb(e_\alpha), vb(e_\beta)) = h(v(b^\alpha_\gamma e_\alpha), v(b^\beta_\mu e_\beta)). \tag{A.14}$$

Using linearity of v, and bilinearity of h, this becomes

$$h(vb(e_\alpha), vb(e_\beta)) = h(v(e_\alpha), v(e_\beta))b^\alpha_\gamma b^\beta_\mu. \tag{A.15}$$

By the definition of $\mathcal{F}(Q)$, (A.11), this is

$$h(vb(e_\alpha), vb(e_\beta)) = \tilde{h}_{\alpha\beta} b^\alpha_\gamma b^\beta_\mu. \tag{A.16}$$

By the definition of $O(s_1, s_2)$, (A.12), this is

$$h(vb(e_\alpha), vb(e_\beta)) = \tilde{h}_{\gamma\mu}. \tag{A.17}$$

Therefore $(y, vb) \in \mathcal{F}(Q)$.

Similarly there is a global coframe of $\mathcal{F}(Q)$ composed of the $\mathbb{R}^s$-valued canonical form

$$\tilde{\phi} = v^{-1} \circ \left(\pi_Q^{\mathcal{F}(Q)}\right)_*, \tag{A.18}$$

and the $\mathfrak{o}(s_1, s_2)$-valued Levi-Civita connection form $\tilde{\boldsymbol{\varphi}}$. The structure equations for this frame are

$$\begin{aligned} d\tilde{\phi} &= -\tilde{\boldsymbol{\varphi}} \wedge \tilde{\phi} \\ d\tilde{\boldsymbol{\varphi}} &= -\tilde{\boldsymbol{\varphi}} \wedge \tilde{\boldsymbol{\varphi}} + \tilde{\boldsymbol{\Phi}} \end{aligned} \tag{A.19}$$

where $\tilde{\boldsymbol{\Phi}}$ is the $\mathfrak{o}(s_1, s_2)$-valued curvature 2-form for $\tilde{\boldsymbol{\varphi}}$.

APPENDIX B
EDS QUOTIENTS WITHOUT TRANSVERSALITY

B.1 Pfaffian Systems

Let G act regularly on a manifold, M, with global cross-section $\mathfrak{s} : M/G \to M$. Suppose also that G is a symmetry group of a Pfaffian system, $\mathcal{I}$. If Γ is the integrable distribution corresponding to the action of G on M, then the semi-basic 1-forms are just those forms in $\Gamma^{\perp}$, and the semi-basic k-forms are $\Omega^k(\Gamma^{\perp})$.

If we let

$$\mathcal{A}^k_{sb} = \mathcal{I} \cap \Omega^k(\Gamma^{\perp}), \tag{B.1}$$

then we get that the differential ideal,

$$\mathcal{I}_{\mathcal{A}} = \langle \mathcal{A}^1_{sb}, \mathcal{A}^2_{sb}, \dots, \rangle, \tag{B.2}$$

can be pulled back by the cross-section, $\mathfrak{s}$, to get the quotient system. That is $\bar{\mathcal{I}} = \mathfrak{s}^* \mathcal{I}_{\mathcal{A}}$ [1].

In [1] they study quotients under a transversality condition,

$$I^{\perp} \cap \Gamma = 0. \tag{B.3}$$

The EDS studied in this thesis does not satisfy this condition, thus we must prove a sufficiency condition that the quotient system, $\bar{\mathcal{I}}$, be Pfaffian without transversality. We start by building a coframe adapted to our situation.

Proposition B.1. *There exists a coframe,* $\{\beta^{i_1}, \theta^{i_2}, \omega^{i_3}, \alpha^{i_4}, \eta^{i_5}, \phi^{i_6}\}$ *such that*

$$\mathcal{I} = \langle \beta^{i_1}, \theta^{i_2}, \alpha^{i_4}, \eta^{i_5} \rangle, \tag{B.4}$$

and

$$\Gamma^{\perp} = \Omega^1(\beta^{i_1}, \theta^{i_2}, \omega^{i_3}), \tag{B.5}$$

where structure equations are

$$\begin{aligned} d\beta^{i_1} &= 0 && \mod I, \\ d\alpha^{i_4} &= 0 && \mod I, \\ d\theta^{i_2} &= A^{i_2}_{j_2 k_2} \omega^{j_2} \wedge \omega^{k_2} && \mod I, \\ d\eta^{i_5} &= B^{i_5}_{j_5 k_5} \omega^{j_5} \wedge \omega^{k_5} + C^{i_5}_{l_5 m_5} \omega^{l_5} \wedge \phi^{m_5} + D^{i_5}_{p_5 q_5} \phi^{p_5} \wedge \phi^{q_5} && \mod I, \end{aligned} \tag{B.6}$$

with the appropriate restriction of the indices.

