Attitude Determination for Small Satellites Using Magnetometer and Solar Panel Data

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Abstract—A low-cost three-axis attitude determination estimator well suited for small spacecraft with relaxed pointing constraints is presented. The new estimator incorporates solar panel data to increase observability and improve convergence of a magnetometer-based extended Kalman filter. In extensive Monte-Carlo simulation, the filter converges from any initial orientation to a total angle error of 1.6° (1σ) within one orbit. Although designed for use aboard Utah State University's USUSAT, the system is generically applicable to spacecraft with body-mounted solar panels in inclined low-earth orbits.

I. INTRODUCTION

The philosophy driving down the size of earth-orbiting satellites is primarily economic: small satellites are cheaper to build and launch into orbit. In keeping with this philosophy, attitude determination (AD) hardware for small spacecraft (s/c) is often limited to a reduced set of inexpensive and non-redundant sensors. One such minimal approach was introduced in the seminal work by Psiaki et al. [2] where a Kalman filtering scheme for three-axis estimation based solely on magnetometer data is developed. Although only two axes of attitude information are simultaneously measurable using a magnetometer, Psiaki demonstrates that for moderately inclined orbits the s/c attitude, rate, and constant disturbance torques are (weakly) observable through proper filtering of the magnetometer data. Application of this filter is limited to nadir-pointing gravity-gradient stabilized s/c, however, since linearization of the s/c dynamics and measurement sensitivity functions, in addition to the weak state observability, leads to instability for wide initial mispointing angles. With the gravity gradient boom, the s/c is able to right itself to within a capture envelope of the assumed initial orientation. Psiaki demonstrates good convergence for mispointings below 45° and possible convergence up to 60°.

P. Landiech [3] later showed that with some modification the magnetometer-based filter could be made to converge from wide mispointing angles. In one test case, the filter converges from nearly a 180° mispointing within roughly one orbit. Extensive Monte-Carlo type simulations are not presented, however, and it is evident from the initial erratic behavior of the filter that pathological cases will surely arise that prevent convergence for a substantial length of time (greater than one orbit). Furthermore, the filter introduced by Landiech includes the full four-element quaternion in the estimated state vector, leading to added computational expense and quaternion normalization. The latter constitutes an interference external to the filter [7].

The filter reported in this monograph is similar to the one originally introduced by Psiaki, but includes several modifications which allow more universal convergence. The structure of the filter is modified to handle mispointings beyond 90° and innovations not conforming to the small-angle assumption of the extended Kalman filter. More significantly, the filter is modified to incorporate solar panel data. Most small s/c are powered by fixed body-mounted solar panels. These are generally not used as attitude sensors since current readings off the solar panels may include significant earth albedo contributions, variations due to coverglass reflection, and variations due to internal s/c charging. Owing to these effects, errors less than 7° cannot be expected on average. But the dubious utility of the solar panels as attitude sensors is enhanced by a shift in roles. Instead of using the panels as a primary attitude sensor, they may be used only to aid initial convergence and as 'watchdog' sensors to assure the estimated attitude lies within the accuracy bounds of the solar panels. Using this methodology, state observability is greatly increased, and rapid convergence from any initial orientation is possible. This benefit comes with minimal additional expense since the technique exploits already available hardware. The result is a robust and widely applicable low-cost attitude estimation system.

In the sequel, reference frames, attitude kinematics, attitude parameterization, and other supporting constructs are introduced. The filter is then developed. These preliminaries are followed by a description of the simulation method and error models. The paper concludes with an evaluation and interpretation of simulation results.

II. ATTITUDE KINEMATICS

A. Reference Frames

For the purposes of this work, a minimal set of reference frames will be introduced. The orientation of s/c body coordinate system (CS) will be determined by the s/c inertia tensor, and its origin will be at the s/c center of mass. When represented in the s/c body CS, the inertia tensor is diagonal. Another reference frame, the s/c body-geometric CS, is aligned with geometric features of the s/c. Due to symmetry, the body CS for USUSAT will be close to the body-geometric CS as depicted in Fig. 1.

A reference frame in which the magnetic field vector and the sun vector are known will be generically referred to as the reference CS, or reference frame. This may be an inertial or non-inertial CS, as long as directional and
rate vectors are modified accordingly. In this work, the Earth-centered-inertial (ECI) CS is chosen as the reference CS. Calculation of the Earth magnetic field is performed in the Earth-centered-fixed (ECF) CS.

B. Attitude Parameterization

The s/c attitude is parameterized by the 4x1 quaternion, \( \tilde{q} \), and the 3x3 direction-cosine matrix, \( A \). The quaternion is composed of a vector and scalar part.

\[
\tilde{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}
\]

with

\[
q = \hat{e} \sin(\theta/2), \quad q_4 = \cos(\theta/2)
\]

Here, \( \hat{e} \) is a unit vector corresponding to the axis of rotation and \( \theta \) is the angle of rotation. The elements of the quaternion possess only three degrees of freedom and satisfy the constraint \( \tilde{q}^T \tilde{q} = 1 \). The direction-cosine matrix \( A \) is related to the quaternion by

\[
A(\tilde{q}) = \begin{bmatrix}
q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\
2(q_1q_2 - q_3q_4) & q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\
2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & q_1^2 + q_2^2 + q_3^2 + q_4^2
\end{bmatrix}
\]

This may also be written

\[
A(\tilde{q}) = (q_4^2 - ||q||^2)I_{3 \times 3} + 2qq^T - 2q_4[q \times]
\]

The skew-symmetric matrix \([q \times]\) defined as

\[
[q \times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_4 \\ -q_2 & q_4 & 0
\end{bmatrix}
\]

is the cross-product equivalent matrix and will be used often in the derivations that follow.

