Bulletin No. 252 - Groundwater: Part I: Fundamental Principles Governing Its Physical Control

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GROUNDWATER

Part I
Fundamental Principles Governing Its Physical Control

WILLARD GARDNER, T. R. COLLIER, and DORIS FARR

Fig. 11—Cross-section showing traces of piezometric surfaces: Well battery.

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FOREWORD

This bulletin embodies the theoretical work of Project 17.

Part I is concerned with the physical control of groundwater and is presented by Willard Gardner, T. R. Collier, and Doris Farr.

Part II will deal with the alkali phase and will be presented by D. S. Jennings, Darrel Peterson, and Delmar Webb.
INTRODUCTION

This article does not purport to present detailed specifications for engineering structures nor to enlighten engineers specifically as to devices or methods appropriate to the control of groundwater. It is the aim rather to discuss fundamental principles and to indicate applications.

Important generalizations in physics as a rule have a basis of profound simplicity, but the analysis of many problems requires a patient adherence to formal methods and the introduction of more or less technical discussion becomes necessary in any adequate presentation. It may seem to impose artificiality into the consideration of practical problems, but attempts to obtain quantitative results lead naturally to this procedure and the serious student is obliged to concede the necessity therefor in some cases.

The primitive laws of physics do not in themselves offer difficulty, but their successful application requires quantitative descriptive terms and relationships with which the casual reader is not familiar. A major part of the paper will be concerned therefore with an attempt to lay a proper foundation so that the reader may follow developments that might without such foundation seem a little out of reach. Contrary to popular belief, mathematical methods are justified because of their great utility and the clearness they introduce into difficult problems.

Acknowledgments: Textbooks have been consulted freely and no attempt is made to give specific credit to authors who have thus been helpful in the preparation of this article. Mr. Wendell O. Rich and Mr. Ellis L. Armstrong deserve credit for the preparation of the diagrams.

1Contribution from Department of Physics, Utah Agricultural Experiment Station.
2Physicist, and graduate assistants, respectively.
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THE FUNDAMENTAL PROBLEM

Primary consideration will be given to such subject matter as will aid in the solution of problems relating to the development of groundwater and the drainage of water-logged lands.

The construction of successful wells often proves expensive and power costs for pumping prohibitive, drainage systems frequently prove inadequate, and legislation concerning water-rights is not all that could be desired. A deeper insight into fundamental principles may help specialists to aid in overcoming these deficiencies.

The meteorological and geological aspects of the question will be dismissed with a brief word. Water evaporates from the soil and from bodies of water into the atmosphere, is transported from place to place in the form of vapor, rain, snow, dew, etc., floods over and seeps into the ground, moves toward the bottom lands, and evaporates again to renew the cycle.

The relatively impervious layers of bedrock underlying the surface debris prevent a rapid sinking of the water into the depths of the earth, and the high temperature with the consequent high pressure constitute a permanent seal at great depths, so that the course of precipitation falling on the watersheds is inevitably toward the lower lands, and the velocity of such lateral flow is obviously determined largely by the character of the soil through which it seeps. Underground "channels" yielding small streams are common and caverns are often traced considerable distances into the mountain sides; usually, however, these streams are fed by slow percolation from the melting snows or rains. It will be taken for granted that water conveyed underground from the watersheds to the agricultural valleys is primarily so conveyed by the slow process of percolation.

Formations in these mountain valleys are frequently such as to lead to the development of hydrostatic pressure in strata near the surface of the ground, and in such cases the moisture tends to seep upward to water-log the surface soil. Attempts to drain such lands have not always succeeded and failure in some cases may be due to inadequate information regarding the laws of movement of water in the soil.

The heterogeneous nature of the subsoils is bound to lead to difficulties at best, but some assistance may be rendered by giving careful consideration to fundamental principles.

TECHNICAL TERMS, SYMBOLS, MATHEMATICAL METHODS, AND PHYSICAL LAWS

The difficulties of students are sometimes greatly reduced if they exercise extreme care in the interpretation of technical expressions and symbols. By means of such abbreviation symbols it is possible at times to express clearly and concisely ideas which when given in ordinary words appear involved and difficult to comprehend. Several such terms and symbols are discussed in the following paragraphs.

Next in importance to the concept of the soil moisture itself is perhaps the velocity characterizing the various elemental quantities into which it may be imagined to be subdivided. If the velocity at a point in the

Velocity soil is multiplied by the density of the fluid at the point a secondary vector quantity, designated flux density, is obtained. For every point throughout the moist soil there belongs a particular value for each of these vectors, and for that reason they are designated point functions. For the case of water flowing through a pipe the product of the actual cross-section of the pipe and the average flux density over the section gives the total flux (or flow). If the pipe is filled with soil grains, however, it is necessary to distinguish between the gross sectional area and the net area. It is evident that the total flow could be expressed as the product of the net area and the velocity
averaged over the net area or as the product of the gross area and the velocity averaged over the gross area. It is found convenient, as a rule, to adopt the latter method, and therefore the velocity is sometimes referred to as the macroscopic velocity, but where a misunderstanding is not likely the modifying adjective will be omitted.

Each point in the soil is characterized by a definite value of the hydrostatic pressure. There is no significant direction to be associated with this quantity and it is therefore classified as a scalar. It may be referred to as a scalar point function. For the case of saturated soils it is ordinarily, though not always, positive.

The region of space occupied by the moist soil is frequently referred to as a vector or scalar field, and it furnishes a proper perspective to think of the vector and scalar quantities as varying continuously throughout the field. This will seem particularly fitting when we come to interpret the meaning of the differential equations to be used.

The term potential, a scalar quantity, has a broad meaning and usefulness. It is a scalar point function and represents the line integral of the field vector from an arbitrary reference point to the point in question. It is best understood by reference to illustrations. The illustration to be presented will itself lead to a definition of the term line integral. If a body is lifted without loss of energy by friction against the force of gravity from one level to another, it is said to have increased in potential energy by an amount equal to the product of its weight and the vertical distance through which it is moved. The force of gravity is given as the product of the mass and the force per unit mass, the latter being the field vector. If, then, a body of unit mass is selected, the amount of potential energy stored in the process of lifting it from one level to another is the product of the force per unit mass times the vertical distance through which it is lifted. This product is the difference in gravitational potential energy for the unit of mass between the two levels and is technically referred to as the measure of the difference in potential.

It is apparent that for a large vertical displacement account must be taken of the decrease in weight per unit mass with distance from the surface of the earth, and in order to do this satisfactorily it is necessary to introduce the calculus, integrating the product gdh rather than taking the simple product g times h, where g represents the magnitude of the gravitational field vector, regarded as a variable, and dh the vertical component of the elementary vector displacement \( \mathbf{dr} \). This integral could be taken along a vertical line or any other straight or curved line, inasmuch as the vertical component of the displacement makes the entire contribution to the integral, the horizontal component contributing nothing. This integral is termed a line integral for the reason that the integration is taken along a line. Its value in this particular case for a selected point depends only upon the point of beginning. It is to be observed therefore that by choosing a particular point for reference, the value of the line integral from this point to any arbitrary point would be characteristic of its position. In this particular case it has the dimensional character of force per unit mass times distance, or velocity squared, but its dimensions in the general case will depend upon the nature of the vector field. The line integral of a velocity field vector, for example, would have the dimensions of velocity times length, or length squared divided by time, but, as will be seen, such a line integral is not always completely defined by the limiting points of integration.

Reversing the path of integration for this gravity field vector would change the sign of the result, and it follows that for such vector fields as are characterized by the existence of a potential the line integral around any arbit-

\[ ^{2} \text{This differential is written in bold-faced type to indicate that it has direction as well as magnitude, a procedure much used and which will be followed throughout this discussion.} \]
trary closed circuit vanishes. For example, the line integral for the path ABC in the diagram of Figure 1 is the same as for the path ADC and the same as for the negative of the path CDA, so that for the closed circuit ABCDA the integral must be zero. As will be observed from the illustration in the following paragraph, while reversing the path of integration of the particular vector chosen would reverse the algebraic sign, the value of the integral depends upon the path of integration as well as upon the end points, and therefore this vector field does not have a potential.

Figure 2 represents a body rotating with an angular velocity $\omega$ about a fixed axis through $O$. The space occupied by the particles of the solid is said to constitute a velocity field because each point has a characteristic velocity. To compute the line integral of this velocity vector from one arbitrary point to another is an intelligible exercise in mathematics even though it is without immediate physical significance. Beginning at a point $A$, the integral along the arc $s$ of the circle centered at $O$ to the point $B$ is given by

$$B\int_{A}^{B} r_1 \omega r_1 d\theta = \omega r_1^2$$

and from $B$ to $C$ it is

$$C\int_{B}^{C} Od\theta = 0$$
the component of the velocity parallel to the displacement vector $ds$ being zero. If, on the other hand, the integration is taken from $A$ to $D$, the one term becomes

$$D \int_A^D 0 ds = 0$$

and the other term from $D$ to $C$

$$C \int_D^C r \omega r_2 d\theta = \theta \omega r_2^2.$$  

In the one case, the aggregate from $A$ to $C$ is $\theta \omega r_2^2$ and in the other case, $\theta \omega r_2^2$. The line integral of this velocity field vector around the closed circuit $ADCBA$ is therefore $\theta \omega (r_2^2 - r_2^1)$. The value of the integral around a closed circuit of this kind is given the name circulation. It may be observed incidentally that the product $\theta (r_2^2 - r_2^1)$ is twice the area enclosed by the circuit. Dividing the area out in order to give a scalar point function representing the circulation per unit area, twice the angular velocity is obtained. The illustration will make clear the mathematical as well as the physical significance of the terms irrotational and rotational to characterize the two kinds of vector fields, namely, those having potentials and those which do not.

**Elementary Vector Algebra**

Elementary textbooks in physics give sufficient explanation of the distinction between scalar and vector quantities and illustrate the meaning of vector addition. Geometrical methods, however, are generally used, and the algebra of vectors is not developed in such texts. An effort will be made therefore to point out some of the elementary rules of vector algebra because of its great utility in this field.

It should be emphasized at this point that much of this development is of formal character. If the reader will make an effort to observe carefully the meaning of the abbreviated notation, as has been suggested, he will discover that he is not called upon to struggle with any sort of abstruse reasoning. Arbitrary significance has at one time or another been given to the primitive symbols of elementary arithmetic and their use has become so common that one is inclined to forget that their meaning is not inherent. The symbols of advanced mathematics, although sometimes more comprehensive, are not in reality more mysterious.