Proof. First, choose a coframe, $\{\beta^{i_1}\}$, $1 \leq i_1 \leq n_1$ for $I' \cap \Gamma^{\perp}$, so that

$$I' \cap \Gamma^{\perp} = \{\beta^{i_1}\} \tag{B.7}$$

Second, extend $\{\beta^{i_1}\}$ to a coframe, $\{\beta^{i_1}, \theta^{i_2}\}$, $1 \leq i_2 \leq n_2$ for $I \cap \Gamma^{\perp}$, so that

$$I \cap \Gamma^{\perp} = \{\beta^{i_1}, \theta^{i_2}\}. \tag{B.8}$$

Third, extend $\{\beta^{i_1}, \theta^{i_2}\}$ to a coframe, $\{\beta^{i_1}, \theta^{i_2}, \omega^{i_3}\}$, $1 \leq i_3 \leq n_3$ for $\Gamma^{\perp}$, so that

$$\Gamma^{\perp} = \{\beta^{i_1}, \theta^{i_2}, \omega^{i_3}\}. \tag{B.9}$$

Fourth, extend $\{\beta^{i_1}\}$ to a coframe, $\{\beta^{i_1}, \alpha^{i_4}\}$, $1 \leq i_4 \leq n_4$ for $\mathcal{I}'$, so that

$$\mathcal{I}' = \{\beta^{i_1}, \alpha^{i_4}\}. \tag{B.10}$$

Fifth, extend $\{\beta^{i_1}, \theta^{i_2}, \alpha^{i_4}\}$ to a coframe, $\{\beta^{i_1}, \theta^{i_2}, \alpha^{i_4}, \eta^{i_5}\}$, $1 \leq i_5 \leq n_5$ for $\mathcal{I}$, so that

$$\mathcal{I} = \{\beta^{i_1}, \theta^{i_2}, \alpha^{i_4}, \eta^{i_5}\}. \tag{B.11}$$

Last, extend $\{\beta^{i_1}, \theta^{i_2}, \alpha^{i_4}, \eta^{i_5}\}$ to a coframe, $\{\beta^{i_1}, \theta^{i_2}, \omega^{i_3}, \alpha^{i_4}, \eta^{i_5}, \phi^{i_6}\}$, $1 \leq i_6 \leq n_6$ for $\Omega^1(M)$, so that

$$\Omega^1(M) = \{\beta^{i_1}, \theta^{i_2}, \omega^{i_3}, \alpha^{i_4}, \eta^{i_5}, \phi^{i_6}\}. \tag{B.12}$$

Since, by equation (B.10), $\beta^{i_1}, \alpha^{i_4} \in \mathcal{I}'$, we get the structure equations

$$\begin{aligned} d\beta^i &= 0 \mod I, \\ d\alpha^j &= 0 \mod I. \end{aligned} \tag{B.13}$$

By equation (B.8), $\theta^{i_2} \in \Gamma^\perp$. Since $\Gamma^\perp$ is integrable we have that $d\theta^{i_2} = 0 \mod \Gamma^\perp$. Using that $\beta^{i_1}, \theta^{i_2} \in \mathcal{I}$ from equation (B.8), and $\beta^{i_1}, \theta^{i_2}, \omega^{i_3}$ span $\Gamma^\perp$ from equation (B.9), we get

$$d\theta^i = A^i_{jk} \omega^j \wedge \omega^k \mod I. \tag{B.14}$$

Finally, since $\beta^{i_1}, \theta^{i_2}, \alpha^{i_4}, \eta^{i_5} \in \mathcal{I}$ from equation (B.11), and $\beta^{i_1}, \theta^{i_2}, \omega^{i_3}, \alpha^{i_4}, \eta^{i_5}, \phi^{i_6}$ spans $\Omega^1(M)$ by equation (B.12),

$$d\eta^i = B^i_{jk}\omega^j \wedge \omega^k + C^i_{lm}\omega^l \wedge \phi^m + D^i_{pq}\phi^p \wedge \phi^q \mod I. \tag{B.15}$$

□

Here we present two examples where the quotient is not Pfaffian.

