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ABSTRACT

Two heuristic algorithms are developed to handle combinatorial and discrete complications persistent in pollution trading markets. The algorithms specifically account for heterogeneous trading ratios innate in environmental problems resulting from the differential solubility and receptor-sensitivity of pollutants. The cost efficiencies of the algorithms are demonstrated both through simple examples and by formal reasoning.

Keywords: advancement algorithm, heterogeneous trading ratio, least cost, retreat algorithm, pollution trading
Matching Traders in Pollution Trading Markets
with Heterogeneous Trading Ratios

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Running Title: “Pollution trading algorithms”

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Abstract

Two heuristic algorithms are developed to handle combinatorial and discrete complications persistent in pollution trading markets. The algorithms specifically account for heterogeneous trading ratios innate in environmental problems resulting from the differential solubility and receptor-sensitivity of pollutants. The cost efficiencies of the algorithms are demonstrated both through simple examples and by formal reasoning.

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1 Introduction

The cost efficiencies associated with pollution trading are by now well known. Beginning with Dales’ [5] basic intuition for trading and Montgomery’s [8] formal characterizations of both emissions and ambient permit market equilibria, the trading literature has mushroomed, encompassing not only numerous theoretical extensions to Montgomery, but also the design, implementation, and evaluation of a wide variety of markets.\(^1\) As Burtraw [3] points out, pollution markets are typically evaluated along one of two lines - the degree of cost efficiency obtained through trading (relative to the traditional command-and-control (CAC) regulatory regime) or the extent of market liquidity (i.e., trading activity), as reflected in trading volume and the time path of permit prices. While the market-liquidity evaluative approach is restricted to assessing the past performance of existing markets, the cost-efficiency approach can also be used to establish benchmarks for future performance, e.g., by actually determining the market’s least-cost allocation of abatement among the grand coalition of polluters.

This paper demonstrates two heuristic algorithms that can be used to establish such benchmarks, particularly for markets with heterogeneous trading ratios (resulting from spatial or mixing differences among polluters) and discrete abatement technology steps. Both algorithms provide a complete matching of buyers and sellers, i.e., they distinguish a specific pattern of trade among market participants which, in turn, provides as detailed a trading benchmark as possible. The first algorithm, which we call the “advancement algorithm,” is shown to obtain the least-cost abatement allocation under uniformly divisible

\(^1\)Tietenberg [12, 13] provides the most comprehensive collection of articles dealing with these issues.
abatement cost levels, or "steps," across all polluters. This would occur, for example, when the possible abatement steps for each source are "flexible," or large in number. The second algorithm, which we call the "retreat algorithm," is shown to enhance the cost efficiency of the advancement algorithm under the more general circumstance of non-uniform cost levels. Simple examples demonstrate the iterative procedures used by each algorithm and illustrate how the algorithms can be used to establish empirical reduced- or least-cost benchmarks for actual pollution markets.

Use of the advancement and retreat algorithms to delineate a specific pattern of trade that reduces the total abatement cost associated with CAC (and establishes a least-cost benchmark in the case of uniformly divisible abatement costs) differs markedly from the traditional simulation (e.g., linear programming) approach used in previous empirical studies. For example, McGartland and Oates [7] simulate a least-cost benchmark for particulate emissions in the Baltimore Air Quality Control Region based on estimates of integer-step abatement cost functions for over 400 polluters. While their simulation approach can be used to compare source-specific emissions under the CAC and least-cost regimes, the specific pattern of trade underscoring the comparison (which would conceivably result through permit trading) is not discernable. Therefore, policy makers have no way of comparing actual trades with the cost-efficient trades that should occur in an optimal trading outcome. Furthermore, McGartland and Oates’ comparison of the CAC and least-cost outcomes is based on estimated marginal cost functions for each source (that are derived

---

from the sources' actual discrete abatement costs), rather than directly on the discrete costs themselves. The algorithms presented in this paper overcome these two limitations.

The next section introduces a basic cost-minimization model of pollution trading with heterogeneous trading ratios and discrete abatement steps. Sec. 3 provides the basic intuition for, and enumerated procedures of, the advancement algorithm. Two simple examples of the algorithm are then provided in Sec. 4. The retreat algorithm is described in Sec. 5. This section demonstrates how the retreat algorithm corrects for cost inefficiencies associated with the advancement algorithm in the case of non-uniformly divisible abatement costs. Sec. 6 concludes.

2 Nomenclature and Optimization Program

Consider a market for trading abatement credits among \( n \) traders, with the set of these traders denoted by \( N = \{1, 2, \ldots, n\} \). (We refer to a trader synonymously with a polluter, a seller, or a purchaser, the use of which depends upon the context.) Define the function \( R : N \rightarrow \mathbb{R}_+ \) by

\[
R(i) = \text{the amount of abatement for polluter } i \text{ required by regulation.}
\]

Each polluter has her own abatement capability and associated cost structure. For computational convenience, we assume that a polluter has at most a countable (and practically

\[^{3}\text{In contrast to these types of simulations, Schmalensee et al. [10] demonstrate an approach using "counterfactual emissions" estimates to establish a CAC benchmark for Phase I electric generating plants participating in the 1990 Clean Air Act's sulfur dioxide emissions trading program. These counterfactual emissions estimates are compared with the plants' actual emissions under the trading program to assess the market's impact on plant-specific emissions. With this approach, neither a least-cost benchmark can be established nor the specific pattern of trade that would conceivably underscore the benchmark.}\]

\[^{4}\text{The C++ code for the advancement and retreat algorithms is available upon request from the authors, as is the executable program used to obtain the results presented in this paper.}\]
finite) number of technology options to abate pollutants. For example, a given polluter’s first-step technology might abate 100 kilograms of pollutants followed by 40 kilograms in the second step, and so on. Define the function $S : \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}$ by

$$S(i) = \text{the number of abatement steps implemented by polluter } i,$$

and the function $A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_+$ by

$$A(i, k) = \text{the abatement achieved by polluter } i \text{ in her } k-\text{th abatement step.}$$

Accordingly, a polluter’s abatement-cost function is defined on a countable domain via the function $C : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_+$ where

$$C(i, k) = \text{the cost incurred by polluter } i \text{ in her } k-\text{th abatement step.}$$

With the above definitions, total abatement and the associated total cost incurred by polluter $i$ are given by $\sum_{k=1}^{S(i)} A(i, k)$ and $\sum_{k=1}^{S(i)} C(i, k)$, respectively. In the absence of trading, the authority’s regulations are met only when

$$R(i) \leq \sum_{k=1}^{S(i)} A(i, k) \text{ for all } i \in \mathbb{N}.$$

However, trading enables polluter $i$ to sell any extra abatement, i.e., abatement credits, $\sum_{k=1}^{S(i)} A(i, k) - R(i) > 0$, to other polluters (who perhaps experience higher abatement costs). In this regard, define the function $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_+$ by

$$P(i, j) = \text{the abatement credits that trader } i \text{ sells to trader } j.$$