The convention here used for \( A \) is that \( A \) casts a vector written in the reference frame into body frame coordinates, i.e.,

\[
b = Ar
\]

The product of two quaternions follows the same ordering convention as the matrix product. Thus,

\[
A(q')A(q) = q' \odot q
\]

The quaternion product operation \( \odot \) is most easily expressed as a matrix product

\[
[q' \odot q] = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_1' & q_2' & q_3' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4'
\end{bmatrix}
\]

or alternatively, as

\[
q' \odot q = \{q\}'q
\]

with the 4 \times 3 matrix \( \Xi(q) \) defined in the next section.

C. Attitude Dynamics

Euler’s equation expresses the fundamental relationship between external moments applied to the s/c and the time rate of change of the angular momentum vector, \( L \).

\[
n_{ext} = \left(\frac{dL}{dt}\right)_I = \left(\frac{dL}{dt}\right)_B + [\omega \times]L
\]

Here the subscripts \( I \) and \( B \) denote that the derivative is taken with respect to the inertial or body frame. The angular momentum vector \( L \) is the product of the \( 3 \times 3 \) inertia matrix \( I \) and the angular velocity vector: \( L = I\omega \). There always exists a reference frame in which \( I \) is a diagonal matrix. This is called a principal reference frame. The s/c body CS is a principal reference frame, and hence \( I \) will always be diagonal with principal moments of inertia \( I_{xx}, I_{yy}, \) and \( I_{zz} \) when expressed in the body CS.

Euler’s equation may be rewritten to isolate the time derivative of \( \omega \):

\[
\dot{\omega} = I^{-1}(-[\omega \times]I\omega + n_c + n_d)
\]

Here, \( n_{ext} \) has been broken down into control and disturbance components

\[
n_{ext} = n_c + n_d
\]

The time evolution of the quaternion is as

\[
\dot{\tilde{q}} = \frac{1}{2} \Omega(\omega)\tilde{q}
\]

with

\[
\Omega(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & -\omega_1 \\ -\omega_3 & 0 & -\omega_1 & \omega_2 \\ \omega_2 & \omega_1 & 0 & -\omega_3 \\ -\omega_1 & -\omega_2 & \omega_3 & 0
\end{bmatrix}
\]
For small sampling intervals $h$, the quaternion may be propagated according to
\[ q_k = \left[ I_{4 \times 4} \cos \left( \frac{\lambda h}{2} \right) + \Omega(\omega) \sin \left( \frac{\lambda h}{2} \right) \right] q_{k-1} \] (17)
where $\lambda = ||\omega||$. This equation is useful for propagation. Also useful is the $4 \times 3$ matrix $\mathcal{E}(\tilde{q})$ defined by
\[ \mathcal{E}(\tilde{q}) = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \] (18)

### III. KALMAN FILTERING

A review of extended Kalman filtering concepts is included here to provide notational consistency.

The state vector $x$ evolves according to the state equation
\[ \dot{x}(t) = f(x(t), u(t), t) + w(t) \] (19)
where $f(x(t), u(t), t)$ is a nonlinear function of the state and control vectors. The process noise $w(t)$ is zero-mean white noise described by the process noise matrix $Q$.
\[ E[w(t)w^T(t')] = Q(t)\delta(t-t') \] (20)

Measurements are assumed to be a nonlinear function of the state, taken at discrete time intervals, and corrupted by measurement noise $v$.
\[ z_k = h(x_k) + v_k \] (21)
The discrete noise sequence $v_k$ is uncorrelated and zero-mean with covariance
\[ E[v_kv^T_s] = R_k\delta_{k,s} \] (22)

In the extended Kalman filter (EKF), nonlinear functions are linearized for use in propagating the matrix Ricatti equations and computing the Kalman gain. If the state error vector is defined as the difference between the true state and the state estimate
\[ \Delta x = x - \hat{x} \] (23)
then a first-order linear approximation is written
\[ \Delta \dot{x}(t) = F(t)\Delta x(t) + G(t)\Delta u(t) + w(t) \] (24)