It is customary in most elementary applications to think of a vector as being made up of three rectangular components along an arbitrarily chosen set of rectangular axes. Three important vectors of unit length taken parallel respectively to the $X$, $Y$, $Z$, axes are designated by the symbols, $i$, $j$, $k$. The symbol $ia$ or $ai$ designates a vector of length $a$ parallel to the $X$-axis and the the other two fundamental unit vectors perform a similar role. A vector $A$

![Fig. 3—Geometric representation of vector sum with rectangular components, ia, jb, kc.](image-url)
having components of magnitudes $a$, $b$, $c$, respectively, is a triple number, and the expression,

$$A = ia + jb + kc$$

constitutes a new definition of the operation addition as it applies to vector quantities. Geometrically interpreted, the single directed segment representing the diagonal of the rectangular parallelepiped, constructed upon the sides, $ia$, $jb$, $kc$, represents the vector sum of the three components. The triple character of the vector as an algebraic number is not so successfully suppressed.

In order to illustrate this vector equation and at the same time present an important vector operator, a special application will be made to a problem of great importance. In elementary arithmetic and algebra extensive use is made of symbols of operation without referring to them specifically as operators. The use of this new term should not suggest anything at all mysterious. The unit vector $i$, for example, placed before the letter $a$ may be regarded as a vector operator transforming the scalar $a$ into a vector along $i$ of magnitude $a$.

Figure 4 represents an infinitesimal element of fluid upon the six faces of which is exerted a pressure due to adjacent fluid. The edges have lengths, $dx$, $dy$, $dz$. The net pressure force exerted upon this element by surrounding fluid will be the vector sum of three components along these coordinate directions. The pressure at the center of the element is designated as $p$, and the three partial derivatives, $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, $\frac{\partial p}{\partial z}$, represent the space rates of change of the pressure in the neighborhood of the center. Aside from infinitesimals of second order, the pressure at the center of the face ABCD is

$$p = (\frac{\partial p}{\partial x})dx/2,$$
and at the center of the opposite face EFGH it is

\[ p + \left( \frac{\partial p}{\partial x} \right) dx / 2. \]

Multiplying these pressures by the common area dydz of the faces and combining them to get the net component of force parallel to the X axis, the following is obtained:

\[ -\left( \frac{\partial p}{\partial x} \right) dx dy dz. \]

By this same process, we obtain for the net component of force parallel to the Y axis

\[ -\left( \frac{\partial p}{\partial y} \right) dx dy dz \]

and for the net component along the Z axis

\[ -\left( \frac{\partial p}{\partial z} \right) dx dy dz. \]

The vector sum may therefore be expressed thus:

\[ ( -i \frac{\partial p}{\partial x} - j \frac{\partial p}{\partial y} - k \frac{\partial p}{\partial z}) dx dy dz. \]

If, now, the volume of the parallelopiped dx dy dz is divided out, there is obtained a vector representing the resultant force F per unit volume as a vector point function:

\[ F = -i \frac{\partial p}{\partial x} - j \frac{\partial p}{\partial y} - k \frac{\partial p}{\partial z}. \]

In order to develop a concise notation it is found convenient to define the symbol of operation \( \nabla \) (called "del") thus:

\[ \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}, \]

thereby condensing the equation to the form

\[ F = -\nabla p. \]

Operating thus upon a scalar quantity such as \( p \) produces the vector called the gradient of the scalar. In the ordinary written language words may have several meanings, depending to some extent upon the context. As a rule, in a technical discussion involving mathematics, it is important to restrict the meaning of certain words.

Any continuously varying scalar point function would lead to a vector by means of this vector operation, the resulting vector being properly designated as the gradient of the scalar.

A careful consideration of the development leads to the inference that this vector is a measure of the space rate of change of the scalar in the direction of the greatest rate of change. The proof of this statement may be found in various places in the literature of physics, for example, in Page's "Theoretical Physics," pp. 17-19.

The operation of multiplying a vector by a scalar seems to suggest nothing new. Vector 2A differs from vector A in an obvious manner. When an attempt is made, however, to multiply one vector by another, invoking the ordinary rule of algebra, the following results:

Vector

\[ (il_1 + jm_1 + kn_1)(il_2 + jm_2 + kn_2) = (iil_1l_2 + jjm_1m_2 + kkn_1n_2) \]

Multiplication

\[ + (jkm_1n_2 + kjn_1m_2 + kn_1l_2 + ikl_1n_2 + ijl_1m_2 + jim_1l_2). \]

The appearance of such bilinear factors as \( ii, jk, \) etc. leaves the result indefinite, and it becomes necessary to define their meaning if a successful vector algebra is to be developed.

To the student who is familiar with solid analytic geometry the two sets of terms on the right-hand side suggest relationships to the cosine and sine,
respectively, of the angle $\theta$ between the two lines whose direction cosines are $l_1, m_1, n_1,$ and $l_2, m_2, n_2$. The product

$$l_1l_2 + m_1m_2 + n_1n_2$$

remains unchanged when new coordinate axes are chosen, and the three quantities,

$$m_2n_1 - n_2m_1, \quad l_1n_2 - l_2n_1, \quad l_1m_2 - m_1l_2,$$

transform in the same way as the three components $ix, jy, kz,$ of the radius vector $r$. Since

$$\sin^2\theta = (m_2n_1 - n_2m_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - m_1l_2)^2$$

it suggests that $\sin \theta$ might be regarded as a vector representing the diagonal of a parallelepiped built upon these three scalar quantities. There is no such suggestion, however, in the triplet of scalars,

$$l_1l_2 + m_1m_2 + n_1n_2,$$

although this sum gives the cosine of the angle $\theta$. If, therefore, the symbol $\varepsilon$ is introduced to represent a unit vector directed along the diagonal of the parallelepiped, it will prove consistent and useful to make the following arbitrary definitions:

$$i.i = j.j = k.k = 1$$

$$(1)$$

$$i \times i = j \times j = k \times k = 0$$

$$(2)$$

$$i \times j = j \times k = k \times i = -i \times k = j.$$  

$$(3)$$

In these definitions the dot and cross evidently indicate two distinct methods of multiplication, which may be designated the dot and cross multiplication, respectively. If two unit vectors, $V_1$ and $V_2$, are defined thus:

$$V_1 = l_1 + jm_1 + kn_1$$

$$V_2 = l_2 + jm_2 + kn_2$$

and the dot product is formed, the following is obtained:

$$V_1 \cdot V_2 = (V_1/V_1) \cdot (V_2/V_2) = (l_1 + jm_1 + kn_1) \cdot (l_2 + jm_2 + kn_2) = l_1l_2 + m_1m_2 + n_1n_2 = \cos \theta$$

$$(7)$$

where $\theta$ is the angle between the two vectors $V_1$ and $V_2$. If, on the other hand, the cross product is formed, the following is obtained:

$$V_1 \times V_2 = (V_1/V_1) \times (V_2/V_2) = (l_1 + jm_1 + kn_1) \times (l_2 + jm_2 + kn_2) = i(m_1n_2 - n_1m_2) + j(n_1l_2 - l_1n_2) + k(l_1m_2 - m_1l_2) = (\sin \theta) \varepsilon.$$  

$$(8)$$

It will be shown that the unit vector $\varepsilon$ is directed along the positive normal to the plane determined by the two vectors.$^{(4)}$

It is to be observed that the choice of reference frame does not influence the result of these two products. Therefore the axes might be so chosen that the vector $V_1$ lies positively along the $X$ axis, $V_2$ making acute angles with the $Y$ and $Z$ axes. The cosines $m_1$ and $n_1$ would both be zero and the $i$ component of the vector product would therefore be zero. (The $X$ axis might equally well have been chosen along $V_2$, which would also have shown $\varepsilon$ to be perpendicular to $V_2$). The $j$ component would consist of one term, $-j1n_2$, and this would be negative, since the method of choosing the reference frame would make $l_1$ and $n_2$ both positive. The $k$ component would consist of the positive term, $k11n_2$. It is concluded therefore that the vector product is a vector directed along the positive normal.

To further illustrate the use of this vector algebra another problem of importance will be considered. For the simple case of fluid flowing through a

$^{(4)}$If the vector $V_1$ is rotated toward $V_2$ the shortest way, the rotation would appear counterclockwise to the observer on the positive side of the plane.
respectively, of the angle $\theta$ between the two lines whose direction cosines are $l_1, m_1, n_1,$ and $l_2, m_2, n_2$. The product

$$l_1 l_2 + m_1 m_2 + n_1 n_2$$

remains unchanged when new coordinate axes are chosen, and the three quantities,

$$m_2 n_1 - n_2 m_1, n_1 l_2 - l_1 n_2, l_1 m_2 - m_1 l_2,$$

transform in the same way as the three components $ix, jy, kz,$ of the radius vector $r$. Since.

$$\sin^2 \theta = (m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - m_1 l_2)^2$$

it suggests that $\sin \theta$ might be regarded as a vector representing the diagonal of a parallelopiped built upon these three scalar quantities. There is no such suggestion, however, in the triplet of scalars,

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$$i.i = j.j = k.k = 1$$

(1)

$$i \times j = j \times k = k \times i = 0$$

(2)

$$i \cdot j = i \cdot k = j \cdot k = j \cdot i = k \cdot i = k \cdot j = 0$$

(3)

$$i \times j = -j \times i = k, j \times k = -k \times j = i, k \times i = -i \times k = j.$$ (4)

In these definitions the dot and cross evidently indicate two distinct methods of multiplication, which may be designated the dot and cross multiplication, respectively. If two unit vectors, $v_1$ and $v_2$, are defined thus:

$$v_1 = V_1 / V_1 = il_1 + jm_1 + kn_1$$

(5)

$$v_2 = V_2 / V_2 = il_2 + jm_2 + kn_2$$

(6)

and the dot product is formed, the following is obtained:

$$v_1 \cdot v_2 = (V_1 / V_1) \cdot (V_2 / V_2) = (il_1 + jm_1 + kn_1) \cdot (il_2 + jm_2 + kn_2)$$

$$= l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos \theta$$

(7)

where $\theta$ is the angle between the two vectors $v_1$ and $v_2$. If, on the other hand, the cross product is formed, the following is obtained:

$$v_1 \times v_2 = (V_1 / V_1) \times (V_2 / V_2) = (il_1 + jm_1 + kn_1) \times (il_2 + jm_2 + kn_2)$$

$$= i(m_2 n_1 - n_2 m_1) + j(n_1 l_2 - l_1 n_2) + k(l_1 m_2 - m_1 l_2) = (\sin \theta) \varepsilon.$$ (8)

It will be shown that the unit vector $\varepsilon$ is directed along the positive normal to the plane determined by the two vectors.\(^4\)

It is to be observed that the choice of reference frame does not influence the result of these two products. Therefore the axes might be so chosen that the vector $v_1$ lies positively along the X axis, $v_2$ making acute angles with the Y and Z axes. The cosines $m_1$ and $n_1$ would both be zero and the $i$ component of the vector product would therefore be zero. (The X axis might equally well have been chosen along $v_2$, which would also have shown $\varepsilon$ to be perpendicular to $v_2$). The $j$ component would consist of one term, $-j l_1 n_2$, and this would be negative, since the method of choosing the reference frame would make $l_1$ and $n_2$ both positive. The $k$ component would consist of the positive term, $k l_1 n_2$. It is concluded therefore that the vector product is a vector directed along the positive normal.