Heterogeneity of pollutant solubility and receptor-sensitivity (i.e., spatiality) among polluters complicates pollution trading. For example, abatement of 100 kilograms by “upstream” polluter 1 may be equivalent to only abatement of 80 kilograms by “downstream”
polluter 2 in terms of a given receptor point. In this case, 80 kilograms of abatement by polluter 2 could account for 100 kilograms of abatement for polluter 1 if abatement credits are exchangeable. Hence, all else equal, it is preferable for “receptor-sensitive” polluters such as polluter 2 to abate excessively and sell their credits to “receptor-insensitive” polluters such as polluter 1. We henceforth assume that such quantity scaling through trading is linear, and trading ratios therefore can be represented by a matrix (Whitehead [14] shows a hypothetical determination of such matrices based on “environmental-equivalence” ratios or transfer coefficients.) Define the function $T : N \times N \rightarrow \mathbb{R}_+$ by

$$T(i, j) = \text{the trading ratio between seller } i \text{ and purchaser } j,$$

which is simply a functional representation of an $n \times n$ matrix. It is natural to assume reciprocal symmetry, $T(i, j)T(j, i) = 1$ for all $i, j \in N$. In particular, this implies $T(i, i) = 1$ for all $i \in N$. Given $T$, we have

$$P(i, j)/T(i, j) = \text{the effective abatement credits that trader } i \text{ sells to trader } j.$$

If $T(i, j) < 1$ (respectively, $> 1$), then the quantity “swells” (respectively, “shrinks”) when trader $j$ purchases abatement credits from trader $i$. Furthermore,

$$T(i, l) = T(i, j)T(j, l),$$

which means that trading from $i$ through $j$ to $l$ is equivalent to trading from $i$ directly to $l$ in terms of the scale of swelling or shrinking.

**Example.** Suppose that polluter 1 is located upstream from polluter 2. Hence, due to spatial differences, 20 percent of pollutants from polluter 1 and 25 percent of pollutants
from polluter 2 transmit to the receptor. In this case, abatement of 100 kilograms by polluter 1 translates to 20 kilograms at the receptor, which, in turn, is equivalent to abatement of 80 kilograms by polluter 2. Relatively speaking then, polluter 2’s trading ratio with polluter 1 is \( T(2,1) = 80/100 \) and polluter 1’s trading ratio with polluter 2 is \( T(1,2) = 100/80 \). The trading ratio matrix therefore looks like

\[
T = \begin{pmatrix}
1 & 5/4 \\
4/5 & 1
\end{pmatrix}.
\]

If the two polluters have the same constant marginal cost of abatement, then it is obviously least-cost for polluter 1 to purchase all of her required abatement from polluter 2.\(^5\) Suppose initially that both polluters are required to abate \( R(i) = 100 \) kilograms. Then, the optimal solution (i.e., least-cost trading) is that polluter 2 abates 180 kilograms and sells her credit of 80 kilograms to polluter 1, i.e., \( P(2,1) = 80 \). This fulfills the abatement requirement for polluter 1 as well since \( P(2,1)/T(2,1) = 100 = R(1) \). Our observation that the abatement “swells” from 80 to 100 by trading will play a key role in the algorithms to be discussed below.

\(^5\)The more general case of unequal, non-constant marginal cost of abatement is explored below in Sec. 4 and 5.
cluding net traded credits. Letting $S$ and $P$ denote the sets of all possible functions $S : N \rightarrow \mathbb{N} \cup \{0\}$ and $P : N \times N \rightarrow \mathbb{R}_+$, respectively, we express the problem as

\[
\min_{P \in \mathcal{P}, S \in \mathcal{S}} \sum_{i=1}^{n} \sum_{k=1}^{n} C(i, k)
\]

subject to

\[
\sum_{k=1}^{n} A(i, k) + \sum_{j=1}^{n} P(j, i)/T(j, i) - \sum_{j=1}^{n} P(i, j) \geq R(i) \quad \text{for each} \quad i \in N
\]

and

\[
P(i, j) > 0 \Rightarrow P(j, i) = 0 \quad \text{for each pair} \quad i, j \in N.
\]

The left-hand side of (3) is polluter $i$'s own-abatement + effective credits purchased - credits sold. Equation (4) prohibits bidirectional trading. Also, (4) implies that the diagonal of $P$ is zero.

Lastly, for convenience define the functional $F : N \times S \times P \rightarrow \mathbb{R}$ by

\[
F(i, S, P) = R(i) + \sum_{j=1}^{n} P(i, j) - \sum_{k=1}^{n} A(i, k) - \sum_{j=1}^{n} P(j, i)/T(j, i).
\]

Constraint (3) then reduces to

\[
F(i, S, P) \leq 0 \quad \text{for each} \quad i \in N.
\]

When $F(i, S, P)$ is positive, polluter $i$ incurs an abatement deficit. Constraint (5) is thus equivalent to the elimination of an abatement deficit for each polluter.

### 3 Advancement Algorithm

The cost minimization problem (2)-(4) is discrete and combinatorial in nature due to $S$ and $P$, thus obviating conventional numerical optimization techniques. Hence, an iterative heuristic algorithm is necessary to solve our problem. We first discuss the intuition for this algorithm in Sec. 3.1. The enumerated procedure for the algorithm is then presented in Sec. 3.2. The name “advancement” algorithm will prove reasonable shortly.
3.1 The Basic Intuition

Start with the null state where no polluters abate (i.e., \( S_0(i) = 0 \) for all \( i \in N \)) and thus no trading has occurred (i.e., \( P_0(i, j) = 0 \) for all \( i, j \in N \)). The algorithm iteratively updates \( S_m \) and \( P_m \) to \( S_{m+1} \) and \( P_{m+1} \), respectively, such that exactly one polluter "advances" her abatement step in each iteration, aiming to reduce abatement deficits \( F(i, S_m, P_m) \).

In other words, in the \((m + 1)\)-st iteration, an increment occurs as \( S_{m+1}(i) = S_m(i) + 1 \) for exactly one \( i \in N \). Polluter \( i \)'s resultant abatement may then be distributed to other polluters through trading, and \( P \) is accordingly modified. The process is iterated so that \( S \) and \( P \) eventually fulfill constraint (5) while keeping the associated total cost (2) as small as possible. The algorithm ends at the \( M \)-th iteration where constraint (5) is first satisfied.

For each iteration, say, in the \((m + 1)\)-st iteration, the primary task is to determine for which polluter \( i \) should the abatement step \( S_m(i) \) be advanced. This selection takes the cost and benefit of each polluter's abatement into account. If polluter \( i \) advances, the resultant abatement and associated cost are \( A(i, S_m(i) + 1) \) and \( C(i, S_m(i) + 1) \), respectively.

It may be intuitively natural to advance the step of the trader \( i \) who has the least average cost \( C(i, S_m(i) + 1)/A(i, S_m(i) + 1) \). However, this idea is cursory; the denominator (or the quantity) may be further swelled (respectively, shrunk) via trading by the trading ratios, thus reducing (respectively, increasing) effective average cost. Recall from Sec. 2 that the quantity of polluter \( i \)'s abatement swells greater as the trading ratio \( T(i, j) \) is smaller. Thus, seller \( i \) should preferably assign the highest precedence of selling her abatement to prospective purchaser \( j \) who has the lowest trading ratio \( T(i, j) \). As such, we need to queue the prospective purchasers of \( i \)'s abatement in the order of increasing trading ratios, which
we call a "derangement."^6

Since by the \( m \)-th iteration those who have fulfilled the constraint, \( F(j, S_m, P_m) \leq 0 \), are not included in the list of prospective purchasers of seller \( i \)' abatement, we consider \( \tilde{N}_m = \{ j \in N \mid F(j, S_m, P_m) > 0 \} \) as the list of prospective purchasers. \( \tilde{N}_m \) can include seller \( i \) herself, but the quantity that she sells to herself is necessarily reinterpreted as unsold abatement. Suppose that \( j_1, j_2, \ldots \) is the deranged list of the potential purchasers \( \tilde{N}_m \). Hence, trader \( i \) assigns the highest precedence of selling her abatement to purchaser \( j_1 \), followed by \( j_2 \), and so on. There are three cases to consider in calculating trader \( i \)'s effective average cost:

**CASE 1:** If trader \( i \)'s next-step abatement \( A(i, S_m(i) + 1) \) is not even enough to meet purchaser \( j_1 \)'s abatement deficit (i.e., \( A(i, S_m(i) + 1)/T(i, j_1) < F(j_1, S_m, P_m) \)), then the resultant swelled quantity is \( A(i, S_m(i) + 1)/T(i, j_1) \). Hence, trader \( i \)'s effective average cost is

\[
\frac{C(i, S_m(i) + 1)}{A(i, S_m(i) + 1)/T(i, j_1)}. \tag{6}
\]

**CASE 2:** If trader \( i \), in her next step, is able to abate an amount sufficient to fulfill the sum of all prospective purchasers’ abatement deficits, then the swelled/shrunk quantity as a result of trading is \( \sum_{j \in \tilde{N}_m} F(j, S_m, P_m) \). (Note that \( F(j, S_m, P_m) \) is already in purchaser \( j \)'s quantity scale, thus there is no need to divide it by \( T(i, j) \).) If we denote the number of purchasers by \( \tilde{n} = |\tilde{N}_m| \), then this swelled/shrunk quantity is rewritten as \( \sum_{\alpha=1}^{\tilde{n}} F(j_{\alpha}, S_m, P_m) \), and trader \( i \)'s effective average cost is

\[
\frac{C(i, S_m(i) + 1)}{\sum_{\alpha=1}^{\tilde{n}} F(j_{\alpha}, S_m, P_m)}. \tag{7}
\]

^6"Derangement" technically refers to reordering of sequence.
CASE 3: If the situation is neither CASE 1 nor CASE 2, then trader \( i \) is able to fulfill at least one prospective purchaser \( j_1 \)'s abatement deficit, but cannot fulfill that of everyone. Thus, let \( \alpha^*(1 \leq \alpha^* < \bar{n}) \) be the number of purchasers (possibly including herself) whose deficits can be fulfilled as a result of trader \( i \)'s selling her next-step abatement. Then, the fulfillment amounts to the swelled/shrunk quantity \( \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) \). Yet, this swelled/shrunk quantity is not all the benefits of trader \( i \)'s abatement. She may still have remnant \( A(i, S_m(i) + 1) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m)T(i, j_{\alpha}) \) even after fulfilling \( \alpha^* \) purchasers' deficits. Even though she cannot completely fulfill the deficit of the \((\alpha^* + 1)\)-st purchaser (by definition of \( \alpha^* \)), she can still sell this remnant to prospective purchaser \( j_{\alpha^*+1} \), the swelled/shrunk quantity of which is

\[
\frac{A(i, S_m(i) + 1) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m)T(i, j_{\alpha})}{T(i, j_{\alpha^*+1})}.
\]

In total, the swelled/shrunk quantity as a result of trader \( i \)'s abatement is

\[
\Omega := \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) + \frac{A(i, S_m(i) + 1) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m)T(i, j_{\alpha})}{T(i, j_{\alpha^*+1})},
\]

and trader \( i \)'s effective average cost is

\[
C(i, S_m(i) + 1)/\Omega
\]

(8)

Recall that the purpose of calculating these effective average costs was to select a trader who, in the next step, has the least effective average cost. The algorithm lets this trader \( i^* \) advance her step, thus \( S_{m+1}(i^*) = S_m(i^*) + 1 \) whereas \( S_{m+1}(i) = S_m(i) \) for all \( i \neq i^* \). The abatement \( A(i^*, S_{m+1}(i^*)) \) will be distributed (i.e., sold) to successive traders in seller \( i^* \)'s derangement, \( j_1, j_2, \ldots, j_{\bar{n}} \). If seller \( i^* \) can fulfill trader \( j_{\alpha} \)'s current
abatement deficit, then the net amount $P(i^*, j_\alpha)$ sold by trader $i^*$ to trader $j_\alpha$ increments by $F(j_\alpha, S_m, P_m)T(i^*, j_\alpha)$. If CASE 3 holds, the remnant sold by trader $i^*$ to the $(\alpha^* + 1)$-st purchaser is $A(i^*, S_{m+1}(i^*)) - \sum_{\alpha=1}^{\alpha^*} F(j_\alpha, S_m, P_m)T(i^*, j_\alpha)$, by which $P(i^*, j_{\alpha^*+1})$ increments.

In the course of repeated iterations, it is possible that a trader who acted as a net seller in earlier iterations of the algorithm turns out to be a net purchaser in later iterations (or vice-versa). This can occur, for example, if a trader in her first step has a low effective average cost but uncomparably high effective average cost in her second step. To comply with (4), the algorithm at the end of each iteration modifies $P$ so that only net sellers have positive values.

### 3.2 Enumerated Procedures of Advancement Algorithm

Define $S_0 \in \mathcal{S}$ and $P_0 \in \mathcal{P}$ by $S_0(i) = P_0(i,j) = 0$ for all $i,j \in N$. Next, let $S_m$ and $P_m$ be updated to $S_{m+1}$ and $P_{m+1}$, respectively, in each $(m+1)$-st iteration, $m = 0, 1, 2, \ldots$

**PROCEDURE 1.** Let $\tilde{N}_m = \{j \in N \mid F(j, S_m, P_m) > 0\}$ and $\tilde{n} = |\tilde{N}_m|$.

**PROCEDURE 2.** For each fixed trader $i \in N$ (as a potential seller), implement the following:

Let $\{j_1, j_2, \ldots, j_{\tilde{n}}\}$ be the derangement of the set $\tilde{N}_m$ such that

$$T(i, j_1) \leq T(i, j_2) \leq \cdots \leq T(i, j_{\tilde{n}}).$$

Let $\alpha^*$ be the largest number such that $0 \leq \alpha^* \leq \tilde{n}$ and

$$\sum_{\alpha=1}^{\alpha^*} F(j_\alpha, S_m, P_m)T(i, j_\alpha) \leq A(i, S_m(i) + 1).$$
Define the effective average cost (EAC) function \( EAC : N \times N \to \mathbb{R} \) by

\[
EAC(i, S_m(i) + 1) = \begin{cases} 
  \frac{C(i, S_m(i) + 1)}{A(i, S_m(i) + 1)/T(i, S_m(i) + 1)} & \text{if } \alpha^* = 0 \\
  \frac{\sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m)}{C(i, S_m(i) + 1)/\Omega} & \text{if } \alpha^* = \tilde{n}. \\
  \frac{A(i, S_m(i) + 1)}{T(i, S_m(i) + 1)} & \text{otherwise}
\end{cases}
\]  

(cf. Equations (6)-(8).)