To arrive at $F$ and $G$, the function $f(x, u, t)$ is linearized about the state estimate. The Kalman filter produces both pre-measurement and post-measurement state estimates, and the philosophy of the extended Kalman filter is to use the best state estimate available at the time linearization is required. For now, this will be denoted generically as $\hat{x}$. Hence,
\[ F(t) = \left. \frac{\partial f(x, u, t)}{\partial x} \right|_{x=\hat{x}}, \quad G(t) = \left. \frac{\partial f(x, u, t)}{\partial u} \right|_{x=\hat{x}} \] (25)
The linearized measurement equation is given by
\[ \Delta z_k = H_k\Delta x_k + v_k \] (26)
where $\Delta x_k$, the innovation, contains the new information provided by the latest measurement, and is defined by
\[ \Delta z_k = z_k - \hat{z}_k = z_k - h(\hat{x}_k) \] (27)
The measurement sensitivity matrix $H_k$ is found by linearizing $h(x_k)$ about the current best state estimate
\[ H_k = \left. \frac{\partial h(x)}{\partial x} \right|_{x=\hat{x}_k} \] (28)
The continuous Kalman filtering equations are now discretized in order to propagate the Ricatti equations at each sampling step. $F(t)$ is assumed constant over the sampling interval, and discretized according to
\[ \Phi(t) = e^{F_t}, \quad \Phi_k = \Phi(T_s) \] (29)
The matrix $\Phi_k$ is called the state transition matrix. Discrete versions of $G(t)$ and $Q(t)$ may be found by
\[ G_k = \int_0^{T_s} \Phi(t)G(t)dt, \quad Q_k = \int_0^{T_s} \Phi(t)Q\Phi^T(t)dt \] (30)

Here again it is assumed that $G$ and $Q$ are approximately constant over the sampling interval $T_s$. Furthermore, $u$ is assumed constant over the sampling interval.

The discrete, linear state space model may now be summarized as follows
\[ \Delta x_{k+1} = \Phi_k\Delta x_k + G_ku_k + w_k \] (31)
\[ \Delta z_k = H_k\Delta x_k + v_k \] (32)
The Kalman filter is applied to this model.

In practice, the state transition matrix is not used in the propagation step (time update) of the Kalman filter. Rather, the nonlinear dynamics equations are numerically integrated with an integration step much smaller than $T_s$.

The state transition matrix is used for the propagation of the discrete Ricatti equations. Because the accuracy of these computations is not needed at the same level as the state vector propagation, the transition matrix is usually approximated using only the first few terms of the Taylor series expansion of $e^{FT_s}$, i.e.,
\[ \Phi_k \approx I + FT_s \] (33)

The extended Kalman filtering equations are summarized as follows:

**Initialization**
- Begin with an initial estimate of the state, $\hat{x}_{0|0}$
- Reflect the uncertainty in the initial estimate in the initial error covariance matrix, $P_{0|0}$

**Prediction (Time Update)**
- Numerically integrate the nonlinear dynamics equations using $\hat{x}_{k|k}$ as the initial condition to obtain a predicted estimate of the state. Call this estimate $\hat{x}_{k+1|k}$. It represents the estimate of the state at step $k + 1$ given the previous $k$ measurements.
- Compute the state transition matrix
\[ \Phi_k \approx I + FT_s \] (34)
$F$ is a linearization of the system dynamics equations about $\dot{x}_{k|k}$.

- Compute the process noise covariance matrix

$$Q_k = \int_0^{T_s} \Phi(t)Q\Phi^T(t)\,dt$$

- Update the error covariance matrix

$$P_{k+1|k} = \Phi_k P_{k|k} \Phi_k^T + Q_k$$

### Filtering (Measurement Update)

- Update the measurement sensitivity matrix by linearizing about the current best state estimate

$$H_{k+1|k} = \frac{\partial h(x)}{\partial x} \bigg|_{x = \hat{x}_{k+1|k}}$$

- Compute the Kalman gain

$$K_{k+1} = P_{k+1|k} H_{k+1|k}^T (H_{k+1|k} P_{k+1|k} H_{k+1|k}^T + R_{k+1})^{-1}$$

- Update the state error estimate

$$\Delta \hat{x}_{k+1|k+1} = \Delta \hat{x}_{k+1|k} + K_{k+1} (\Delta x_{k+1} - H_{k+1|k} \Delta x_{k+1|k})$$

This may be simplified by noting that by definition

$$\Delta \hat{x}_{k+1|k+1} = \hat{x}_{k+1|k+1} - \hat{x}_{k+1|k+1} = 0$$

and rewriting

$$\Delta \hat{x}_{k+1|k+1} = K_{k+1} \Delta x_{k+1} = K_{k+1} (\Delta x_{k+1} - h(\hat{x}_{k+1|k}))$$

- Add the state error estimate to the predicted state estimate to obtain the filtered (post-measurement) state estimate

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \Delta \hat{x}_{k+1|k+1}$$

- Update the measurement sensitivity matrix using the filtered state estimate. Note that this second update of $H$ is not a part of the traditional EKF. It is included to bring about a more rapid decrease in the value of the error covariance matrix $P$.

- Update the error covariance matrix

$$P_{k+1|k+1} = (I - K_{k+1} H_{k+1|k+1}) P_{k+1|k} (I - K_{k+1} H_{k+1|k+1})^T + K_{k+1} R_{k+1} K_{k+1}^T$$

### IV. An EKF for Spacecraft Attitude Determination Based on Magnetometer and Solar Panel Measurements

#### A. Derivation

The dependence of the four quaternion elements given by $q^T q = 1$ gives rise to an error covariance matrix $P$ that is singular. This follows from the fact that since $\hat{q}$ and $\bar{q}$ are each of euclidean length 1, their difference, $\Delta \bar{q}$ must be orthogonal to both $\hat{q}$ and $\bar{q}$ as $||\Delta \bar{q}|| \to 0$. Hence, $\Delta \bar{q}^T \bar{q} \approx 0$, and

$$\begin{bmatrix} \hat{q} \\ 0_{6\times1} \end{bmatrix}$$

is a null vector of $P$. Maintaining the singularity of $P$ is made difficult because of round-off error accumulation.