To further illustrate the use of this vector algebra another problem of importance will be considered. For the simple case of fluid flowing through a

\(^4\)If the vector $v_1$ is rotated toward $v_2$ the shortest way, the rotation would appear counterclockwise to the observer on the positive side of the plane.
tube (Figure 5), the law of conservation of mass leads to a simple equation of the form,

$$A_1 \rho v_1 = A_2 \rho v_2$$

where $A$ represents the cross-sectional area, $\rho$ the density of the fluid (regarded as variable), and $v$ the component of the velocity normal to the section $A$. This is known as the equation of continuity, but in this form it is limited in usefulness. For many purposes it is appropriately expressed in the form of a differential equation. It is developed in this form below:

Let an infinitesimal parallelepiped of fluid be isolated in a fixed region of space and let us consider the net change of its mass due to flow across the fixed boundaries. The flux density at the center will be represented by the vector $\rho v$, with components, $\rho v_x, \rho v_y, \rho v_z$. On the basis of the law of conservation of mass, the net inward flux across the bounding faces is equal to the time rate of increase of fluid within the boundary. The equation is developed on the basis of these facts, thus:

For the left-hand face, the entering flux is

$$[\rho v_x - \partial / \partial x (\rho v_x) dx / 2] dy dz$$

and for the right-hand face that which leaves is

$$[\rho v_x + \partial / \partial x (\rho v_x) dx / 2] dy dz.$$

The net flux in from this pair of faces is

$$-\partial / \partial x (\rho v_x) dx dy dz,$$
with similar expressions for the other pairs of faces,

$$-\partial / \partial y(\rho v_y) dx dy dz,$$

and

$$-\partial / \partial z(\rho v_z) dx dy dz.$$

By dividing out the volume element $dx dy dz$ in order to obtain the total rate of entrance of fluid per unit volume as a scalar point function, the following is obtained:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v_x) - \frac{\partial}{\partial y}(\rho v_y) - \frac{\partial}{\partial z}(\rho v_z). \quad (10)$$

By extending the meaning of the term multiplication to justify a procedure that is strictly formal, the following equation involving the vector operator $\nabla$ may be written,

$$\nabla \cdot \rho v = (i \partial / \partial x + j \partial / \partial y + k \partial / \partial z)(i \rho v_x + j \rho v_y + k \rho v_z)$$

$$= \partial(\rho v_x) / \partial x + \partial(\rho v_y) / \partial y + \partial(\rho v_z) / \partial z \quad (11)$$

and the equation of continuity in its general form may be written concisely,

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho v). \quad (10)$$

We recall that the result of operating with $\nabla$ upon a scalar produced a vector termed the gradient of the scalar. Since in this case it operates upon a vector it is important to indicate which type of multiplication is used. The dot is therefore used and the result is obviously a scalar. This scalar is termed the divergence of the vector $\rho v$.

Without at this point making an attempt to illustrate by means of a physical example, it will be pointed out parenthetically that by substituting the cross for the dot in equation (10) a new vector is obtained which is known as the curl of the original vector $\rho v$. The term rotation is also used as a synonym to curl. The operation is given below in the expanded and in the condensed form:

$$(i \partial / \partial x + j \partial / \partial y + k \partial / \partial z) \times (i \rho v_x + j \rho v_y + k \rho v_z)$$

$$= i[\partial(\rho v_z) / \partial y - \partial(\rho v_y) / \partial z]$$

$$+ j[\partial(\rho v_x) / \partial z - \partial(\rho v_z) / \partial x]$$

$$+ k[\partial(\rho v_y) / \partial x - \partial(\rho v_x) / \partial y]$$

$$= \nabla \times (\rho v). \quad (12)$$

**Fundamental Laws**

Newton's second law states that for a particle of mass $m$ the force $F$ acting on it is proportional to the product of the mass and the acceleration, $\frac{d^2 r}{dt^2}$, leading to a single second-order vector differential equation

**Newton's second law**

$$F = m \frac{d^2 r}{dt^2}.$$

It will prove advantageous to subdivide the forces into two groups: (1) Those which are derivable from potentials and (2) those which are not. In a previous paragraph it has been shown that the negative gradient of the pressure as a vector point function represents the pressure force per unit volume. Dividing this by the density of the water reduces the result to force per unit mass, and for the case of incompressible fluids the density is constant and may be included behind the derivative sign and therefore behind the $\nabla$ operator. The force of gravity per unit mass is also the negative gradient of the gravitational potential $\phi$ as will be shown in the next paragraph, but the frictional forces are somewhat more evasive. It is convenient, as a matter of theory, to define another symbol of operation $\psi$ known as the stress dyadic. By computing the dot product of the operator $\nabla$ and this new symbol an expression for the force per unit volume is obtained which takes proper
account of the shearing or resisting stresses. The elements of the dyadic, however, include the components of the shearing stresses, which are difficult to measure; for that reason this method seems thus far to offer no special practical assistance in handling soil-moisture problems. By introducing the hypothesis that the forces resisting flow are proportional to the relative velocity of the fluid element and neglecting the acceleration, a relatively simple differential equation is obtained by including in addition to the forces due to gravity and to the pressure gradient a force proportional to the velocity, thus:

\[-\nabla \left( \frac{p}{\rho} \right) - \nabla \phi - \left( \frac{1}{k} \right) v = 0 \]  

(13)

the proportionality factor \(1/k\) depending upon the nature of the medium through which the water moves. Solving explicitly for \(v\), this takes the form

\[ v = -k \nabla \left( \frac{p}{\rho} + \phi \right) \]  

(14)

Darcy has shown by experimental means that for horizontal movement of water through porous media in one dimension the velocity is directly proportional to the loss of pressure head per unit distance, and it is to be observed that this equation is consistent with Darcy's experiment, inasmuch as the vertical component of the gravity gradient is balanced out by vertical reactions from the pipe. This general form has been referred to as Darcy's law and will be so used in this discussion.

The algebraic statement of the law of conservation of mass was developed in a previous paragraph and is repeated below:

**Law of conservation of mass**

\[ \frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho v) \]  

(10)

These two laws constitute the fundamental basis upon which the developments of this paper are founded, although the equations may be written in special form to suit the conditions of the problem.

**SPECIAL DEVELOPMENTS, DIFFERENTIAL EQUATIONS, BOUNDARY CONDITIONS**

It has been taken for granted that the line integral of the gravitational field vector about an arbitrary closed circuit vanishes and that therefore this field has a potential. This may be seen to be a direct consequence of Newton's law of gravitation. The gravitational field vector \(\mathbf{f}\) is directed toward the center of the earth and its magnitude is \(k/R^2\), where \(R\) represents the distance from the center of the earth. The element of the line integral (Figure 7) may be expressed

\[ f \, dr = (k/R^2) \cos \theta \, dr = (k/R^2) dR \]
where \( r \) is the vector drawn from the center of the earth (its magnitude being represented by \( R \) in the equation) and \( dr \) is the element of vector displacement. Upon integration, the right-hand member becomes
\[
\frac{R_2}{R_1} \frac{dR}{R^2} = k (1/R_1 - 1/R_2).
\]
It is obvious therefore that if the path of integration is a closed circuit, \( R_2 = R_1 \) and the integral will vanish.

A somewhat more direct proof may be made by showing that the expanded element of the line integral is an exact differential. If the magnitudes of the three components of the field vector are expressed briefly by \( X, Y, Z \), this expanded element becomes
\[
X dx + Y dy + Z dz.
\]

The separate vanishing of the expressions,
\[
\left( \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right), \left( \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right), \left( \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right),
\]
which are seen to represent the components of the curl of the vector constitutes the necessary and sufficient condition that this element is an exact differential. In expanded form the vector becomes
\[
f = iX + jY + kZ = k \left\{ \frac{ix}{(x^2 + y^2 + z^2)^{3/2}} + \frac{iy}{(x^2 + y^2 + z^2)^{3/2}} + \frac{kz}{(x^2 + y^2 + z^2)^{3/2}} \right\}.
\]

It will be found that the components of the curl of this vector vanish separately and that therefore the element of the line integral is an exact differential.

It is possible now to combine the two equations (14) and (10) representing Darcy’s law and the law of conservation of mass to deduce an important equation known as Laplace’s equation. For the case of an incompressible fluid the left-hand member of equation (10) is zero. If \( v \) is eliminated from this equation by means of equation (14), Laplace’s equation results. It is written in the condensed and also in the expanded form:
\[
\nabla \cdot \nabla (p/\rho + \phi) = \frac{\partial^2}{\partial x^2}(p/\rho + \phi) + \frac{\partial^2}{\partial y^2}(p/\rho + \phi) + \frac{\partial^2}{\partial z^2}(p/\rho + \phi) = 0. \tag{15}
\]

It may be of help to the reader at this point to be reminded that in the main what has been attempted thus far is a definition of terms, a presentation of mathematical technique, and a brief discussion of two important physical laws concerning the movement of groundwater which have led to a second order partial differential equation. If he can now convince himself that this equation is to form the basis for the solution of important practical problems he will have succeeded in properly interpreting the purpose of that which has preceded.