PROCEDURE 3. Choose \( i^* = \min_{i \in N} EAC(i, S_m(i) + 1) \). Then, define \( S_{m+1} \in \mathcal{S} \) by

\[
S_{m+1}(i) = S_m + \delta_{i, i^*},
\]  

where \( \delta_{i, i^*} \) is the kronecker delta. For \( i^* \), let \( \{j_1, j_2, \ldots, j_n\} \) be the derangement of the set \( \tilde{N}_m \) such that

\[
T(i^*, j_1) \leq T(i^*, j_2) \leq \cdots \leq T(i^*, j_{\tilde{n}}),
\]

and let \( \alpha^* \in N \) be the largest number such that

\[
\sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) T(i^*, j_{\alpha}) \leq A(i^*, S_{m+1}(i)).
\]

Define \( \hat{P}_{m+1} \in \mathcal{P} \) by

\[
\hat{P}_{m+1}(i^*, j_{\alpha}) = P_m(i^*, j_{\alpha})
\]

\[
\begin{cases} 
  F(j_{\alpha}, S_m, P_m) T(i^*, j_{\alpha}) & \text{if } \alpha \leq \alpha^* \\
  A(i^*, S_{m+1}(i^*)) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) T(i^*, j_{\alpha}) & \text{if } \alpha = \alpha^* + 1 \leq \tilde{n} \\
  0 & \text{otherwise}
\end{cases}
\]

for each \( \alpha = 1, 2, \ldots, n \) and

\[
\hat{P}_{m+1}(i, j) = P_m(i, j) \text{ for all } i \in N \setminus \{i^*\} \text{ and } j \in N.
\]

PROCEDURE 4. Lastly, correct \( \hat{P}_{m+1} \) to obtain \( P_{m+1} \) by

\[
P_{m+1}(i, i) = 0 \text{ for all } i \in N
\]
and

\[ P_{m+1}(i,j) := \begin{cases} \hat{P}_{m+1}(i,j) - \hat{P}_{m+1}(j,i)/T(j,i) & \text{if } \hat{P}_{m+1}(i,j) > \hat{P}_{m+1}(j,i)/T(j,i) \\ 0 & \text{otherwise} \end{cases} \] (12)

for all \( i, j \in N \).

In each \((m + 1)\)-st iteration, we implement PROCEDURES 1-4 sequentially. This is repeated until the required abatement for each polluter is fulfilled, i.e., \( F(i, S_m, P_m) \leq 0 \) for all \( i \in N \) as in (5). The resultant \( S_M \) and \( P_M \) will be the finalized trading policy, \( S \) and \( P \), heuristically solving (2)-(5). See the appendix for a proof of the optimality of this algorithm under a particular condition.

4 Examples of The Advancement Algorithm

4.1 Two Traders

This section provides a simple example to elucidate the enumerated procedures of Sec. 3.2, as well as to validate the algorithm. The setting is a minor modification of the example in Sec. 2.

Consider a two-polluter community \( N = \{1, 2\} \) where both polluters are supposed to abate \( R(1) = R(2) = 100 \) kilograms. As in Sec. 2, let the trading ratios be \( T(1, 2) = 5/4 \) and \( T(2, 1) = 4/5 \). Suppose that polluter 1 has only one abatement step whose attributes are \( A(1, 1) = 100 \) and \( C(1, 1) = 100 \). Suppose that polluter 2 has two abatement steps whose attributes are \( A(2, 1) = 90, C(2, 1) = 90, A(2, 2) = 90, \) and \( C(2, 2) = 100 \).

ITERATION 1: Currently, \( S_0(1) = S_0(2) = 0, P_0(1, 2) = P_0(2, 1) = 0, \) and \( F(1, S_0, P_0) = F(2, S_0, P_0) = 100 \). First, for the possible seller \( i = 1 \), the deranged list of possible pur-
chasers is \{1, 2\} and \( \alpha^* = 1 \). By (9), trader 1’s effective average cost is

\[
EAC(1, S_0(1) + 1) = \frac{C(1, S_0(1) + 1)}{F(1, S_0, P_0) + \frac{A(1, S_0(1) + 1) - F(1, S_0, P_0)T(1, 1)}{T(1, 2)}} = \frac{100}{100 + \frac{100 \cdot 1}{5/4}} = 1
\]

Second, for the possible seller \( i = 2 \), the deranged list of possible purchasers is also \{1, 2\} and \( \alpha^* = 1 \). By (9), trader 2’s effective average cost is

\[
EAC(2, S_0(2) + 1) = \frac{C(2, S_0(1) + 1)}{F(1, S_0, P_0) + \frac{A(2, S_0(2) + 1) - F(1, S_0, P_0)T(2, 2)}{T(2, 2)}} = \frac{90}{100 + \frac{90 - 100 \cdot 4/5}{1}} = \frac{9}{11}\]

Since \( EAC(1, S_0(1) + 1) > EAC(2, S_0(2) + 1) \), we choose \( i^* = 2 \) and advance her step as \( S_1(2) = S_0(2) + 1 = 1 \) by (10). But we retain \( S_1(1) = S_0(1) = 0 \). Since \( \alpha^* = 1 \) and the deranged list of purchasers is \{1, 2\} for seller \( i^* = 2 \), trading fulfills purchaser 1’s abatement deficit. For this, the seller \( i^* = 2 \) sells \( F(1, S_0, P_0)T(2, 1) = 100 \cdot 4/5 = 80 \) to purchaser 1, and

\[
\hat{P}_1(2, 1) = P_0(2, 1) + F(1, S_0, P_0)T(2, 1) = 0 + 80 = 80
\]

by (11). The seller \( i^* = 2 \) sells its remnant \( A(2, S_1(2)) - F(1, S_0, P_0)T(2, 1) = 90 - 80 = 10 \) to herself, and

\[
\hat{P}_1(2, 2) = P_0(2, 2) + [A(2, S_1(2)) - F(1, S_0, P_0)T(2, 1)] = 0 + 10 = 10
\]

again by (11). There is no update to \( P(1, \cdot) \), so \( \hat{P}_1(1, 1) = P_0(1, 1) = 0 \) and \( \hat{P}_1(1, 2) = P_0(1, 2) = 0 \).

By PROCEDURE 4, we correct \( \hat{P}_1 \) to \( P_1 \). The only change is \( \hat{P}_1(2, 2) = 10 \Rightarrow P_1(2, 2) = 0 \). For other trading pairs \( (i, j) \in N \times N \backslash \{(2, 2)\} \), we simply set \( P_1(i, j) = \hat{P}_1(i, j) \).
ITERATION 2: Currently, $S_1(1) = 0$, $S_1(2) = 1$, $P(1,2) = 0$, $P(2,1) = 80$, $F(1,S_1,P_1) = 0$, and $F(2,S_1,P_1) = 90$. First, for the possible seller $i = 1$, the deranged list of possible purchasers is $\{2\}$ and $\alpha^* = 0$. By (9), trader 1’s effective average cost is

$$EAC(1, S_1(1) + 1) = \frac{C(1, S_1(1) + 1)}{A(1, S_1(1) + 1)/T(1, 2)} = \frac{100}{100/(5/4)} = \frac{5}{4}.$$ 

Second, for the possible seller $i = 2$, the deranged list of possible purchasers is also $\{2\}$ and $\alpha^* = 1$. By (9), trader 2’s effective average cost is

$$EAC(2, S_1(2) + 1) = \frac{C(2, S_1(2) + 1)}{F(2, S_1, P_1)} = \frac{100}{90} = \frac{10}{9}.$$ 

Since $EAC(1, S_1(1) + 1) > EAC(2, S_1(2) + 1)$, we choose $i^* = 2$ again, and advance her steps as $S_2(2) = S_1(2) + 1 = 2$ by (10). But we retain $S_2(1) = S_1(1) = 0$. Since $\alpha^* = 1$ and the deranged list of purchasers is $\{2\}$ for the seller $i^* = 2$, trading fulfills the purchaser 2’s own abatement deficit. For this, the seller $i^* = 2$ sells $F(2, S_1, P_1)T(2, 2) = 90 \cdot 1 = 90$ to herself, and

$$\hat{P}_2(2, 2) = P(2, 2) + F(2, S_1, P_1)T(2, 2) = 0 + 90 = 90$$ 

by (11). Since no other tradings occur, $\hat{P}_2(i, j) = P(i, j)$ for all $(i, j) \in N \times N \setminus \{(2, 2)\}$.