There are several ways to deal with this issue. One may simply ignore the singularity of $P$, and treat each of the quaternion elements as independent in the filtering process. Normalization of the quaternion external to the filter becomes necessary, and this represents an outside interference which must be taken into account. No effort is made to maintain the singularity of $P$. This method works reasonably well in practice, although propagation of the outside interference constitutes an additional computational expense [7].

Another method described in Lefferts et al. [1] is adapted for use in the sequel.

Typical attitude determination is concerned with estimating the s/c attitude (as parameterized by the quaternion) and angular rate. In the absence of rate gyros, both the quaternion and angular rate vector are included in the state to be estimated. For added robustness and accuracy in the face of slowly varying disturbance torques, an estimate of the disturbance torque vector, $n_d$, is also included in the state estimate [2]. The full 10-dimensional state is then

$$x = \begin{bmatrix} \hat{q} \\ \omega \\ n_d \end{bmatrix}$$

In order to represent the state without the quaternion redundancy, a $9 \times 1$ body-referenced state vector is defined as

$$\tilde{x} = \begin{bmatrix} \delta q \\ \omega \\ n_d \end{bmatrix}$$

The quantity $\delta q$ is called the vector component of the error quaternion. The error quaternion is defined implicitly by

$$\bar{q} = \delta \bar{q} \otimes \hat{q}$$

Because the error quaternion corresponds almost certainly to a small rotation, the fourth component will be close to unity. But during initial convergence, this approximation is often violated. This will be accounted for in a later section. For now, it is assumed that $\theta$ is sufficiently small. Hence all attitude information of interest is contained in the vector part of the error quaternion, $\delta q$. Using (11), the quaternion composition is rewritten as a matrix product

$$\bar{q} = \delta \bar{q} \otimes \hat{q} = [\Xi(\hat{q})] \delta \bar{q}$$

The normalization constraint on the quaternion gives rise to the following three properties involving $\Xi(\delta q)$:

$$\Xi^T(q)\bar{q} = 0$$
\( \dot{q}^T \Xi(q) = 0 \)  \hspace{2cm} (51)
\( \Xi^T(q) \Xi(q) = I_{3 \times 3} \)  \hspace{2cm} (52)

Using these properties, it follows easily that
\( \delta q = \Xi^T(\dot{q}) \ddot{q} \)  \hspace{2cm} (53)
\( \delta q_{ik} = \ddot{q}_{ik} \)  \hspace{2cm} (54)

The body-referenced state vector may now be related to the standard state vector
\[
\begin{bmatrix}
\delta q \\
\omega \\
n_d
\end{bmatrix} =
\begin{bmatrix}
\Xi^T(\dot{q}) & 0_{3 \times 6} \\
0_{6 \times 3} & I_{6 \times 6}
\end{bmatrix}
\begin{bmatrix}
\ddot{q} \\
\omega \\
n_d
\end{bmatrix}
\]
or
\( \ddot{x} = S^T(\dot{q}) \dot{x} \)  \hspace{2cm} (56)

By noting that
\( \ddot{x} = S^T(\dot{q}) \dot{x} = 
\begin{bmatrix}
0 \\
\omega \\
n_d
\end{bmatrix} \)  \hspace{2cm} (57)

the vector \( \Delta \ddot{x} \equiv \ddot{x} - \dot{x} \) becomes
\( \Delta \ddot{x} = 
\begin{bmatrix}
\delta q \\
\Delta \omega \\
\Delta n_d
\end{bmatrix} \)  \hspace{2cm} (58)

This 9-dimensional body-referenced state error vector is the state vector for the linearized dynamics and measurement equations
\[ \Delta \ddot{x}(t) = F(t) \Delta \ddot{x}(t) + G(t) \Delta u(t) + w(t) \]  \hspace{2cm} (59)

\[ \Delta x_k = H_k \Delta \ddot{x} + v_k \]  \hspace{2cm} (60)

Attention now turns to finding explicit forms for \( F(t) \) and \( H_k \). It isn’t necessary to find \( G(t) \) since numerical integration is used to propagate the state, and only the discretized version of \( F(t) \) is necessary for propagating the matrix Ricatti equations.