Example B.2. Let M be a dimension 3 manifold with coframe $\{\omega^1, \omega^2, \eta^1\}$ as in proposition B.1. That is

$$\Gamma^\perp = \Omega^1(\omega^1, \omega^2), \tag{B.16}$$

$$\mathcal{I} = \{\eta^1\}, \tag{B.17}$$

and

$$d\eta^1 = \omega^1 \wedge \omega^2 \mod I. \tag{B.18}$$

Then the semi-basic form, $d\eta^1 = \omega^1 \wedge \omega^2$ is in $\mathcal{A}^2_{sb}$, and

$$\mathcal{I}_{\mathcal{A}} = \{\omega^1 \wedge \omega^2\}. \tag{B.19}$$

Example B.3. Let M be a dimension 5 manifold with coframe $\{\omega^1, \omega^2, \omega^3, \eta^1, \phi^1\}$ as in proposition B.1. That is

$$\Gamma^\perp = \Omega^1(\omega^1, \omega^2, \omega^3), \tag{B.20}$$

$$\mathcal{I} = \{\eta^1\}, \tag{B.21}$$

and

$$d\eta^1 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \phi^1 \mod I. \tag{B.22}$$

Then the semi-basic form, $d\eta^1 \wedge \omega^3 = \omega^1 \wedge \omega^2 \wedge \omega^3$ is in $\mathcal{A}^3_{sb}$, and

$$\mathcal{I}_{\mathcal{A}} = \{\omega^1 \wedge \omega^2 \wedge \omega^3\}. \tag{B.23}$$

In both of these examples the differential ideal

$$\mathcal{I}_{\mathcal{A}} = \langle \mathcal{A}^1_{sb}, \mathcal{A}^2_{sb}, \dots \rangle \tag{B.24}$$

is not Pfaffian. However, if $B^{i_5}_{j_5k_5} = 0$, then neither could occur.

Lemma B.4. *For a coframe as in proposition B.1, with $B^{i_5}_{j_5k_5} = 0$, the ideal generated by the semi-basic forms of $\mathcal{I}$ is*

$$\mathcal{I}_{\mathcal{A}} = \{\beta^{i_1}, \theta^{i_2}, A^{i_2}_{j_2k_2}\omega^{j_2} \wedge \omega^{k_2}\}. \tag{B.25}$$

Proof. for $B^{i_5}_{j_5k_5} = 0$ the algebraic generators of $\mathcal{I}$ are

$$\mathcal{I} = \{\beta^{i_1}, \theta^{i_2}, \alpha^{i_4}, \eta^{i_5}, A^{i_2}_{j_2k_2}\omega^{j_2} \wedge \omega^{k_2}, C^{i_5}_{l_5m_5}\omega^{l_5} \wedge \phi^{m_5} + D^{i_5}_{p_5q_5}\phi^{p_5} \wedge \phi^{q_5}\}. \tag{B.26}$$

Therefore for $\rho \in I^k$, $k > 2$, we have

$$\rho = \mu^1_{i_1} \wedge \beta^{i_1} + \mu^2_{i_2} \wedge \theta^{i_2} + \mu^4_{i_4} \wedge \alpha^{i_4} + \mu^5_{i_5} \wedge \eta^{i_5} + \nu^2_{i_2} \wedge (A^{i_2}_{j_2k_2}\omega^{j_2} \wedge \omega^{k_2}) + \nu^5_{i_5} \wedge (C^{i_5}_{l_5m_5}\omega^{l_5} \wedge \phi^{m_5} + D^{i_5}_{p_5q_5}\phi^{p_5} \wedge \phi^{q_5}). \tag{B.27}$$

Where we assume, without loss of generality, that

$$\begin{array}{rcll} \mu^1_{j_1} & \in & \Omega^{k-1}(\beta^{i_1}, \omega^{i_3}, \phi^{i_6}) & 1 \leq j_1 \leq n_1, \\ \mu^2_{j_2} & \in & \Omega^{k-1}(\beta^{i_1}, \theta^{i_2}, \omega^{i_3}, \phi^{i_6}) & 1 \leq j_2 \leq n_2, \\ \mu^4_{j_4} & \in & \Omega^{k-1}(\beta^{i_1}, \theta^{i_2}, \omega^{i_3}, \alpha^{i_4}, \phi^{i_6}) & 1 \leq j_4 \leq n_4, \\ \mu^5_{j_5} & \in & \Omega^{k-1}(M) & 1 \leq j_5 \leq n_5, \\ \nu^2_{j_2} & \in & \Omega^{k-2}(\omega^{i_3}, \phi^{i_6}) & 1 \leq j_2 \leq n_2, \\ \nu^5_{j_5} & \in & \Omega^{k-2}(\omega^{i_3}, \phi^{i_6}) & 1 \leq j_5 \leq n_5. \end{array} \tag{B.28}$$