By PROCEDURE 4, we correct $\hat{P}_2$ to $P_2$. The only change is $\hat{P}_2(2, 2) = 90 \Rightarrow P_2(2, 2) = 0$. For other trading pairs $(i, j) \in N \times N \setminus \{(2, 2)\}$, we simply set $P_2(i, j) = \hat{P}_2(i, j)$.

END OF ITERATIONS: Currently, $S_2(1) = 0$, $S_2(2) = 2$, $P_2(1, 2) = 0$, $P_2(2, 1) = 80$, $F(1,S_2,P_2) = 0$, and $F(2,S_2,P_2) = 0$. Since constraint (5) is satisfied, we terminate the iterations. The final policy is, therefore, $S(1) = 0$, $S(2) = 2$, $P(1, 2) = 0$, and $P(2, 1) = 80$. In other words, polluter 1 abates nothing while polluter 2 abates $\sum_{k=1}^{S(2)} A(2, k) = 180$
kilograms, out of which \( P(2, 1) = 80 \) kilograms are sold to polluter 1, which actually amounts to \( P(2, 1)/T(2, 1) = 100 \) kilograms of swelled quantity equaling the required abatement \( R(1) = 100 \) for polluter 1. Compare this result with the example in Sec. 2. The resultant total cost is \( \sum_{i=1}^{2} \sum_{k=1}^{S(i)} C(i, k) = C(2, 1) + C(2, 2) = 190 \), which is lower than the total cost 290 that would arise in the absence of trading. In this example, it is also easy to see that the advancement algorithm solves the cost-minimization problem (2)-(5).

### 4.2 Multiple Traders

We now demonstrate the advancement algorithm with a little more realistic example from Whitehead [14]. In this example, we consider a water quality trading market with five polluters, whose basic data is summarized in Table 1. Though the table shows only the first three steps of abatement processes, we assume that \( A(i, k) = a \) for all \( i \in N \) and \( k > 3 \). Table 2 summarizes the pre-determined trading ratios among the five traders. The iteration-wise results of the advancement algorithm (of Sec. 3.2) with respect to this example are shown in Tables 3 and 4.\(^7\)

A glance at Table 1 suggests highest efficiency for trader 1 in the first step. Hence, in the absence of huge variations in trading ratios, it is reasonable to expect that trader 1 makes the first advancement of \( S \), which is indeed true in Iteration 1 in Table 3. This abatement is sold to herself, which is reinterpreted as unsold abatement. Since her first-step abatement \( A(1, 1) \) falls short of her own required reduction \( R(1) \), she is not done yet (i.e., \( F(1, S_1, P_1) > 0 \)). In the next iteration, trader 4 is nominated as the most efficient seller.

\(^7\)Due to the rough digital representations of trading ratios, the results contain nontrivial numerical inconsistency.
Since trader 4 (as a seller) has the lowest trading ratio $T(4, j)$ with trader 1 (as a purchaser), trader 4's abatement $A(4, 1) = 16$ is sold to trader 1, which amounts to the swelled quantity $A(4, 1)/T(4, 1) \approx 69.57$, thus reducing polluter 1's abatement deficit by this amount, from 255.00 to 185.44. Similar iterative processes ensue. Notice that trader 2 turns from a net purchaser to a net seller against trader 5 between Iterations 5 and 6 (c.f. Table 4). This is a case forecasted in the last paragraph of Sec. 3.1. Its consequential modification is implemented by PROCEDURE 4 of Sec. 3.2.

The repetition ends at Iteration 6, as requirement (5) is first satisfied. With the final $S = (2, 1, 0, 2, 1)$, the resultant total abatement cost for the community is

$$C(1, 1) + C(1, 2) + C(2, 1) + C(4, 1) + C(4, 2) + C(5, 1) = 3,454,464$$

compared with the total cost that would arise in the absence of trading:

$$C(1, 1) + C(1, 2) + C(2, 1) + C(3, 1) + C(4, 1) + C(4, 2) + C(5, 1) = 9,762,715$$

for $S = (2, 1, 1, 2, 1)$.

Table 1: A set of information about five traders.
<table>
<thead>
<tr>
<th>(i) (\times) (j)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>2.58</td>
<td>2.86</td>
<td>3.87</td>
<td>3.82</td>
</tr>
<tr>
<td>2</td>
<td>0.39</td>
<td>1.00</td>
<td>1.11</td>
<td>1.50</td>
<td>1.48</td>
</tr>
<tr>
<td>3</td>
<td>0.35</td>
<td>0.90</td>
<td>1.00</td>
<td>1.35</td>
<td>1.33</td>
</tr>
<tr>
<td>4</td>
<td>0.23</td>
<td>0.67</td>
<td>0.74</td>
<td>1.00</td>
<td>0.99</td>
</tr>
<tr>
<td>5</td>
<td>0.26</td>
<td>0.68</td>
<td>0.75</td>
<td>1.01</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 2: Trading ratio matrix \(T(i, j)\) among five traders.

### Iteration 1:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(EAC(i, S_1(i)))</th>
<th>(S_1(i))</th>
<th>(S_2(i))</th>
<th>(P_1(i, j))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>547.51</td>
<td>* 1</td>
<td>91</td>
<td>1 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>2</td>
<td>2551.53</td>
<td>0</td>
<td>0</td>
<td>2 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>3</td>
<td>8307.98</td>
<td>0</td>
<td>0</td>
<td>3 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>4</td>
<td>805.46</td>
<td>0</td>
<td>0</td>
<td>4 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>5</td>
<td>1303.24</td>
<td>0</td>
<td>0</td>
<td>5 0.00 0.00 0.00 0.00 0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(i)</th>
<th>(EAC(i, S_1(i)))</th>
<th>(S_1(i))</th>
<th>(S_2(i))</th>
<th>(P_1(i, j))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1168.01</td>
<td>1</td>
<td>91</td>
<td>1 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>2</td>
<td>2751.31</td>
<td>0</td>
<td>0</td>
<td>2 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>3</td>
<td>9009.86</td>
<td>0</td>
<td>0</td>
<td>3 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>4</td>
<td>805.46</td>
<td>* 2</td>
<td>16</td>
<td>4 16.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>5</td>
<td>1487.66</td>
<td>0</td>
<td>0</td>
<td>5 0.00 0.00 0.00 0.00 0.00</td>
</tr>
</tbody>
</table>

### Iteration 2:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(EAC(i, S_2(i)))</th>
<th>(S_2(i))</th>
<th>(S_3(i))</th>
<th>(P_2(i, j))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1308.17</td>
<td>2</td>
<td>91+623</td>
<td>1 0.00 437.57 0.00 0.00 0.00</td>
</tr>
<tr>
<td>2</td>
<td>2926.46</td>
<td>0</td>
<td>0</td>
<td>2 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>3</td>
<td>9631.91</td>
<td>0</td>
<td>0</td>
<td>3 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>4</td>
<td>2098.96</td>
<td>1</td>
<td>16</td>
<td>4 16.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>5</td>
<td>1668.10</td>
<td>0</td>
<td>0</td>
<td>5 0.00 0.00 0.00 0.00 0.00</td>
</tr>
</tbody>
</table>