\( F(t) \) is formed by linearizing the state dynamics equations about a filtered estimate of the state, \( \hat{x}_{k|k} \). The nonlinear dynamics equations for propagating \( \ddot{x} \) are based on those used for the propagation of \( x \), which are
\[ \ddot{q} = \frac{1}{2} \Omega(\omega) \ddot{q} \]  \hspace{2cm} (61)
\[ \dot{\omega} = I^{-1}(-[\omega \times] \dot{\omega} + n_d + n_c) \]  \hspace{2cm} (62)
\[ \dot{n}_d = 0 \]  \hspace{2cm} (63)

Focusing first on the quaternion update, an expression must be found for the linear time evolution of \( \delta q \). In other words, \( F_1(t) \) in the equation
\[ \delta q(t) = F_1(t) \Delta \ddot{x}(t) + G_1(t) \Delta u(t) + w_1(t) \]  \hspace{2cm} (64)
is sought. Equations useful for deriving \( F_1 \) are repeated here for convenience
\[ \ddot{q} = \delta q \otimes \ddot{q} \]  \hspace{2cm} (65)

Also useful are the following properties of quaternion composition:
- Association
\[ (a \otimes b) \otimes c = a \otimes (b \otimes c) \]  \hspace{2cm} (66)
- Commutative relation
\[ a \otimes b = b \otimes a + 2 \begin{bmatrix} b \times a \\ 0 \end{bmatrix} \]  \hspace{2cm} (67)
- Product rule for quaternion composition
\[ \frac{d}{dt} (a \otimes b) = \left( \frac{d}{dt} a \right) \otimes b + a \otimes \left( \frac{d}{dt} b \right) \]  \hspace{2cm} (68)

Applying the product rule to (65) yields
\[ \ddot{q} = \delta \dot{q} \otimes \ddot{q} + \delta q \otimes \ddot{q} \]  \hspace{2cm} (69)

into which the definitions for the derivatives are substituted
\[ \frac{1}{2} \Omega(\omega) \ddot{q} = \delta \dot{q} \otimes \ddot{q} + \frac{1}{2} \delta q \otimes \Omega(\omega) \ddot{q} \]  \hspace{2cm} (70)

Rearranging, and using the quaternion inverse \( q^{-1} \) defined by
\[ q \otimes q^{-1} = [0,0,0,1]^T \]  \hspace{2cm} (71)
yields
\[ \delta \ddot{q} = \frac{1}{2} \Omega(\omega) \delta q \otimes q^{-1} - \frac{1}{2} \delta \dot{q} \otimes \Omega(\omega) [0,0,0,1]^T \]  \hspace{2cm} (72)

But, by definition,
\[ \delta \dot{q} = \ddot{q} \otimes \ddot{q}^{-1} \]  \hspace{2cm} (73)
yielding
\[ \delta \ddot{q} = \frac{1}{2} \Omega(\omega) \delta q \otimes q^{-1} - \frac{1}{2} \delta \dot{q} \otimes \Omega(\omega) [0,0,0,1]^T \]  \hspace{2cm} (74)

Let
\[ \ddot{\omega} = \begin{bmatrix} \omega \\ 0 \end{bmatrix} \]  \hspace{2cm} (75)
then further simplification yields
\[ \delta \ddot{q} = \frac{1}{2} \Omega(\omega) \delta q \otimes q^{-1} - \frac{1}{2} \delta \dot{q} \otimes \ddot{\omega} \]  \hspace{2cm} (76)

Noting that \( \Omega(q) \) is linear in its elements,
\[ \Omega(\omega) = \Omega(\ddot{\omega} + \Delta \omega) = \Omega(\ddot{\omega}) + \Omega(\Delta \omega) \]  \hspace{2cm} (77)
and invoking the commutative relation yields, after some cancellation
\[ \delta \ddot{q} = \left[ -[\omega \times] \delta q \right] + \frac{1}{2} \Omega(\Delta \omega) \delta q \]  \hspace{2cm} (78)
The following is observed about the second term on the right hand side:

\[
\Omega(\Delta \omega) \delta \hat{q} = \begin{bmatrix} \Delta \omega \\ 0 \end{bmatrix} \delta q_4 + \text{HOT} \tag{81}
\]

where \( \delta q_4 \approx 1 \) and \( \text{HOT} \) is made up of negligible second-order terms. With this approximation,

\[
\delta \hat{q} = \begin{bmatrix} -[\hat{\omega} \times] \delta \hat{q} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Delta \omega \\ 0 \end{bmatrix} \tag{82}
\]

from which the desired expression for \( F_1 \) is extracted

\[
F_1 = \begin{bmatrix} -[\hat{\omega} \times] \frac{1}{2} I_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \tag{83}
\]

The second component of the dynamics matrix, \( F_2 \) defined by

\[
\Delta \hat{\omega}(t) = F_2(t) \Delta \hat{x}(t) + G_2(t) \Delta \hat{u}(t) + w_2(t) \tag{84}
\]

is found by straightforward linearization of

\[
f_2(\hat{x}) = I^{-1}(-[\hat{\omega} \times]J \omega + n_d + n_e) \tag{85}
\]

so that

\[
F_2 = \frac{\partial f_2(\hat{x})}{\partial \hat{x}} \bigg|_{\hat{x}=\hat{x}_{\text{est}}} = \begin{bmatrix} 0_{3 \times 3} \Theta(\hat{\omega}) I^{-1} \end{bmatrix} \tag{86}
\]

where

\[
\Theta(\hat{\omega}) = \frac{df_2(\hat{x})}{d\omega} \bigg|_{\hat{x}=\hat{x}_{\text{est}}} \tag{87}
\]

may be written explicitly for a diagonal inertia tensor \( I \) as

\[
\Theta(\hat{\omega}) = \begin{bmatrix} \frac{\omega_3 (I_{yy} - I_{zz})}{I_{yy}} & \frac{\omega_3 (I_{yy} - I_{zz})}{I_{yy}} & 0 \\
\frac{\omega_2 (I_{zz} - I_{xx})}{I_{zz}} & \frac{\omega_2 (I_{zz} - I_{xx})}{I_{zz}} & 0 \\
0 & 0 & 0 \end{bmatrix} \tag{88}
\]