Inherent difficulties arising in attempts to solve problems that are encountered in connection with wells, drains, springs, swamps, etc., tend to confuse the novice, and without reasonable care he stands a great chance of going astray in his conclusions. Certainly, he stands a much better chance of avoiding difficulty if he is able to recognize and appreciate these simple basic principles. Furthermore, it is probable that a proper understanding of these principles will aid in shaping the perspective of the research worker, bringing to light important facts that would be expensive if acquired by direct experimental means.

To comprehend the far-reaching significance of a partial differential equation, particularly when it is of second order, is not an easy task for the student who has had only a first course in calculus. To make a direct application of such an equation to a practical problem is admittedly even more
difficult, but the important feature of such application is in no sense beyond
the comprehension of a patient student.

The potential \( \Phi \) at a particular instant is a function of the three coor-
dinates, \( x, y, z \). Such an expression as \( \frac{\partial \Phi}{\partial x} \) gives the measure of the space
rate of change of \( \Phi \) at a given point and along a given axis or direction,
whether or not it satisfies a particular equation. This partial derivative,
\( \frac{\partial \Phi}{\partial x} \), and also the second partial derivative, \( \frac{\partial^2 \Phi}{\partial x^2} \), as well as those of
higher order, are themselves scalar point functions. If, for example,
\[ \Phi = k \sqrt{x^2 + y^2 + z^2} \]
then
\[ \frac{\partial \Phi}{\partial x} = -kx / (x^2 + y^2 + z^2)^{3/2} \]
A similar set of functions would be obtained by computing the partial deri-
atives with respect to each of the other two independent variables, \( y \) and \( z \).
Adding together the three second partial derivatives gives the following:
\[
3kx^2 \left( x^2 + y^2 + z^2 \right)^{3/2} + 3ky^2 \left( x^2 + y^2 + z^2 \right)^{3/2} + 3kz^2 \left( x^2 + y^2 + z^2 \right)^{3/2} \\
-3k \left( x^2 + y^2 + z^2 \right)^{3/2} -k \left( x^2 + y^2 + z^2 \right)^{3/2} -k \left( x^2 + y^2 + z^2 \right)^{3/2} = 0.
\]
Therefore this function,
\[ \Phi = k \sqrt{x^2 + y^2 + z^2} \]
when substituted in the expression
\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \]
will make it vanish, and it is therefore a solution of Laplace's equation.

There are many functions that will satisfy this equation. The important
problem, however, is to discover the particular solution that satisfies the
conditions of the particular problem involved, these conditions being spoken
of as the boundary conditions. It will be shown that if a particular solution
is found which satisfies the boundary conditions it will be unique, that is,
it will be the only solution and therefore the correct solution. Before under-
taking to make this proof, which is a little long and tedious, an illustration is
presented.

Let us assume that water converges through uniform sand from a spheri-
cal surface along radial lines toward the center of the sphere and is removed
by means of a pump. Since the velocity vector is proportional to the gradient
of the potential function, the potential levels are concentric spherical sur-
faces. If the value of the potential \( \Phi \), is given for the surface of radius \( R \),
the constant \( k \) has the value \( \Phi / R \), and the function takes the form,
\[ \Phi = \Phi / R \sqrt{x^2 + y^2 + z^2} \]
which may therefore be computed without ambiguity for any point \( x, y, z \).

To make this illustration a little more concrete, the following substitution
is introduced.
\[ \Phi = p / \rho + hg, \]
where the pressure \( p \) and the height \( h \) are measured from arbitrary reference
values. If, in this case therefore \( p = p_0 \) when \( h = h_0 \), the values belonging
to the center of the sphere \( (5) \), an expression for the pressure \( p \) as a function
of \( x, y, z \) is obtained, thus:
\[ p / \rho + hg = \frac{(p_0 / \rho + h_0 g) R}{\sqrt{x^2 + y^2 + z^2}}, \]
where \( R \) is the radius of the small sphere taken so small that \( h_0 \) repre-
sents with sufficient precision the elevation of any point on it. If the reference

\[ ^5 \text{Strictly speaking, the potential becomes infinite at the center; these values should there-
fore represent the surface of a small sphere. In fact, any finite sphere will serve the purpose}
quite as well. \]
sphere is large the expression in the parentheses would represent a constant potential but \( p_o \) and \( h_o \) would vary.

By means of Gauss' theorem it is possible to give a clear concise proof that the solution which satisfies Laplace's equation and at the same time satisfies the boundary conditions is unique. The proof of Gauss' theorem is therefore presented first.

This theorem states that the volume integral of the divergence of a vector is equal to the surface integral of the normal component of the vector taken over the surface bounding the volume \( T \) considered, thus:

\[
\int \nabla \cdot \mathbf{v} \, dT = \int_{\partial T} \mathbf{v} \cdot d\sigma
\]

\( d\sigma \) representing the element of surface regarded as a vector directed along its outward normal. By substituting \( dxdydz \) for \( dT \) and expanding the integrand of the left-hand member, we obtain

\[
\int \nabla \cdot \mathbf{v} \, dT = \int \int \left( \frac{\partial v_x}{\partial x} \right) dxdydz + \int \int \left( \frac{\partial v_y}{\partial y} \right) dxdydz + \int \int \left( \frac{\partial v_z}{\partial z} \right) dxdydz.
\]

Consider the first term on the right. It may be integrated with respect to \( x \), which takes account of a horizontal filament, as indicated in Figure 8. We may substitute \( d\sigma_x \) for the projection in the YZ plane of the right end of the filament and \(-d\sigma_x \) for the projection in this plane of the left end. The coordinates at the right end are \( x_2, y, z \) and at the left end \( x_1, y, z \). The integral therefore becomes

\[
\int \int \frac{\partial v_x}{\partial x} \, dx dy dz = \int v_x(x_2, y, z) \, d\sigma_x + \int v_x(x_1, y, z) \, d\sigma_x,
\]

Fig. 8—Illustrating method of proof of Gauss’ theorem.
the first surface integral on the right taking account of the right end of the filament and the second one the left end, so that the equation may be expressed

$$\int \int \int \frac{\partial v_x}{\partial x} \, dx \, dy \, dz = \int v_x \, d\sigma,$$

with the understanding that the surface integral on the right is to be extended over both ends of all the filaments.

Taking account now of the two additional terms similar in form, the following result is finally obtained:

$$\int \nabla \cdot v \, dT = \int (v_x \, d\sigma_x + v_y \, d\sigma_y + v_z \, d\sigma_z) = \int v \cdot d\sigma.$$

Turning now to the proof of the uniqueness theorem, let it be assumed that there are two distinct solutions of the Laplace equation, \( \Phi_1 \) and \( \Phi_2 \), which satisfy the same boundary conditions. As may be readily shown, the difference between the two, \( \Phi = \Phi_1 - \Phi_2 \), will also satisfy the equation. We may write down the identity

$$\nabla \cdot \nabla \Phi = \nabla \cdot (\Phi \nabla \Phi) = \Phi \nabla \cdot \nabla \Phi \tag{1}$$

which may be checked by direct expansion. Integrating both sides over the volume \( T \),

$$\int (\nabla \Phi)^2 \, dT = \int \nabla \cdot (\Phi \nabla \Phi) \, dT = \int \Phi \nabla \cdot \nabla \Phi \, dT \tag{2}$$

is obtained. Since \( \Phi \) satisfies Laplace’s equation the integrand of the second integral on the right is zero. The other one may be transformed to a surface integral by means of Gauss’ theorem, leading to

$$\int (\nabla \Phi)^2 \, dT = \int (\Phi \nabla \Phi) \cdot d\sigma \tag{3}$$

the surface integral being taken over the bounding surface where \( \Phi_1 = \Phi_2 \). This term must therefore vanish, and since the integrand in the left-hand member is essentially positive it must vanish, so that

$$(\nabla \Phi)^2 = \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0$$

results and

$$\frac{\partial \Phi}{\partial x} = 0, \frac{\partial \Phi}{\partial y} = 0, \frac{\partial \Phi}{\partial z} = 0.$$

Therefore \( \Phi \) must remain constant, and since it vanishes at the boundary it is everywhere zero, proving that throughout the region \( \Phi_1 = \Phi_2 \).

It is beyond the scope of this presentation to undertake to go very far into a vast literature of mathematical physics having to do with the solution of Laplace’s equation. It should be reassuring to the reader to know that, although much may have been left undone, substantial progress has been made by the mathematical physicists. If this brief outline proves sufficient to enable the reader to understand the fundamental background of the applications presented it will have served its purpose.

**APPLICATION OF PRINCIPLES**

In attempting to make direct application of this simple theory to problems which have to do with the control of groundwater it should be recognized that the character of the subsoil varies greatly from place to place, and it is for this reason, no doubt, that many practical investigators have avoided this attack. The excuse is frequently offered that the factors are so numerous and the mathematical processes so difficult as to render the task insurmountable. It should be pointed out for the sake of emphasis that the reliability of this theory is dependent entirely upon the reliability of Darcy’s law, and if
this law furnishes a good first approximation then also this procedure deserves a considerable degree of confidence. It is not so much, as some investigators would wish to insist, new experimental information that is required to build in this way a successful foundation theory but rather a bold acceptance of fundamental mechanical principles and a determination to surmount analytical difficulties.

It is scarcely to be expected, in spite of their somewhat ambitious program, that the authors should present a complete digest of applications. They are willing to acknowledge that most problems encountered are difficult. They do propose, however, to present a number of simple illustrations with the confident hope that others may help to carry forward a modest beginning.

Seepage of Water Under Pressure

Darcy’s conclusions were based upon results obtained for horizontal flow in one dimension, that is, through a horizontal tube of uniform section. In this simple case the pressure will be found to decrease linearly, the velocity will be uniform along the tube, and its magnitude will be proportional to the uniform rate of change of pressure per unit length.

Another simple case consists of horizontal movement toward an axis, as in the case of a cylindrical well completely penetrating a stratum of uniform water-bearing gravel. The component of the gravitational field in the direction of flow at any point is zero and Darcy’s law reduces to

\[
v = -(k/p)(dp/dr),
\]

where \( p \) is the hydrostatic pressure and \( r \) is the radius vector drawn from the axis of the well. The pressure levels are here assumed to be circular and the equation of continuity takes the simple form:

**Horizontal radial flow toward an axis**

\[
Q = -2\pi rl\rho v = 2\pi rlk(dp/dr),
\]

where \( Q \) is the mass of water flowing in the well per unit time and \( l \) is the thickness of the gravel. Upon integration between the limits \( p \) and \( p_1 \) and \( r \) and \( r_1 \), this becomes

\[
p - p_1 = C(ln r/r_1),
\]

the constant \( C \) being introduced for brevity to replace \( Q/2\pi lk \). For this case the pressure is a linear function of the logarithm of the distance from the axis of the well.