For ρ to be semi-basic, it must be that $\mu^4_{j_4} = \mu^5_{j_5} = \nu^5_{j_5} = 0$. We must also have

$$\begin{array}{rcll} \mu^1_{j_1} & \in & \Omega^{k-1}(\beta^{i_1}, \omega^{i_3}) & 1 \leq j_1 \leq n_1, \\ \mu^2_{j_2} & \in & \Omega^{k-1}(\beta^{i_1}, \theta^{i_2}, \omega^{i_3}) & 1 \leq j_2 \leq n_2, \\ \nu^2_{j_2} & \in & \Omega^{k-2}(\omega^{i_3}) & 1 \leq j_2 \leq n_2. \end{array} \tag{B.29}$$

This shows that

$$\mathcal{A}^k_{sb} \subset \{\beta^{i_1}, \theta^{i_2}, A^{i_2}_{j_2k_2}\omega^{j_2} \wedge \omega^{k_2}\}, \tag{B.30}$$

proving the lemma.

□

As noted above (see [1]), we can determine if $\bar{\mathcal{I}}$ is Pfaffian by checking that $\mathcal{I}_{\mathcal{A}}$ is Pfaffian.

Theorem B.5. *If there exists a coframe, as in proposition (B.1), such that* $B^{i_5}_{j_5 k_5} = 0$ *then* $\bar{\mathcal{I}}$ *is Pfaffian.*

Proof. Since Γ is integrable, $d\theta^{i_2} = 0 \mod \Gamma^{\perp}$. Also, from the structure equations in proposition B.1,

$$d\theta^{i_2} = A^{i_2}_{j_2 k_2} \omega^{j_2} \wedge \omega^{k_2} \mod I \cap \Gamma^{\perp}. \tag{B.31}$$

Using equation (B.8), we see that this implies $\mathcal{I}_{\mathcal{A}}$ is Pffafian. Therefore $\bar{\mathcal{I}}$ is Pfaffian as well.

□

B.2 Non-Pfaffian Systems

Using the same hypotheses as in section B.1, we obtain a similar result for systems generated by 1-forms and 2-forms.

Theorem B.6. *Let* $\{\theta^{i_1}, \omega^{i_2}, \eta^{i_3}, \phi^{i_4}\}$ *be a basis for* $\Omega^1(M)$ *such that,*

$$\mathcal{I} = \{\theta^{i_1}, \eta^{i_3}, A^i_{jk} \omega^j \wedge \omega^k, C^r_{lm} \omega^l \wedge \phi^m + D^r_{pq} \phi^p \wedge \phi^q\} \tag{B.32}$$

and

$$\Gamma^{\perp} = \{\theta^{i_1}, \omega^{i_2}\}. \tag{B.33}$$

Then

$$\mathcal{I}_{\mathcal{A}} = \langle \theta^{i_1}, A^i_{jk} \omega^j \wedge \omega^k \rangle, \tag{B.34}$$

and $\bar{\mathcal{I}}$ *is generated by 1-forms and 2-forms.*

Proof. The proof of lemma B.4 proves equation (B.35). Pulling back by the cross-section, $\mathfrak{s}$, we get

$$\bar{\mathfrak{I}} = \langle \mathfrak{s}^*\theta^{i_1}, \mathfrak{s}^*(A^i_{jk}\omega^j \wedge \omega^k)\rangle. \tag{B.35}$$

Therefore $\bar{\mathfrak{I}}$ is generated by 1-forms and 2-forms.

□

APPENDIX C
PROLONGATIONS AND QUOTIENTS

We now use the results from appendix B to show the relationship of the associated singular differential systems for $\mathcal{E}$ and $\bar{\mathcal{E}}$.

C.1 The Quotient of the Associated Singular Differential Systems

We can show that the quotient of the associated singular system for $\mathcal{E}^{(k)}$, $\hat{V}$ is Pfaffian. Using this it is easy to show that $\hat{V}/\mathcal{G} = \hat{W}$, the associated singular system for $\bar{\mathcal{E}}^{(k)}$.