Finally, \( F_3 \), defined by

\[
\Delta \hat{u}(t) = F_3(t) \Delta \hat{x}(t) + G_3(t) \Delta \hat{u}(t) + w_3(t) \tag{89}
\]

is simply

\[
F_3 = [0_{3 \times 9}] \tag{90}
\]

by (63). Combining \( F_1 \), \( F_2 \) and \( F_3 \), yields the linearized dynamics matrix

\[
F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} -[\hat{\omega} \times] \frac{1}{2} I_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & \Theta(\hat{\omega}) I^{-1} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \tag{91}
\]

Attention now turns to finding a linearization for the measurement equation. That is, \( H_k \) is sought such that to first order

\[
\Delta z_k = H_k \Delta \hat{x} + v_k \tag{92}
\]

As mentioned previously, \( \Delta z_k \) is referred to as the innovation and is defined for the classical EKF as

\[
\Delta z_k = z_k - h(\hat{x}_k|k-1) \tag{93}
\]

This definition differs from the innovation used in [2], which is based on a cross-product. The present definition is preferred where mispointings may exceed 90° and ambiguity would arise using the cross-product. The physical significance of the cross-product innovation as reported in [2] is useful for interpretation, but provides no advantage over the classical innovation for overall filter accuracy. For the present filter, the measurement \( z_k \) contains the normalized magnetic field reading from the magnetometer, and may be augmented by scalar readings from the solar panels. Scalar solar panel readings are based on the relation

\[
\frac{i(\alpha)}{i(0)} = \cos(\alpha) = p^T s_B \tag{94}
\]

which equates a normalized panel current reading to the inner product of the unit vector normal to the panel, \( p \), and the normalized sun vector in body coordinates, \( s_B \), where \( \alpha \) is the sunlight incidence angle. The measurement \( z_k \) then becomes

\[
z_k = h(\hat{x}_k) + v_k = \begin{bmatrix} A(\hat{q}) s_k^B \\
\vdots \\
p^T(s_B \hat{q}) \end{bmatrix} + v_k \tag{95}
\]

where \( b_k \) and \( s_k \) are the magnetic field and sun vectors in the reference CS, and \( N \) is the number of sunlit solar panels. To find \( H_k \), \( h(\hat{x}_k) \) is linearized about the current best state estimate, \( \hat{x}_k \)

\[
H_k = \frac{\partial h(\hat{x})}{\partial \hat{x}} \bigg|_{\hat{x}=\hat{x}_k} \tag{96}
\]

To this end, \( A(\hat{q}) \) is rewritten as the product of factors

\[
A(\hat{q}) = \hat{A}(\hat{q}) \hat{A}(\hat{q}) \tag{97}
\]

The estimated magnetic field and sun vectors in body coordinates

\[
\hat{b}_k^B \equiv \hat{A}(\hat{q}) b_k, \quad \hat{s}_k^B \equiv \hat{A}(\hat{q}) s_k \tag{98}
\]

do not depend on any of the elements of the state \( \hat{x} \), and may be regarded as multiplicative constants. The rotation matrix \( \hat{A}(\delta \hat{q}) \) does depend on state elements, and is linearized by neglecting second-order terms

\[
\hat{A}(\delta \hat{q}) \approx I_{3 \times 3} - 2[\delta \hat{q} \times] \tag{99}
\]

The derivative of

\[
h(\hat{x}) = \begin{bmatrix} (I_{3 \times 3} - 2[\delta \hat{q} \times]) \hat{b}_k^B \\
\vdots \\
(I_{3 \times 3} - 2[\delta \hat{q} \times]) \hat{s}_k^B \end{bmatrix} \tag{100}
\]

is now affected by simple extraction of the linear terms

\[
H_k = \frac{\partial h(\hat{x})}{\partial \hat{x}} \bigg|_{\hat{x}=\hat{x}_k} = \begin{bmatrix} 2[b_k^B \times] & 0_{3 \times 6} \\
2[p^T(s_k^B \times) & 0_{1 \times 6} \\
\vdots \\
2[p^T(s_k^B \times) & 0_{1 \times 6} \end{bmatrix} \tag{101}
\]


Implementation follows the pattern outlined in section III with slight modification. The state vector $\hat{x}_{k|k}$ is propagated as usual with numerical integration to yield $\hat{x}_{k+1|k}$. However, when the body-referenced state error estimate, $\Delta \hat{x}_{k+1|k}$, is to be combined with $\hat{x}_{k+1|k}$ to yield an updated state estimate, care must be taken to combine the quaternions properly. The rate and disturbance torque components of the estimate are added as usual:

$$\dot{\omega}_{k+1|k+1} = \dot{\omega}_{k+1|k} + \Delta \dot{\omega}_{k+1|k+1} \tag{102}$$

$$\hat{n}_{d k+1|k+1} = \hat{n}_{d k+1|k} + \Delta \hat{n}_{d k+1|k+1} \tag{103}$$

but the updated quaternion is formed by

$$\hat{q}_{k+1|k+1} = \left[ \frac{\delta \hat{q}_{k+1|k+1}}{1 - ||\delta \hat{q}_{k+1|k+1}||^2} \right] \otimes \hat{q}_{k+1|k} \tag{104}$$