In the case of flow toward a circular battery of wells, at a considerable distance from the battery the pressure approaches a corresponding logarithmic distribution, but near the battery the pressure function is not so simple (Figure 9). At distances remote from the axis of the battery the potential levels are effectively circular cylinders. The levels near the axes of the individual small wells also approximate circular cylinders. The potential at a random point may be regarded as the sum of the separate potentials “due” to the individual small wells, and these potentials will be linear functions of the logarithms of the respective distances of the random point from the axes of these small wells. In order to present a graphical representation of this problem it is convenient to define the term *piezometric surface* as the locus of points to which water would rise above an arbitrary reference plane in static pressure tubes for any arbitrary state of horizontal flow in the gravel stratum. Traces of two such surfaces in a vertical plane containing the axis of the well battery and also two of the individual wells of the battery are shown in Figure 11 (cover cut). The one surface corresponds to a single central well, the water being pumped from the upper boundary of the gravel; the other corresponds to an equivalent aggregate flow into a series of wells of the same diameter arrayed in a circle, the water
Fig. 9—Showing contours on well battery piezometric surface.

Fig. 10—Cross-section showing traces of corresponding piezometric surfaces: Single central well.
being pumped from the elevation of the broken horizontal line. The piezometric surface for the single well could be generated by rotating the curve about the axis of the well. The corresponding surface for the battery is evidently much more complicated. Contours in this latter surface are illustrated in Figure 9. It is evident that the contours for the case of the single well are concentric circles.

Careful consideration of this geometrical representation leads to the conclusion that the function \( \Phi \) which satisfies Laplace’s equation and the boundary conditions of this case is correctly represented by the elevations of the points of this piezometric surface. This function \( \Phi \) is the sum of a set of functions \( \Phi_i \) representing solutions of Laplace’s equation for the \( n \) small wells of the battery which take the form

\[
Q_i = Q/n = 2\pi r_i \rho k (d\Phi_i / dr_i),
\]

where \( r_i \) is the distance measured from the axis of well \( i \) to the random point. An attempt will be made to obtain the equation of this surface, using cylindrical coordinates with the axis lying along the axis of the well battery and the origin taken where it cuts the upper plane boundary of the gravel stratum. To obtain this equation, the following equation may be written

\[
(\Phi_i - \Phi_1) = (h_i - h_1) g
\]

and a new constant defined

\[
c = C/(\rho g),
\]

and the \( n \) contributions, \( (\Phi_i - \Phi_1) \), summed thus:

\[
h = \sum (h_i - h_1) = c/n \sum \ln r_i / r_i.
\]

As Figure 10 will show, we may now eliminate the distances \( r_i \) by means of the equations,

\[
r_i = \sqrt{R^2 + R^2 - 2RR \cos(\theta + \frac{i2\pi}{n})}
\]

where \( R \) is the distance from the coordinate axis to the random point and \( R_a \) is the radius of the circle in which the individual wells of the battery are located. We thus obtain finally

\[
h = c/n \left[ n \ln (R_a/r_i) + \frac{1}{2} \ln G \right]
\]

where

\[
G = \prod \left[ 1 + (R/R_a)^2 - 2(R/R_a) \cos(\theta + \frac{i2\pi}{n}) \right].
\]

The equation for the piezometric surface for the single central well is

\[
H = c \ln (R/R_a) + h'
\]

\( R_a \) being the horizontal distance from the axis to the vertical line cutting this piezometric surface at the elevation \( h' \) above the reference plane. This corresponds to the height of the surface of the water in the small wells of the battery. It may be shown that for large values of the ratio \( R/R_a \) the value of the expression \( G \) tends toward the value \( (R/R_a)^n \). By substituting this value in the right-hand member of equation (1) and putting \( h' = c \ln (R_a/r_i) \) in the right-hand member of equation (2), it may be observed that \( H \) becomes equal to \( h', \) or, in other words, that the two piezometric surfaces come together at remote distances from the center of the well. The quantity \( 2R \) has therefore been called the effective diameter of the well battery, inasmuch as a single well of this diameter when pumped to the same depth as the wells of the battery with the same distribution of potential at points remote from the well will produce the same quantity of water.
A case of interest arises in connection with a single cylindrical well drawing water through a horizontal gravel stratum from a river regarded as a long straight source. It is possible to compute the diameter of an equivalent source of supply in the form of a circle centered at the well. From the development which follows this effective radius is found to be twice the distance $s$ from the well to the river.

Line source

It is possible to compute the diameter of a line source an equivalent source of supply in the form of a circle centered at the well. From the development which follows this effective radius is found to be twice the distance $s$ from the well to the river.

**Fig. 12**—Showing river "line" source and corresponding circular source.

By an elementary procedure, we are led to the conclusion that the potential distribution for the well and "line" source is exactly the same as it would be with the "line" source removed and a "negative" well placed at a distance $s$ on the opposite side of the "line." This is known as the method of images. The potential, $p/\rho + \phi$, at any point, measured from the potential of the river taken as a reference potential, would be given by superposing the effects of the positive and the negative well, thus:

$$p/\rho + \phi = (p'/\rho + \phi) - (p_o'/\rho + \phi) - \left[ (p''/\rho + \phi) - (p_o''/\rho + \phi) \right] = c \ln r'/r_i - c \ln r''/r_i.$$  

(3)

For the substitute "circular" source, we may write

$$P/\rho + \phi - (P_o/\rho + \phi) = c \ln R/r_i.$$  

(4)

By putting $r'' = 2s$ and $r' = r$, we obtain $c \ln 2s/r_i$, the potential at the well with reference to the river taken as zero potential. Substituting $R_e$, the radius of the equivalent well, in the right-hand member of equation (4) and equating to $c \ln 2s/r_i$, we have

$$c \ln R_e/r_i = c \ln 2s/r_i.$$  

(5)

results, leading to,

$$R_e = 2s.$$  

A case of practical importance arises in connection with the design of drainage systems. It follows as a direct result of this analytical procedure that where there is an appreciable pressure in a water-bearing stratum near the surface of the ground, the installation of tile drains for effective lowering of the water-table is an expensive procedure.

When the weather is damp and foggy, the water-table tends to rise to maximum heights and careful design should take proper account of this fact. In order to provide adequate protection under these conditions, gravity drains should be installed at sufficient depth and sufficiently close together to reduce the pressure to zero (or to atmospheric pressure) midway between the drains at the surface of the ground.
The curves of Figure 13 have been drawn to represent the streamlines resulting from the combination of drains with a uniform flow vertically upward, and the algebraic development will make clear the basis for the figures presented in the table. Two conditions will be satisfied by the equations: (1) The total flow upward from the gravel, computed for the one unit of width s and length l, on the assumption that the available head H is utilized in determining a uniform vertical flow will be equated to the aggregate flow into a single drain, computed on the assumption that the flow is inward along the radial lines drawn from the axis of the drain; (2) the resultant pressure will be zero (or atmospheric) midway between drains at the surface of the ground and at the circumference of the drain.

In order to avoid unnecessary mathematical difficulties, it will be assumed that the soil extends to great distances above and below the level of the drains. Figure 13 suggests the possible amount of the error thus committed. The flow through the two central "tubes" will reach the surface of the ground to be evaporated rather than to be diverted into the drains. This will lead to a magnitude for the drain spacing slightly large and on that account an inequality sign will be used in the final formula resulting from the analysis.
The equation of continuity for the flow into the drain may be written
\[ Q = 2\pi r l p v \] (6)
where \( Q \) is the amount of water flowing upward from the selected area per unit of time and \( \rho \) is the density of the water. The elimination of \( v \) from this equation by means of Darcy's equation will lead to the simple form of Laplace's equation
\[ Q = 2\pi r l p k (d\Phi_s/d\tau), \] (7)
the subscript \( s \) being introduced in order to emphasize that the potential \( \Phi_s \) is that part of the final total potential \( \Phi \) which is due to the presence of the drain regarded as a sink independently of the uniform flow.

The corresponding equation for the uniform vertical flow from the gravel may be written
\[ Q = sp l k (H_g/w), \] (8)
with a uniform drop in potential along the vertical lines of flow. In this equation \( H \) represents the available head, \( g \) the acceleration due to gravity, and \( w \) the depth of soil overlying the gravel.

By means of this equation \( Q \) may be eliminated from the differential equation (Equation 7), and it is readily integrated, giving after substituting appropriate limits.

\[ \Phi_s_2 - \Phi_s_1 = \frac{[(sH_g/w)/(2\pi)]}{(ln r_2/r_1)} \] (9)

\( r_1 \) being the distance from the axis of the drain to the point midway between drains at the surface of the ground and \( r \), the radius of the drain. The aggregate potential at a point due to various sources is the algebraic sum of the separate potentials. If therefore the terms \( \Phi_s_2 \) and \( \Phi_s_1 \) are added to the respective contributions due to the uniform vertical flow, it is evident that these contributions will differ by an amount equal to \( (h/w)(H_g) \), inasmuch as \( H_g \) represents the total potential drop from the upper boundary of the gravel along the vertical streamline midway between drains to the surface of the ground. The difference in total potential therefore at the drain and at the surface of the ground midway between drains will be \(- (h/w)(H_g)\) more than the difference indicated in equation (9). Adding this quantity therefore to the right-hand member of equation (9) and then introducing the second of the two conditions mentioned in a previous paragraph, the following equation results:

\[ -(h/w)(H_g) + \frac{[(sH_g/w)/(2\pi)]}{(ln r_2/r_1)} = h_g, \] (10)
inasmuch as the pressure is to be zero at both places and the potential difference will be due entirely to difference in elevation of the drain and the surface of the ground.