Proposition C.1. *For $k \geq 0$, if $\hat{V}$ and $\hat{W}$ are associated singular systems for $\mathcal{E}^{(k)}$ and $\bar{\mathcal{E}}^{(k)}$ respectively, then $\hat{V}/\mathcal{G} = \hat{W}$.*

Proof. For $k > 2$, section 5.3, equation (5.91), gives

$$\hat{V} = \{\beta_1, \beta_2, \alpha_1, \ldots, \alpha_k, \tilde{\alpha}_{k+1}, \zeta_1, \ldots, \zeta_k, \eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_k, \tau\}. \tag{C.1}$$

Theorem 5.1, and equation (5.98) give the structure equations,

$$
\begin{aligned}
d\beta_1 &= 0 && \text{mod} \quad \hat{V}, \\
d\beta_2 &= 0 && \text{mod} \quad \hat{V}, \\
d\alpha_i &= 0 && \text{mod} \quad \hat{V} \quad 1 \leq i \leq k, \\
d\zeta_i &= 0 && \text{mod} \quad \hat{V} \quad 1 \leq i < k, \\
d\eta_i &= 0 && \text{mod} \quad \hat{V} \quad 1 \leq i \leq k, \\
d\xi_i &= 0 && \text{mod} \quad \hat{V} \quad 1 \leq i < k, \\
d\zeta_k &= \sigma \wedge \tilde{\zeta}_{k+1} && \text{mod} \quad \hat{V}, \\
d\xi_k &= \sigma \wedge \tilde{\xi}_{k+1} && \text{mod} \quad \hat{V}, \\
d\tilde{\alpha}_{k+1} &= 0, \\
d\tilde{\eta}_{k+1} &= 0 && \text{mod} \quad \hat{V}.
\end{aligned}
\tag{C.2}
$$

This satisfies the hypothesis of theorem B.5, therefore $\hat{V}/\mathcal{G}$ is Pfaffian. The quotient system is then

$$
\hat{V}/\mathcal{G} = \{\eta_1, \ldots, \eta_k, \tilde{\eta}_{k+1}, \xi_1, \ldots, \xi_k, \tau\}. \tag{C.3}
$$

From section 5.3, equation (5.92), we see that this is just $\hat{W}$. This completes the proof for $k > 2$; using equation (3.52) and equation (4.34) gives analogous results for $k = 0$ and $k = 1$ respectively.

□

C.2 Quotients of Prolongations and Prolongations of Quotients

We now justify our disrespect for the placement of the bar in our notation for both the quotient manifolds and quotient EDS.

Theorem C.2. *Quotienting the prolonged manifold, $U^{(k+1)}$, by $\mathcal{G}$ gives the same result as prolonging the quotient manifold, $U^{(k)}/\mathcal{G}$. That is*

$$U^{(k+1)}/\mathcal{G} = (U^{(k)}/\mathcal{G})^{(1)}. \tag{C.4}$$

The same result holds for the EDS:

$$\mathcal{E}^{(k+1)}/\mathcal{G} = (\mathcal{E}^{(k)}/\mathcal{G})^{(1)}. \tag{C.5}$$

Proof. By theorem 5.5, the quotient manifold is

$$\bar{U}^{(k)}/\mathcal{G} = \bar{U} \times \mathbb{R}^{2k}, \tag{C.6}$$

and the quotient EDS is a Pfaffian system,

$$\mathcal{E}^{(k)}/\mathcal{G} = \langle \eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_k \rangle. \tag{C.7}$$

The integral 2-planes of $\mathcal{E}^{(k)}/\mathcal{G}$ with independence condition $\sigma \wedge \tau \neq 0$ and no characteristic directions, $\phi^1_2 = 0$, are

$$E_{(s_{k+1}, t_{k+1})} = \text{span}\{X + s_{k+1}\tilde{W}_{k+1}, Y + t_{k+1}\tilde{V}_{k+1}\}, \tag{C.8}$$

where $s_{k+1}, t_{k+1} \in \mathbb{R}$. Therefore the prolonged manifold is

(C.9)

$$(U^{(k)}/\mathcal{G})^{(1)} = G_2(\mathcal{E}^{(k)}/\mathcal{G}) = \{(x, y, v, q_0, r_0, s_0, t_0, \ldots, q_k, r_k, s_k, t_k, E_{(s_{k+1}, t_{k+1})})\} = U^{(k+1)}/\mathcal{G}.$$

The pullback of $E^{\perp}_{(s_{k+1},t_{k+1})}$ by the projection $\pi : \bar{U}^{(k+1)} \to \bar{U}^{(k)}$ gives the Pfaffian system on $\bar{U}^{(k+1)}$,

$$(\mathcal{E}^{(k)}/\mathcal{G})^{(1)} = \langle \pi^*(E^{\perp}_{(s_{k+1},t_{k+1})}) \rangle = \langle \eta_1, \ldots, \eta_{k+1}, \xi_1, \ldots, \xi_{k+1} \rangle = \mathcal{E}^{(k+1)}/\mathcal{G}. \tag{C.10}$$

Thus quotienting and prolongation commute.

$\square$