During initial convergence, the argument of the square root in (104) may become negative, meaning the small angle assumption has been violated by a large mispointing. When this condition is detected, the estimated error quaternion is written instead as

$$\frac{1}{\sqrt{1 + ||\delta \hat{q}_{k+1|k+1}||^2}} \left[ \frac{\delta \hat{q}_{k+1|k+1}}{1} \right] \tag{105}$$

The more accurate update (104) is again adopted as the filter settles and $||\delta \hat{q}_{k+1|k+1}||^2$ decreases below unity.

Also during initial convergence, the error covariance matrix $P$ may become very large due to violations of the small angle approximation. This is mitigated by starting the algorithm with an initial measurement update before performing the first time update. Also, the re-calculation of the $H_k$ matrix using the filtered state estimate as mentioned in section III helps reduce the size of $P$ at each sample step. Simulation has demonstrated that for some initial conditions, however, these countermeasures are not failsafe, and it becomes necessary to reset $P$ and $\hat{x}$ to their initial values, i.e., $P_{k+1|k+1} = P_{0|0}$, $\hat{x}_{k+1|k+1} = \hat{x}_{0|0}$. This reset is effected when the trace of $P$ exceeds a predetermined threshold.

Solar panel data is roughly an order of magnitude less accurate than magnetometer data, and is only incorporated as needed. Necessity is established by observing at each time step the elements of the innovation vector $\Delta \hat{x}_k$ corresponding to the scalar solar panel measurements. If these innovations exceed expected albedo contributions by a predetermined threshold, a flag is set. While the flag is set, solar panel data is incorporated into the EKF. The flag remains set until the sun is no longer available, or the innovations become sufficiently small over a sufficiently long span, at which time the flag is cleared and the filter uses magnetometer data only. Incorporation of the solar panel data serves a dual purpose: Initial convergence time is decreased and attitude anomalies arising in steady-state may be detected more easily using this second independent reference.

It should be noted that within the construct of the EKF, even one sunlit solar panel can provide useful data. At least two sunlit solar panels are required, in addition to the magnetometer data, to uniquely determine the spacecraft attitude, but one panel often reduces the estimate error significantly, allowing the magnetometer-based EKF to converge. Also, the EKF structure lends itself readily to additional vector or scalar measurements, should other sensors be available.

V. Simulation

A high fidelity simulation was chosen as means to test the new EKF. Analytical analysis of the linearized system is limited in its ability to predict filter accuracy and stability for the varying biases, initial conditions, and disturbances encountered in practice. For a thorough linear analysis of the magnetometer-based EKF, the reader is referred to [2].

A. Simulation Structure and Error Modeling

Simulations were carried out in Matlab Simulink. Attitude and ephemerides were generated with a three-degrees-of-freedom satellite rotational model and a two-body orbit propagator. Solar ephemerides were calculated using the algorithm presented in [9] to a precision of 0.01°. Albedo impingent on the solar panels was calculated assuming a diffusely radiating sphere [11] and a time varying albedo factor [10]. Shadowing effects were also taken into account for USUSAT's particular dual-boom structure.

Probable magnetic measurement related errors were calculated as shown in table I.

<table>
<thead>
<tr>
<th>TABLE I</th>
<th>MAGNETIC FIELD ESTIMATION AND MEASUREMENT ERROR SOURCES.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source</td>
<td>RMS Value(deg)</td>
</tr>
<tr>
<td>Modeling error (10th order)</td>
<td>0.1</td>
</tr>
<tr>
<td>In-track orbit uncertainty</td>
<td>0.384</td>
</tr>
<tr>
<td>Onboard magnets</td>
<td>0.5 (calibrated)</td>
</tr>
<tr>
<td>TAM noise</td>
<td>0.0077</td>
</tr>
<tr>
<td>12-bit quantization</td>
<td>0.027</td>
</tr>
<tr>
<td>Scale Factor</td>
<td>negligible</td>
</tr>
<tr>
<td>Orthogonality and alignment</td>
<td>0.5 (calibrated)</td>
</tr>
<tr>
<td>RSS Total</td>
<td>0.81</td>
</tr>
</tbody>
</table>

It would be at best ingenious to simulate the above errors by simply adding an uncorrelated error source producing an equivalent total RMS value of 0.81° to the simulation. Many of the above error sources are highly time-correlated. The Kalman filter deals much less effectively with time-correlated noise than with white noise sources. To approximate the autocorrelation of the above sources, two IGRF field models were used. The truth model was chosen as a 10th order IGRF model. The estimated field was a 6th order IGRF model with coefficients offset from the truth epoch by 5 years. This results in time-correlated
magnetic field errors with an RMS value close to 0.81°, as seen in Fig. 2.

Fig. 2. Typical magnetic field model error using a 10th order truth model and 6th order, 5 year offset estimation model.