If we now introduce the approximation of equating \( s/2 \) to \( r_2 \) and also the inequality sign for the reason explained, the final formula becomes

\[ s < \frac{(2\pi w h)}{(l + H/w)} / (H ln s/2r_1). \] (11)

\( H \) and \( w \) are quantities determined by the field conditions, \( h \) represents the arbitrary depth of the drain, and \( r_1 \) the arbitrary radius of the tile. The accompanying table of computed values of \( s \) for various values of the other quantities has been prepared in order to illustrate the meaning of the inequality. It will be seen, for example, that for an excessive value of \( h \) (\( h = 10 \) ft.), an excessive value of \( r_1 (r_1 = 1 \) ft.), and a comparatively small value of \( H \) (\( H = 5 \) ft.), \( w \) taken as 50 ft., the drain spacing should not exceed 15 ft. Where \( h = 3 \) ft., \( r_1 = \frac{1}{4} \) ft., \( H = 30 \) ft., and \( w = 50 \) ft., the drain spacing should not exceed 15 ft.
Table of computed values of $s$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$r_1$</th>
<th>$H$</th>
<th>$w$</th>
<th>$s$</th>
</tr>
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<td>50</td>
<td>46</td>
</tr>
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<td>15</td>
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<td>1</td>
<td>30</td>
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</tr>
</tbody>
</table>

(Distances expressed in feet)

For many practical cases therefore the drain spacing is so close as to make the cost of effective drainage by means of tile drains prohibitive.

On the other hand, provided wells be designed to reduce power costs sufficiently, pumping offers promise for effective drainage. It is important to recall that for the hypothetical case of a uniform stratum of artesian gravel the piezometric surface is a logarithmic function of the distance from the axis of the well. This fact has an important bearing upon the design of the well.

Increasing the diameter of the well decreases the necessary height of lift, as will be seen in Figure 14. For example, if the well whose diameter is represented by $d$ were so enlarged as to have a diameter $D$, the height of lift would be reduced from $h$ to $H$. The problem of determining the appropriate size resolves itself into a balancing of the saving in power costs against the cost of enlarging the well. The cost of the pump and pump equipment also will enter into the problem. By reducing the flow from a single well the height of lift could be reduced, but this would require more wells and more pumps to handle a given amount of water. The well battery is a device for reducing the flow from an individual small well and using a single pump to operate a series of such small wells.

Fig. 14—Showing saving in height of “lift” by enlarging well.
If we recall that the quantity $G$ which appears in equation (1) approaches the value $(R/R_a)^{2n}$ when $R/R_a$ approaches infinity, by means of equations (1) and (2), we may determine the value of $R_e$, obtaining

$$R_e = (G_\beta)^{1/2n} R_a$$

the subscript $\beta$ appearing to indicate that the value of $G$ is to be computed for the case where

$$R/R_a = (R_a + r_i)/R_a = \beta. \quad (13)$$

In Figure 15 is shown this quantity $(G_\beta)^{1/2n}$ plotted as a function of $n$ with $\beta$ taken equal to $(20 + 1/6)/20$.

![Graph showing relation of the geometric mean distance to individual wells in terms of number of wells.](image)

It is possible to design various kinds of well batteries, but there seems to be some intuitive justification for arraying the small wells on a circle. The problem of choosing the appropriate number of small wells presents itself, but no rational method for determining this number has been devised. The same thing may be said of the choice of $\beta$.

If it may be assumed as a first approximation that for the relatively small individual wells of the battery the cost is directly proportional to the product of the radius $r_i$ and the length $l$ of the pipe, the cost of the battery itself will be a linear function of its effective radius $R_e$. In order to develop an expression for the cost $z$ per unit quantity of water pumped, we make the following definitions:

- $a$ is the aggregation of well costs assumed to be independent of $R_e$ and $H$, such, for example, as the cost of transformer, pump, and the vertical pipes for the small individual wells if they are to be of constant diameter. It has the "economic" dimensions of capital multiplied by a factor (of the dimensions of reciprocal of time) representing interest and depreciation rate. It will be unnecessary for our purpose to assign an order of magnitude to it.
- $c$ is a secondary constant of proportionality determined by the respective sizes of pipe for the vertical small wells and for conveying the water to the pump at the center of the well battery and also upon the number of wells $n$ of the battery. Its relation to other constants is shown in equation (14). For the case of $n = 6$ and $\beta = 121/120$, it will have an order of magnitude of $4 \times 10^{-9}$ dollars per second per centimeter with 4-inch pipe installed taken as one dollar per foot, interest and depreciation taken as about 40 per cent per year. This high rate is selected in order to introduce safety into conclusions to be drawn.
- $a$ is the cost of pipe per unit length per unit radius.
- $\gamma$ includes $a$ plus the cost per unit length per unit radius for driving the pipe.
- $P$ is determined by the power cost. Its dimensions differ from those of $z$ by a factor having the dimensions of reciprocal of length. It has an order of magnitude of $10^{-12}$ dollars per centimeter per gram when the power charge is 2 cents per kilowatt-hour and the motor has an efficiency of 50 per cent.
- $r_a$ is the radius of the horizontal radial pipes.
- $b$ is a constant having the same dimensions as $z$ and representing terms in well costs that are proportional to $Q$. 
The combined cost of the \( n \) radial and \( n \) vertical pipes of radii \( r_a \) and \( r_l \) \( (= (\beta - 1)R_e) \), respectively, would be expressed thus:

\[
[nar_a + n\gamma(\beta - 1)l]R_a,
\]

and if we introduce a factor \( (G_\beta)^{1/2n} \) into the numerator and denominator, thus:

\[
\frac{[nar_a + n\gamma(\beta - 1)l](G_\beta)^{1/2n}R_a}{(G_\beta)^{1/2n}}
\]

and then introduce the abbreviated notation

\[
c = \frac{[nar_a + n\gamma(\beta - 1)l]}{(G_\beta)^{1/2n}},
\]

the expression for \( z \) in terms of the radius of the well \( R_e \), the quantity of flow \( Q \), and the height of lift \( H \), together with the parameters, would be given by the following:

\[
z = \frac{a + cR_e}{Q} + P(H + \epsilon) + b
\]

the term \( \epsilon \) being introduced as a correction to take account of the friction in the radial horizontal pipes. The factor \( R_e \) has been introduced from equation (12).

For the case of a single well in a large area the quantity of water \( Q \) is determined by \( R_e \) and \( H \), according to the equation

\[
Q = CH/(\ln R/R_e)
\]

and if \( Q \) be regarded as constant this equation determines a relation between \( H \) and \( R_e \) such that

\[
dH/dR_e = -Q/(C R_e).
\]

If, then, we differentiate equation (15) with respect to \( R_e \) and introduce equations (16) and (17) we obtain, upon equating this derivative to zero,

\[
R_e = Q^P/(cC)
\]

to represent the most efficient value of \( R_e \).

To illustrate, if \( Q \) is taken as 225 gallons per minute (which reduces to 14,200 cu. cm. per second), the magnitudes introduced for the constants \( P \) and \( c \), together with a magnitude of 200 cgs. units for \( C \), will lead to a value of 250 cm. (approximately 8 ft.) for \( R_e \). It should be emphasized, however, that the choice of 225 gallons per minute is wholly arbitrary and was chosen for the reason that it represents a comparatively small well. Choosing a large value leads to a large value for the economic radius, and this analysis leads qualitatively only therefore to the conclusion that the well should be large.

It is to be observed that equation (16) is derived on the supposition that the horizontal water-bearing stratum is completely isolated by impermeable bounding layers. While this equation is useful for many calculations, it should not be overlooked that leakage to or from relatively impervious strata overlying the water-bearing gravel may exert considerable influence upon the performance of any actual well. The development to follow may throw some light on this question. Disregarding the effect of leakage from or into the clay underlying the gravel and making certain simplifying approximations, a modified form of this equation may be derived thus:
Let \( h \) represent the elevation from the top of the gravel of the actual piezometric surface at the distance \( R \) from the axis of the well,

\( k \) the transmission constant for the gravel,

\( k_e \) the transmission constant for the overlying clay,

\( Q \) the horizontal flow in the gravel at the cylindrical surface of radius \( R \), regarded now as a function of \( R \),

\( w \) the thickness of the clay,

\( l \) the thickness of the gravel.

The upward leakage from the gravel into the element of overlying clay of differential area \( 2\pi R \, dR \) will then be

\[
dQ = (2\pi R \, dR) \rho k_e g \left( \frac{h-w}{w} \right),
\]

the factor \( k_e g \left( \frac{h-w}{w} \right) \) representing the velocity obtained from Darcy's equation. This will correspond with the increase in horizontal flow in the gravel at the cylinder of radius \( R + dR \) over that of radius \( R \), thus:

\[
dQ = d[2\pi R \rho g k (dh/dR)] = 2\pi \rho g k \left( \frac{Rdh}{dR^2} + \frac{dh}{dR} \right).
\]

Equating right-hand members of equations (19) and (20), we obtain

\[
\frac{R k_e}{lk} \left[ \frac{h-w}{w} \right] = \left[ \frac{Rdh}{dR^2} + \frac{dh}{dR} \right].
\]

This equation is linear and its solution involves the solution of the abridged equation

\[
\frac{d^2 h}{dR^2} + \frac{1}{R} \frac{dh}{dR} - \frac{k_h}{lkw} = 0.
\]
This is a special case of Bessel's equation and the following converging series is found to satisfy it:

\[ h_i = 1 + \frac{a r^2}{2^2} + \frac{a r^4}{2^4} + \frac{a r^6}{2^4 6^2} + \ldots \]  

(23)

where

\[ a = k_c / (l k w). \]  

(24)

This leads to the following approximate solution of equation (21):

\[ h = \lambda / 4 + w + \frac{a w R^2}{12} + \frac{\lambda / 6 + (2/9) w}{1/x} \]

\[ + (\lambda / 2)x + [C - (\lambda / 2) \ln (x / R^2)] x^2 \]  

(25)

where \( x \) is defined thus:

\[ x = [1 + (a/2) R^2] \]  

(26)

and \( \lambda \) and \( C \) are integration constants having the dimensions of length.

The constants of equations (16) and (25) were given such arbitrary values as to make their graphs pass through the same point at the well, determine the same flow into the well, and to cut the surface of the ground near the well. The curves are shown in Figure 16. From the well to the point where these surfaces cut the surface of the ground the quantity \( h - w \) is negative and the leakage is negative, that is, it is into the gravel from the clay. Beyond this point it is positive and at some critical point the slopes of the actual and the virtual curves will be the same. Beyond this critical point the slope of the actual curve will remain larger than for the other. If the actual surface does not cut the surface of the ground the leakage is everywhere negative.