Table 3: Iteration-wise results of the advancement algorithm.
Iteration 4:

<table>
<thead>
<tr>
<th>i</th>
<th>$EAC(i, S_4(i))$</th>
<th>$i^*$</th>
<th>$S_4(i)$</th>
<th>$\sum_{k=1}^{S_4(i)} A(i,k)$</th>
<th>$P_4(i,j)$</th>
<th>$F(i, S_4, P_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\infty$</td>
<td>2</td>
<td>91+623</td>
<td>1</td>
<td>0.00</td>
<td>437.57</td>
</tr>
<tr>
<td>2</td>
<td>3920.16</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>12903.00</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>6114.36</td>
<td>1</td>
<td>16</td>
<td>4</td>
<td>16.00</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>2591.57 *</td>
<td>1</td>
<td>163</td>
<td>5</td>
<td>0.00</td>
<td>77.79</td>
</tr>
</tbody>
</table>

Iteration 5:

<table>
<thead>
<tr>
<th>i</th>
<th>$EAC(i, S_5(i))$</th>
<th>$i^*$</th>
<th>$S_5(i)$</th>
<th>$\sum_{k=1}^{S_5(i)} A(i,k)$</th>
<th>$P_5(i,j)$</th>
<th>$F(i, S_5, P_5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\infty$</td>
<td>2</td>
<td>91+623</td>
<td>1</td>
<td>0.00</td>
<td>437.57</td>
</tr>
<tr>
<td>2</td>
<td>6814.40</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>20724.20</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>6753.18 *</td>
<td>2</td>
<td>16+24</td>
<td>4</td>
<td>16.00</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>$\infty$</td>
<td>1</td>
<td>163</td>
<td>5</td>
<td>0.00</td>
<td>77.79</td>
</tr>
</tbody>
</table>

Iteration 6:

<table>
<thead>
<tr>
<th>i</th>
<th>$EAC(i, S_6(i))$</th>
<th>$i^*$</th>
<th>$S_6(i)$</th>
<th>$\sum_{k=1}^{S_6(i)} A(i,k)$</th>
<th>$P_6(i,j)$</th>
<th>$F(i, S_6, P_6)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\infty$</td>
<td>2</td>
<td>91+623</td>
<td>1</td>
<td>0.00</td>
<td>437.57</td>
</tr>
<tr>
<td>2</td>
<td>7627.05 *</td>
<td>1</td>
<td>622</td>
<td>2</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>23195.70</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>7849.78</td>
<td>2</td>
<td>16+24</td>
<td>4</td>
<td>16.00</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>$\infty$</td>
<td>1</td>
<td>163</td>
<td>5</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The End of Advancement Algorithm

Table 4: Continued from Table 3.
5 Retreat Algorithm

5.1 Need of Retreat

Sec. 4.2 demonstrated how the advancement algorithm reduces total abatement cost. Nevertheless, the algorithm still falls short of strict optimality (except under the condition described in the appendix). In particular, the two abatement steps implemented by trader 4 were, in fact, redundant; trader 2 ended up with a huge abatement surplus, \( F(2, S, P) = -271.59 \), which could have compensated for the two abatement steps of trader 4. Similarly, the step implemented by trader 5 was redundant in relation to the surplus of trader 2. However, the final abatement surplus of trader 2 was not anticipated until the very last iteration, prior to which traders 4 and 5 had already made abatement steps due to their lower time-consistent effective average costs. Since the advancement algorithm is prone to ending with such abatement surpluses, we need to remedy the result with a retroactive operation, namely a “retreat” of redundant steps, as opposed to an advancement.

In the context of our example in Sec. 4.2, those polluters who have made abatement steps during the advancement algorithm, \( \tilde{N} := \{i \in N \mid S(i) > 0\} = \{1, 2, 4, 5\} \), are eligible to retreat. Among them, we rule out those traders whose last-step abatement cannot be compensated for by the current surpluses. Specifically, we remove 1 and 2 from \( \tilde{N} \), resulting in \( \tilde{N} = \{4, 5\} \). Though both traders 4 and 5 are eligible to retreat, we see from Table 1 that retreat of trader 5 is more cost effective, as \( C(4, 1) + C(4, 2) < C(5, 1) \). Hence, we choose trader \( i^* := 5 \) for a retreat. This retreat is possible only if trader 5 purchases the equivalent effective abatement from a surplus holder, like trader 2. When trader 5 retreats her step, the retreated amount, \( A(5, 1) = 163 \), translates to trader 2.
having to sell quantity \( A(5, 1)T(2, 5) = 241.24 \). We thus set \( S(5) = 1 \Rightarrow S(5) = 0 \), which is compensated by incrementing \( P(2, 5) \) by 241.24. Accordingly, \( F(2, S, P) \) automatically increments from -271.59 to -30.35. This example illuminates the enumerated procedures for the retreat algorithm presented below in Sec. 5.2.

### 5.2 Enumerated Procedures of Retreat Algorithm

To begin, we note that the notation defined in Sec. 3 generally does not carry over to this section. Start with the policy \( S \in \mathcal{S} \) and \( P \in \mathcal{P} \) obtained by the advancement algorithm (cf. Sec. 3.2).

**PROCEDURE 1:** Temporarily set \( \hat{S} := S \) and \( \hat{P} := P \). Let \( \hat{N} = \{ i \in N \mid S(i) > 0 \} \) be the set of those traders who implement at least one abatement step.

**PROCEDURE 2:** Let \( N^+ = \{ i \in N \mid F(i, S, P) < 0 \} \) be the set of surplus holders, and \( n^+ = |N^+| \) be their number. Choose \( i^* = \arg \max_{i \in \hat{N}} C(i, S(i)) \), the polluter whose last abatement step incurred the largest cost.

**PROCEDURE 3:** Check if trader \( i^* \)'s last abatement can be compensated for by other traders' surpluses, i.e.,

\[
A(i^*, S(i^*)) \leq \sum_{j \in N^+} -\frac{F(j, S, P)}{T(j, i^*)}.
\]

If (13) holds, then execute the following:

(i) Decrement \( \hat{S}(i^*) \) by one. If \( \hat{S}(i^*) = 0 \), then remove \( i^* \) from \( \hat{N} \).

(ii) Write the set of surplus holders in terms of indices as \( N^+ = \{ j_1, j_2, \ldots, j_{n^+} \} \). Let \( \alpha^* \leq n^+ \) be the largest number such that \( \sum_{\alpha=1}^{\alpha^*} -\frac{F(j_\alpha, S, P)}{T(j_\alpha, i^*)} \leq A(i^*, S(i^*)) \).

For each \( \alpha \leq \alpha^* \), increment \( \hat{P}(j_\alpha, i^*) \) by \( -F(j_\alpha, S, P) \). If \( \alpha^* < n^+ \), then increment
\[
P(j_{\alpha+1}, i^*) = A(i^*, S(i^*)) + \sum_{\alpha=1}^{\alpha} F(j_{\alpha}, S, P)/T(j_{\alpha}, i^*) \cdot T(j_{\alpha+1}, i^*).
\]

Intuitively, (i) means a retreat of trader \(i^*\) by one step, and (ii) means compensating trader \(i^*\) via trading. If (13) does not hold, then remove \(i^*\) from \(\hat{N}\) and go back to PROCEDURE 2.