The filter was applied successfully to several different s/c models, but most extensively tested using the specifications for USUSAT (15 kg, 51°, 400km circular orbit, $I_{xx} = I_{yy} = 0.85$, $I_{zz} = 1.6$ kg-m$^2$).

B. Filter Tuning

The parameters $P_{00}$, $R$, and $Q$ within the EKF may be modified to optimize its performance for a given application. These parameters must be chosen judiciously to balance inherent tradeoffs involved. For a linear Kalman filter, $P_{00}$ may be chosen arbitrarily large, with the rate of convergence increasing with larger $P_{00}$. When nonlinear dynamics and measurement equations are linearized in the EKF, however, it is implicitly assumed that the initial state estimate is close to the actual initial state, and a large $P_{00}$ causes the filter to diverge. Steady-state performance of the EKF is most directly linked to the process noise covariance matrix $Q$, which reflects disturbances and possible uncertainty in the s/c dynamics model. In the face of white Gaussian process and measurement noise, one would increase the value of the diagonal elements of $Q$ to add robustness and increase the bandwidth of the filter, and decrease the values to improve accuracy. When noise sources are non-Gaussian and non-white, changing the values in $Q$ has a less predictable effect. For the present filter, tuning proceeded as follows: The diagonal elements of $P_{00}$ were chosen slightly less than the square of the expected errors in the initial state vector $x_{00}$. All off-diagonal elements are set to zero. The diagonal elements of $R$ are chosen to reflect measurement error. For the magnetic field vector, these values are obtained by comparing the magnetic field truth model against the estimation model. For the scalar solar panel measurements, corresponding diagonal elements of $R$ reflect expected albedo contributions. Off-diagonal elements are set to zero. The elements of $Q$ corresponding to the vector part of the error quaternion are set to zero. The remaining six diagonal elements are initialized with the square of expected rate and torque errors, and then tuned to balance robustness and accuracy objectives.

C. Filter Evaluation

Extensive Monte-Carlo simulation was performed on the filter. Initial attitude and rates were varied, as well as simulation epoch and RAAN. Initial rates were bounded between 0.03 and 3 deg/s. No knowledge of initial attitude or rates was assumed.

Using magnetometer data alone, the filter converged in most cases, usually in less than one orbit. A typical example of this is given in Fig. 3.

Fig. 3. Typical convergence of magnetometer-only EKF

Rapid convergence using magnetometer data only is by no means guaranteed, however, as demonstrated by Fig. 4. Cases such as this arose with an average frequency of 1 in 10 when initial rates were varied near the orbital rate.

Fig. 4. Pathological case, magnetometer data only.

With the initial conditions of Fig. 4, solar panel data was added to the filter. The result is displayed in Fig. 5, along with a plot of sun availability and intensity for each panel. With the increased observability, the filter converges very rapidly. Extensive simulations of this sort were carried out, and for each case tested, the filter converged to less than 5° within one orbit.

Fig. 5. Convergence with solar panel data. The lower figure indicates availability and intensity of light incident on the solar panels.
A zoomed view of the latter half of Fig. 5 is provided in Fig. 6 to demonstrate steady-state accuracy. The filter performs within a 1.6° (1σ) envelope.

![Graph showing steady-state accuracy](image)

Fig. 6. A zoomed view of the second orbit of Fig. 5 (degrees).

Filter robustness was evaluated by subjecting the dynamic model of the s/c to a slowly varying external disturbance torque. Given the geometry and altitude of USUSAT, the only non-negligible disturbance torque will be aerodynamic. Assuming a 2cm offset between the center of pressure of the largest panel and the s/c center of mass, a 1μN-m disturbance torque is not unreasonable. A sinusoidally varying disturbance torque with such an amplitude was applied to the body X-axis. The results of this are displayed in Figs. 7 and 8. They demonstrate good estimation of the input torque, and little effect on overall accuracy.

![Graph showing estimation of input torque](image)

Fig. 7. Estimation of a 1μN-m amplitude slowly varying sinusoidal input torque on the body X-axis.

![Graph showing steady-state error](image)

Fig. 8. Steady state error for case with varying 1μN-m disturbance torque (degrees).

Robustness of the filter was also demonstrated by adding uncertainty to the s/c inertia tensor. A 10 percent moment of inertia variation on each axis produced negligible changes in steady-state accuracy. This is due to the filter’s ability to model inertia mismatching as a disturbance torque.

VI. Conclusion

A three-axis extended Kalman filter using magnetometer and solar panel data has been developed. A high-fidelity simulation was created to model s/c motion and error sources. The increased state observability due to the addition of solar panel data allows the filter to converge in simulation from any initial orientation and modest angular rates within one orbit. In steady-state, total angular error is below 1.6° (1σ), and is lower limited by magnetic field modeling error, orbit uncertainty, and measurement corruption from onboard magnets. The filter is robust against modest disturbance torques and inaccuracy in the s/c dynamic model, as these are estimated as part of the state vector. An AD system designed around this filter would be very low-cost and light-weight since a three-axis magnetometer is the only dedicated hardware component. Applications of the filter would include any s/c in inclined, low-earth orbit with body-mounted solar panels and modest pointing constraints.

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REFERENCES


[10] *NASA TM 48277*, pp. 9-10