The flow into the well represents either a direct contribution from the overlying clay or a cutting down of the leakage into the clay; the definite integral for the flow into the clay taken over the surface of contact between the two strata may therefore be differentiated in order to estimate the relation between \( Q \) and \( R_e \). The integrand of this integral is obtained from equation (19) and the derivative becomes

\[ \frac{dQ}{dR_e} = - \frac{d}{dR_e} \left[ (2\pi \rho k_c g) / w \right] \int_{R_e}^{L} (h - w) RdR \]

\[ = - \left[ (2\pi \rho k_c g) / w \right] \int_{R_e}^{L} \left( \frac{\partial h}{\partial R_e} \right) RdR - R_e (h - w). \]  

(27)

The partial derivative \( \partial h / \partial R_e \) is negative. When \( R = 0 \) it appears to be infinite and when \( R \) is large it becomes extremely small. It would seem logical therefore as a first approximation, to make the substitution

\[ \frac{\partial h}{\partial R_e} = - C / R. \]  

(28)

With this substitution, equation (27) reduces to

\[ \frac{dQ}{dR_e} = \frac{2\pi \rho k_c g C L}{w} + \frac{2\pi \rho k_c g}{w} (h - w - C) R_e. \]  

(29)

Introducing now the abbreviated notation

\[ p = \frac{2\pi \rho k_c g C L}{w} \]  

(30)
and

\[ B = \frac{(C + w - h)2\pi\rho k_r g}{2w} \]

we obtain

\[ \frac{dQ}{dR_e} = p - 2BR_e, \]

which, when integrated, becomes

\[ Q = A + p R_e - B R_e^2. \]

Obviously, the constant \( A \) is zero and \( B \) is small compared with \( p \). The upper limit \( L \) of the definite integral of equation (27) is presumed to be taken large. If it represents the radius of what has been called a “circle of influence”, then it would seem that the validity of equation (33) rests primarily on the assumption introduced in equation (28), and if this assumption is approximately near the truth, then also equation (33), with \( B \) small compared with \( p \), must be a reasonably close approximation.

If therefore we write

\[ Q = p R_e \]

and by means of this equation eliminate \( Q \) from equation (15), we obtain

\[ z = \frac{a + cR_e}{pR_e} + P(H + \varepsilon) + b. \]

It is evident from this equation that, aside from the influence of the small correction term \( \varepsilon \), \( z \) will continue indefinitely to decrease with increasing \( R_e \).

This analysis seems to point with reasonable certainty therefore to the inference that there is opportunity in some cases for improving the efficiency of design of drainage systems by substituting large well batteries for surface drains.

Sub-irrigation seems to be accomplished by seepage through layers of coarse soil containing a considerable amount of sand. If these strata are underlaid by clay and the clay in turn overlies saturated gravel without appreciable hydrostatic pressure, the water seeps downward slowly into the clay so that at some distance from the source of supply the gross seepage downward will equal the influx from the sub-irrigation ditch. In order to illustrate the case a specific problem is considered.

![Fig. 17—Illustrating sub-irrigation from a canal.](image_url)

In Figure 17 is shown the cross-section of a canal from which seepage takes place into a layer of sand of thickness \( s \) having a transmission constant \( k_s \), with a pressure head \( h \) measured from the lower boundary of the sand. The clay has a transmission constant \( k_c \). We undertake to compute the maximum distance water may be expected to “sub” by determining the loss downward through the clay on the supposition that the pressure in the clay is uniformly zero, or approximately so, the gradient of the pressure being taken equal to zero. At some point at distance \( x \), from the canal the gross downward flow will have become equal to the flow in from the canal.
and beyond this point the pressure in the sand will be zero or less, that is, actually under capillary tension, and the horizontal flow may be neglected. If we assume that the sand stratum is completely saturated to the point $x_1$ and that there is no loss into the loam overlying the sand, the computed value of $x_1$ will obviously be too high, but our interest is to determine a safe upper limit and therefore these assumptions will be made. The potential gradient in the clay will be the vertical gravitational field $g$ and the horizontal component of the gradient in the sand will be $\left(\frac{1}{\rho}\right)\left(\frac{dp}{dx}\right)$, with a value at the intake or canal end of the stratum $\left(\frac{1}{\rho}\right)\left(\frac{dp}{dx}\right)_0$. The width of the stratum measured along the canal will be designated by $w$. Combining Darcy's law with the equation of continuity, we obtain

$$\left(-\frac{wspk_s}{\rho}\right)\frac{dp}{dx} + wxpkcg = \left(-\frac{wspk_s}{\rho}\right)\left(\frac{dp}{dx}\right)_0.$$  (36)

For the case where $p = 0$, $x = x_1$, we have

$$\left(-\frac{wsk_s}{\rho}\right)\left(\frac{dp}{dx}\right)_0 = wxpkcg.$$  (37)

If we integrate equation (36) and put $p = 0$ to correspond with $x = x_1$, we obtain

$$p_{x_1} = \left(\frac{dp}{dx}\right)_0 x_1 + \left\{\frac{(pkcg)}{(sk_s)}\right\} x_1^2/2 + p_0 = 0.$$  (38)

Eliminating $\left(\frac{dp}{dx}\right)_0$ from equations (38) and (37), we obtain

$$x_1 = \sqrt{\left(\frac{2p_0sk_s}{sk_c}\right)/\left(k_cpg\right)}.$$  (39)

To illustrate this result, we take $s = 30$ centimeters, $p_0 = 60(\rho g)$, $k_s/k_c = 100,000$, and obtain for $x_1$ approximately 20,000 centimeters, or 632 feet.

In connection with the problem of water-logging, it is interesting to inquire as to the pressure distribution in cases where the boundary conditions are defined in a simple way. If the top surface of a uniform column of uniform soil is kept saturated but with negligible excess pressure head at the ends of the column, the pressure would remain uniform, the water moving in response to gravity direct. A slight negative pressure or suction at the bottom would transmit negative pressure throughout the tube and in such case there would be no tendency for water to stand in an open hole bored vertically into the soil. If therefore the pressure in the soil is less than atmospheric pressure it would not be rated as water-logged, whereas if the pressure is even slightly above that of the atmosphere water would stand in such an open hole and the soil would exhibit the character of water-logging. In considering the question of sub-irrigation, the stratum of sand was taken of uniform thickness, but if we consider a converging wedge (Figure 18) the potential gradient must be higher at points where the velocity is greater. If the axis of the wedge is itself inclined to the horizontal the gravity gradient has a component along the tube. The piezometric surface characterizing the tube in this case is not a plane. Soil lying on top of this surface would not be water-logged from

![Fig. 18—Showing piezometric surface (or water-logging boundary) for uniformly converging artesian stratum.](image)
the tube. The equation of this surface involving the shape of the tube and its inclination is developed here.

If \( v \) is the average velocity over the cross-section at a distance \( z \) from the top (the velocity being sensibly uniform over the section), \( H \) representing the total length of the wedge and \( A \) the area of the cross-section at \( z \), \( \theta \) the angle of inclination with the vertical, \( A_o \) the area of the top end of the tube, the equation of continuity becomes

\[
Av = [(H-z)A_o/H] (\rho v) = Q \cos \theta \quad \text{(a constant).} \tag{40}
\]

The factor \( \cos \theta \) enters the equation for the reason the velocity and therefore the flow is proportional to \( \cos \theta \). For this case the approximation for Darcy's law is written

\[
v = -k \frac{d\Phi}{dz} \tag{41}
\]

and Laplace's equation takes the simple form

\[
(H-z) \left( \frac{d\Phi}{dz} \right) = a_o \tag{42}
\]

where \( a_o \) is defined as follows:

\[
a_o = \frac{(QH \cos \theta)}{(\rho A_o k)}. \tag{43}
\]

Integrating equation (42) between the limits 0 and \( z \), we obtain

\[
\Phi - \Phi_o = a_o \ln \left[ \frac{H-z}{H} \right]. \tag{44}
\]

In mountainous country with high rainfall much of the precipitation floods over the surface of the ground, giving rise to excessive erosion, the soil being insufficiently deep to absorb and carry away the moisture. It is possible to develop a simple relation between the minimum depth of soil required and the rainfall intensity \( i \) on the mountain slope, the transmission constant of the soil appearing as a parameter. Figure 19 will make clear the meaning of the formula. It is developed by equating the flux integrals over the slope \( oz \) and over the arc \( ab \), thus:

\[
\int_0^x i \cos \theta \, dx = \int_{\theta_1}^{\theta_2} kg \sin \theta \, dz = \int_{\theta_1}^{\theta_2} kg \sin \theta \, xd\theta. \tag{45}
\]

This leads to the final formula

\[
\cos \theta_2 = \left( 1 - \frac{i}{kg} \right) \cos \theta_1. \tag{46}
\]

where \( i \) is the rainfall intensity and \( k \) is the transmission constant for the soil. The symbol \( x \) appearing as an upper limit in the first integral and in the argument of the last integral is the length of the section considered measured along the slope.
In many cases, such as the flow toward a vertical axis in a horizontal stratum, but two independent variables are required, as has been the case in most of the illustrations thus far presented, and Laplace's equation takes the simple form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (47)$$

If we select two functions, \(u(x, y)\) and \(v(x, y)\), in such a way that the complex quantity \(w\), defined thus:

$$w = u + iv = f(x+iy) = f(z) \quad (48)$$

has an unambiguous derivative with respect to the independent complex variable \(z = x + iy\), \(w\) is said to be an analytic function of the complex quantity \(z\). An attempt is made to show that any two such functions, \(u, v\), having these properties are solutions of this form of Laplace's equation, and, furthermore, that the two families of curves

\[ u = c_1 \]

and

\[ v = c_2 \]

are orthogonal, that is, they intersect at right angles.

From what has preceded, we are prepared to observe that the potential levels and the velocity lines constitute such an orthogonal set of curve families and the pair of functions of the coordinates which are constant over these respective curves also satisfy Laplace's equation. We have already shown that the solution which satisfies this equation and at the same time satisfies the boundary conditions is unique. If therefore we have some degree of familiarity with the nature of the transformations corresponding with various types of functions of the complex variable it should aid in finding appropriate solutions of problems in two-dimensional flow.