PROCEDURE 4: We now set back \(S := \hat{S}\). For \(P\), we comply with (4) by the following:

\[
P(i, i) = 0 \text{ for all } i \in N
\]

and

\[
P(i, j) := \begin{cases} 
P(i, j) - \hat{P}(j, i)/T(j, i) & \text{if } \hat{P}(i, j) > \hat{P}(j, i)/T(j, i) \\
0 & \text{otherwise}
\end{cases}
\]

for all \(i, j \in N\).

Repeat the PROCEDURES 2-4 sequentially as long as \(\hat{N} \neq \emptyset\).

5.3 Performance of Retreat Algorithm

Recall that our motivation for the retreat algorithm arose from the redundant abatement step of trader 5 in the example of Sec. 4.2 (see Sec. 5.1). We therefore apply the retreat algorithm to the final result of the advancement algorithm for this example (i.e., Iteration 6 of Table 4). Table 5 shows the iteration-wise results of the retreat algorithm applied to the final result of the advancement algorithm.\(^8\) As shown in Table 5, there is one retreat made by trader 5, which was actually forecasted by our discussion in Sec. 5.1. Comparing the overall total costs incurred by the community of five traders, we have

\(^{8}\)Note that in this last result trader 5 sells \(P(5, 3) = 85.21\) to trader 3 even though she does not implement any abatement herself, i.e., \(S(5) = 0\). This is not a flaw in the algorithm, as trader 5's sale comes from the extra abatement she has effectively purchased from trader 2 \((P(2, 5)/T(2, 5) = 193.70)\) less her required abatement \(R(5) = 108\), i.e., the surplus of 85.7.
The final result (at Iteration 6) from the advancement algorithm:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$S(i)$</th>
<th>$\sum_{k=1}^{S(i)} A(i, k)$</th>
<th>$P(i, j)$</th>
<th>$F(i, S, P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>91+623</td>
<td>1</td>
<td>0.00 437.57 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>622</td>
<td>2</td>
<td>0.00 0.00 157.57 33.00 45.44</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0.00 0.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>16+24</td>
<td>4</td>
<td>16.00 0.00 0.00 0.00 0.00</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>163</td>
<td>5</td>
<td>0.00 0.00 85.21 0.00 0.00</td>
</tr>
</tbody>
</table>

Retreat 1:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$C(i, S(i))$</th>
<th>$i^*$</th>
<th>$S(i)$</th>
<th>$\sum_{k=1}^{S(i)} A(i, k)$</th>
<th>$P(i, j)$</th>
<th>$F(i, S, P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>464,444</td>
<td>2</td>
<td>91+623</td>
<td>1</td>
<td>0.00 437.57 0.00 0.00 0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>2,074,237</td>
<td>1</td>
<td>622</td>
<td>2</td>
<td>0.00 0.00 157.57 33.00 45.44</td>
<td>-30.35</td>
</tr>
<tr>
<td>3</td>
<td>N/A</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0.00 0.00 0.00 0.00 0.00 0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>219,022</td>
<td>2</td>
<td>16+24</td>
<td>4</td>
<td>16.00 0.00 0.00 0.00 0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>590,906</td>
<td>*</td>
<td>0</td>
<td>5</td>
<td>0.00 0.00 85.21 0.00 0.00</td>
<td>-0.50</td>
</tr>
</tbody>
</table>

The End of Retreat Algorithm

Table 5: Iteration-wise results of the retreat algorithm.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Steps: $S$</th>
<th>Total Cost: $\sum_{i=1}^{5} \sum_{k=1}^{S(i)} C(i, k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Trading</td>
<td>(2,1,1,2,1)</td>
<td>$9,762,715$</td>
</tr>
<tr>
<td>Advancement Algorithm</td>
<td>(2,1,0,2,1)</td>
<td>$3,454,464$</td>
</tr>
<tr>
<td>Retreat Algorithm</td>
<td>(2,1,0,2,0)</td>
<td>$2,863,558$</td>
</tr>
</tbody>
</table>

As shown in Sec. 4.2, the advancement algorithm made a large improvement vis-a-vis the case of no trading ($9,762,715 \Rightarrow 3,454,464$). As shown above, a further remedial improvement is made by the retreat algorithm ($3,454,464 \Rightarrow 2,863,558$). Hence, this demonstration shows the cost efficiency gains attributable to the retreat algorithm. Indeed, the total abatement cost of $2,863,558 is the lowest possible cost attainable in this example.

The combination of the advancement and retreat algorithms, however, does not always produce a strictly optimal policy. For example, consider $N = \{1,2\}$, $T(1,2) = T(2,1) = 1$, $R(1) = R(2) = 100$, $A(1,1) = A(1,2) = 100$, $C(1,1) = 60$, $C(1,2) = 150$, $A(2,1) = 25$.
200, and \( C(2,1) = 200 \). In this case, large differences exist in abatement costs across polluters. The advancement algorithm produces \( S = (2,0) \), leaving no possibility of retreat. (To see this, start with \( S_0 = (0,0) \). In Iteration 1, \( EAC(1, S_0(1) + 1) = 60/100 < 200/200 = EAC(2, S_0(2) + 1) \), thus we choose \( i^* = 1 \) and set \( S_1 = (1,0) \). In Iteration 2, \( EAC(1, S_1(1) + 1) = 150/100 < 200/100 = EAC(2, S_1(2) + 1) \), thus we again choose \( i^* = 1 \) and set \( S_2 = (2,0) \).) The resultant policy \( S = (2,0) \) incurs the total cost of $210, whereas the least-cost policy would be \( S = (0,1) \) incurring a cost of only $200. This type of imperfectness is more likely the case when large differences exist in abatement costs. See the appendix for a proof of why the algorithm is optimal in the absence of such differences.

6 Concluding Remarks

We have developed two algorithms in this paper which heuristically achieve cost efficiencies in pollution trading markets. While the "advancement" algorithm alone solves the cost minimization problem under the condition of uniformly divisible abatement costs, its general limitations (for cases of non-uniform divisibility) are remedied by the "retreat" algorithm, which corrects for redundancies in abatement effort. The example used in this paper (presented in Sec. 4.2, 5.1, and 5.3) to illustrate the properties of the advancement and retreat algorithms specifically concerns watershed pollution. However, the algorithms can easily be applied to more general empirical problems, such as those of the previous studies mentioned in Sec. 1. Source-specific data requirements are minimal, consisting only of required abatement levels (e.g., current minus target load), abatement achieved and associated costs for each technology step, and where necessary a matrix of trading ra-
tios (e.g., see Tables 1 and 2). Finally, the advancement and retreat algorithms represent a marked departure from the simulation approaches used in previous studies to compare CAC and least-cost outcomes. In particular, the algorithms distinguish a specific pattern of trade among market participants and accommodate discrete abatement steps without the need to estimate marginal cost functions on a continuum.

As demonstrated in Sec. 5.3, the combination of the advancement and retreat algorithms does not necessarily guarantee a least-cost solution to the problem of non-uniformly divisible abatement costs. This is a limitation common to combinatorial and discrete problems. For example, the well-known traveling salesperson problem, whereby a shortest-distance travel strategy is incalculable in a polynomial computational time (see Cormen et al. [4]), shares basic features with the environmental problem studied in this paper. Developing an iterative algorithm in future research (similar to ones discussed in this paper) to solve the cost minimization problem under any circumstance may therefore be infeasible. However, scope remains for developing alternative algorithms that might outperform the retreat algorithm under different degrees of non-uniform divisibility in abatement costs.