We write

$$w = u + iv \quad (49)$$

and express the derivative

$$\frac{dw}{dz} = \frac{(\partial w/\partial x)dx + (\partial w/\partial y)dy}{dx + idy}$$

$$= \frac{(\partial w/\partial x) + (\partial w/\partial y)(dy/dx)}{1 + i(dy/dx)}$$

or

$$\frac{dw}{dz} = \frac{(\partial w/\partial x)(dx/dy) + (\partial w/\partial y)}{dx/dy + i} \quad (51)$$

In order that this derivative shall be unambiguous, the right-hand members of these equations must be independent of \(dy/dx\) and of \(dx/dy\), respectively. From this, we conclude by taking these derivatives equal to zero that

$$\partial w/\partial x = dw/dz \quad (52)$$

and

$$\partial w/\partial y = idw/dz. \quad (53)$$

This same reasoning may be repeated for a second differentiation on the supposition that these derivatives are themselves analytic functions of the complex argument \(z\), leading to the result

$$\frac{\partial^2 w}{\partial x^2} = d^2w/dz^2 \quad (54)$$

and

$$\frac{\partial^2 w}{\partial y^2} = -d^2w/dz^2 \quad (55)$$

and therefore finally to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \quad (56)$$
If we replace \( w \) in equations (55) and (56) by \( u + iv \), we obtain
\[
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]  
(57)
and
\[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.
\]  
(58)

If now we form the scalar product of the two vectors, \( \nabla u \) and \( \nabla v \), we obtain, by taking account of equations (57) and (58),
\[
(\nabla u) \cdot (\nabla v) = (\frac{\partial u}{\partial x})(\frac{\partial v}{\partial x}) + (\frac{\partial u}{\partial y})(\frac{\partial v}{\partial y}) = 0,
\]  
(59)
from which it is concluded that the level lines of these two functions are at right angles.

Weaver (Journal of Mathematics and Physics, Vol. 11, No. 2, June, 1932) presents an article "Uplift Pressure on Dams"; from this article is quoted an illustration of the use of this method of conjugate functions in computing the pressure under a dam with sheet piling driven into the ground a distance \( d \) at the heel. The upper part of Figure 20 has been photographed from an illustration presented by Weaver. It will be recalled that it requires

Lines of seepage flow and lines of equal pressure under dam with sheet-piling.

Fig. 20—Showing velocity lines: Above: For pressure field alone; Below: Corrected for gravity.
two dimensions to represent a complex number geometrically and therefore two planes to illustrate the functional relationship

\[ w = f(z). \]

If we choose the special function

\[ w = \cosh^{-1}\left[ \frac{2z-b}{b} \right], \]  

where \( b \) is the width of the dam at the base, and substitute the \( u, v \) coordinates of a series of points along some arbitrary horizontal line lying between the two lines, \( v = 0 \) and \( v = \pi \), the \( x, y \) coordinates of these points will be found to lie along an hyperbola. The vertical lines, on the other hand, of the \( u, v \) plane map out into ellipses in the \( x, y \) plane. The two families of parallel straight lines in the one plane transform into the two families of orthogonal conics in the other plane.

The ellipses cross the \( x \) axis at right angles, suggesting that at the surface of the ground above the dam they would represent the velocity lines, and inasmuch as the functions which are constant over these curves satisfy Laplace's equation this transformation seems to offer direct help in completing the solution of the problem. In order, however, to satisfy the boundary conditions with the sheet piling at the heel Weaver has developed the function:

\[ w = 1 + \frac{\sqrt{1+b/d}}{2} (\cosh w + 1) - 1 = \sqrt{1 + (z/d)^2}. \]  

He has associated the \( v \) function with the velocity lines and the \( u \) function with the pressure lines. It is evident, however, that the potential function \( (p/\rho + \phi) \) rather than the pressure \( p \) should be associated with the \( u \) function inasmuch as the direct action of gravity must be taken into account in writing Darcy's equation for this general case. Inasmuch as the function \( \phi \) satisfies Laplace's equation if the gravitational field \( g \) is taken as constant, the function \( p/\rho \) must also be a solution and for that reason \( \phi \) may be ignored in the determination of the pressure levels. This cannot be done, however, in determining the velocity lines. In this publication an attempt has been made to superpose on the lines of Weaver's diagram a uniform flow due to gravity, with the result shown at the bottom of Figure 20.

**Application to Unsaturated Soils**

It is at once evident that the transmission constant depends upon the amount of moisture in the soil in the case of movement under the influence of capillary forces. As a first approximation it has been proposed to make this factor directly proportional to the moisture content \( \rho \), writing the modified form of Darcy's law:

\[ v = -k \rho \nabla \phi. \]  

The potential function \( \phi \) (which cannot be now construed as a velocity potential) must include what has been called the capillary potential \( \psi \). If the pressure \( p \) is now construed as capillary pressure (or tension), this potential may be defined by means of the differential equation

\[ \nabla \psi = (1/\rho) \nabla p. \]  

The moisture percentage of the soil is more readily measured in some cases than the capillary pressure \( p \), and experimental data are required in order to eliminate one or the other of these quantities. To express this relationship with precision would require mathematics somewhat involved. It seems necessary therefore to sacrifice precision in the interest of expediency and the following tentative proposal has been made:

\[ \psi = c/\rho + b, \]
where $c$ and $b$ are empirical constants characteristic of the soil. Combining equations (2) and (3), we obtain

$$\nabla p = -(\eta / \rho) \nabla \rho.$$  \hfill (4)

If we eliminate the potential from the modified form of Darcy's law for the case of horizontal capillary flow by means of these equations, we obtain

$$v = -k / \rho (\nabla \rho).$$  \hfill (5)

If we multiply through by $\rho$, we obtain an expression for the flux density

$$\rho v = -k (\nabla \rho),$$  \hfill (6)

which conforms with the familiar diffusion law. It appears therefore that whatever may be the justification for the two tentative assumptions introduced they combine to conform with experience. There seems to be some evidence in the literature of soil physics that for extremely dry soils there is a resistance to wetting, and it seems probable that for application to such soils a correction term $\xi$, representing a thin film of moisture over the soil grains, should be subtracted from $\rho$. This film would seem to form a part of the solid phase. The term "lento-capillary constant" has been suggested for this quantity. It would be characteristic of the soil.

The equations are somewhat more complicated when corrected for the influence of gravity. For convenient reference the equation of continuity is rewritten, thus:

$$\partial p / \partial t = - \nabla \cdot (\rho v).$$  \hfill (7)

Eliminating $v$ by means of equation (1), we obtain

$$\partial p / \partial t = \nabla \cdot (\rho k \nabla \Phi).$$  \hfill (8)

If $\nabla \Phi$ is replaced by the two terms

$$\nabla \Phi = (1 / \rho) \nabla p + g$$  \hfill (9)

and the expression simplified, we have

$$\partial p / \partial t = \{ \rho \nabla^2 p + 2 \rho \nabla \rho \cdot g + (\nabla \rho) \cdot (\nabla p) \}.$$  \hfill (10)

Eliminating $p$ by means of equation (4), this reduces, for the case of a steady state, to

$$\partial \rho / \partial t = kc [ \nabla^2 p - (2 \rho / c)(\nabla \rho) \cdot g] = 0.$$  \hfill (11)

For the case of a canal in which the water has run for sufficient time to establish a more or less steady condition in the underlying soil this equation should serve to determine the moisture distribution, as well as the velocity lines. The reference frame may be chosen with the $z$ axis vertically downward, the $x$ axis horizontal to the right, and the $y$ axis forward along the canal. The factors, $\partial \rho / \partial y$ and $\partial^2 \rho / \partial y^2$, will be zero and equation (11) will reduce to the simpler form

$$\partial^2 \rho / \partial x^2 + \partial^2 \rho / \partial z^2 - (2 \rho \rho / c) \rho / \partial z = 0.$$  \hfill (12)

Fig. 21—Showing soil moisture contours in neighborhood of canal flowing over deep soil.
This is a non-linear equation for which there appears to be no general solution. The authors are indebted, however, to Mr. Harry G. Romig and Dr. G. R. Stiblitz of the Bell Telephone Laboratories for a particular solution from which we have computed approximate moisture percentage levels as shown in Figure 21.

The moisture gradient is evidently large near the canal, decreasing rapidly to relatively small values tending to approximate uniformity of moisture distribution at points not very far away from the source. This seems to conform qualitatively to experimental facts.

Another application of interest may be noted. If the amount of moisture Q in a given section of soil is considered as a function of the time (taking a simple case where the velocity lines are parallel vertical lines), the following results:

**Loss of soil moisture by downward percolation**

\[ Q = \int_{0}^{a} A \rho dz \quad (13) \]

where \( A \) is the cross-sectional area and \( a \) is the depth of the section considered. This becomes upon differentiating partially with respect to the time

\[
\frac{\partial Q}{\partial t} = A \int_{0}^{a} \left( \frac{\partial \rho}{\partial t} \right) dz = A \int_{0}^{a} -kc \left[ \frac{\partial^2 \rho}{\partial z^2} - \left( \frac{2\rho}{c} \right) \frac{\partial \rho}{\partial z} \right] dz
\]

\[ = -k \left\{ c \left( \frac{\partial \rho}{\partial z} \bigg|_{z=a} - \frac{\partial \rho}{\partial z} \bigg|_{z=0} \right) - g(\rho_a - \rho_0) \right\} \quad (14) \]

We shall introduce as a first approximation the relation

\[ \rho = \rho_0 + \alpha e^{-\beta t} \quad (15) \]

where \( \rho_0 \) is independent of \( z \) and \( t \), and \( \alpha \) is independent of \( t \).
This will satisfy Laplace's equation and will be taken therefore as a first approximation to the solution of equation (11). The limits only of the $z$ variable will appear in the right-hand member of equation (14). It may be written therefore in the form

$$\frac{\partial Q}{\partial t} = B e^{-Bt} + C e^{-2Bt}, \quad (16)$$

which, when integrated, takes the form

$$Q = K + L e^{-Bt} + M e^{-2Bt}. \quad (17)$$

The constants may be treated as empirical.

In Figure 23 are plotted the mean values of the water content in the first 6 feet of a field plot at the Greenville Experiment Farm as a function of the time after a long-continued thorough sprinkling of the plot. The dots indicate experimental data, whereas the curve represents an empirical fit of the above equation.

**SUMMARY**

The authors have undertaken to present in considerable detail the mathematical background required in the presentation of the principles of the movement of groundwater.

They have shown that Newton's second law of motion, together with elementary hypotheses concerning frictional forces resisting the flow of water through soils, leads to Darcy's experimental velocity law generalized for flow in three dimensions.

Applications are made to the solution of practical problems in the design of drainage structures, flow into wells, water-shed erosion, leakage from canals, sub-irrigation, etc.

A modified approximation form of Darcy's law is presented for the solution of problems in capillary flow.

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