Appendix

Condition Under Which The Advancement Algorithm is Optimal

Sec. 5.1 revealed a limitation of the advancement algorithm. However, this limitation is only peculiar to data sets containing large enough differences in abatement costs across polluters, as shown in Table 1 (ranging from $49,823 to $6,308,251). These differences are typically imputed to inflexible or indivisible abatement technology. If instead abatement
levels are divisible into uniform cost intervals for each abatement step across all polluters, then the retreat algorithm is, in fact, unnecessary, i.e., the advancement algorithm alone produces a strictly optimal policy \((S^*, P^*)\).

**Proposition 1.** Suppose that, for some constant \(\gamma > 0\), \(C(i, k) = \gamma\) for all \(i \in N\) and all \(k \in N\). Then, the advancement algorithm yields an optimal policy (that minimizes the total cost of abatement).

**Proof.** Note that, by (1), if \(T(i', j') \leq T(i'', j'')\), then

\[
T(i'', j') = T(i', i')T(i', j') \leq T(i'', i')T(i', j'') = T(i'', j''),
\]

and vice versa. This implies that the derangement \(\{j_1, j_2, \ldots, j_n\}\) of \(N\) such that \(T(i, j_1) \leq T(i, j_2) \leq \cdots \leq T(i, j_n)\) is identical across all \(i \in N\). Thus, without loss of generality, assume that

\[
T(i, 1) \leq T(i, 2) \leq \cdots \leq T(i, n) \quad \text{for all } i \in N. \tag{14}
\]

Let \(c = \sum_{i \in N} F(i, S_0, P_0)/T(i, 1)\). In other words, \(c\) is the sum of the initial abatement deficits of all polluters interpreted in terms of trader 1's trading ratio. Since (1) says that trading through trader 1 is equivalent to direct trading between any pair of traders, constraint (5) is equivalent to

\[
c \leq \sum_{i \in N} \sum_{k=1}^{S(i)} A(i, k)/T(i, 1) \tag{15}
\]
as a requirement for the optimal \(S \in S\). We now consider the following auxiliary claim.

**Claim 1.** For any iteration, \(m\), in the advancement algorithm except the very last iteration, \(EAC(i_1, k_1) \leq EAC(i_2, k_2)\) implies \(A(i_1, k_1)/T(i_1, 1) \geq A(i_2, k_2)/T(i_2, 1)\), where \(k_i = S_m(i) + 1\).
(Proof of Claim 1). Regardless of the state of the current iteration \((S_m \text{ and } P_m)\) in the advancement algorithm, the derangement \(\{j_1, j_2, \ldots, j_{\tilde{n}}\}\) of \(\tilde{N}\) as in Sec. 3.2 is the same for both traders \(i_1\) and \(i_2\) because of (14). Let \(\alpha_1^*\) and \(\alpha_2^*\) be the numbers corresponding to \(\alpha^*\) of Sec. 3.2 for traders \(i_1\) and \(i_2\), respectively. Note that since our assumption rules out the very last iteration, the next abatement step cannot satiate the abatement deficits, and we therefore have \(\alpha_1^* < \tilde{n}\) and \(\alpha_2^* < \tilde{n}\). In view of (9), we see that the effective average costs of \(i_1\) and \(i_2\) depend on \(\alpha_1^*\) and \(\alpha_2^*\), respectively. In particular, the fact that \(\alpha_1^* < \tilde{n}\) and \(\alpha_2^* < \tilde{n}\) allows only the first and third cases of (9). Assume \(\text{EAC}(i_1, k_1) \leq \text{EAC}(i_2, k_2)\). Then, by definition of \(\alpha_1^*\) and \(\alpha_2^*\) and the fact that \(C(i, k) = \gamma\) for all \(i \in N\), we have \(\alpha_1^* \geq \alpha_2^*\).\(^9\)

Hence, the three cases, (i) \(\alpha_2^* < \alpha_1^* < \tilde{n}\), (ii) \(\alpha_1^* = \alpha_2^* = 0\), and (iii) \(0 < \alpha_1^* = \alpha_2^* < \tilde{n}\), exhaust all possibilities.

Case (i): Suppose \(\alpha_1^* > \alpha_2^*\). In this case, by definition of \(\alpha_1^*\) and \(\alpha_2^*\),

\[
A(i_1, k_1) \geq \sum_{\alpha=1}^{\alpha_1^*} F(j_{\alpha}, S_m, P_m) T(i_1, j_{\alpha})
\]

and

\[
\sum_{\alpha=1}^{\alpha_1^*} F(j_{\alpha}, S_m, P_m) T(i_2, j_{\alpha}) > A(i_2, k_2).
\]

Multiply the first inequality by \(T(1, i_1)\) and the second inequality by \(T(1, i_2)\), and then use (1) to get

\[
A(i_1, k_1)T(1, i_1) \geq \sum_{\alpha=1}^{\alpha_1^*} F(j_{\alpha}, S_m, P_m) T(1, j_{\alpha})
\]

\[
> A(i_2, k_2)T(1, i_2).
\]

\(^9\)Intuitively, the condition that trader 1 incurs a smaller EAC (i.e., greater swelled quantity) than trader 2 implies that trader 1 sells her abatement credits to at least as many potential purchasers as trader 2 does, as the deranged list of potential purchasers is the same for both traders 1 and 2 by (14).
Since \(T(1,i_1) = 1/T(i_1,1)\) and \(T(1,i_2) = 1/T(i_2,1)\), this inequality implies

\[A(i_1,k_1)/T(i_1,1) > A(i_2,k_2)/T(i_2,1),\]

as required.

Case (ii): Suppose \(\alpha^*_1 = \alpha^*_2 = 0\). By (1), (9), and \(C(i,k) = \gamma > 0\) for all \(i \in N\),

\[EAC(i_1,k_1) \leq EAC(i_2,k_2)\]

\[\implies A(i_1,k_1)/T(i_1,j_1) \geq A(i_2,k_2)/T(i_2,j_1)\]

\[\implies A(i_1,k_1)/[T(i_1,j_1)T(j_1,1)] \geq A(i_2,k_2)/[T(i_2,j_1)T(j_1,1)]\]

\[\implies A(i_1,k_1)/T(i_1,1) \geq A(i_2,k_2)/T(i_2,1),\]

as required.

Case (iii): Suppose \(0 < \alpha^*_1 = \alpha^*_2 < \bar{n}\). Let \(\alpha^*\) denote \(\alpha^*_1 = \alpha^*_2\). By (9),

\[EAC(i_1,k_1) \leq EAC(i_2,k_2) \implies \]

\[\sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) + \frac{A(i_1,k_1) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m)T(i_1,j_{\alpha})}{T(i_1,j_{\alpha^*+1})}\]

\[\geq \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) + \frac{A(i_2,k_2) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m)T(i_2,j_{\alpha})}{T(i_2,j_{\alpha^*+1})},\]

i.e.,

\[A(i_1,k_1)/T(i_1,j_{\alpha^*+1}) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) \frac{T(i_1,j_{\alpha})}{T(i_1,j_{\alpha^*+1})}\]

\[\geq A(i_2,k_2)/T(i_2,j_{\alpha^*+1}) - \sum_{\alpha=1}^{\alpha^*} F(j_{\alpha}, S_m, P_m) \frac{T(i_2,j_{\alpha})}{T(i_2,j_{\alpha^*+1})}.\]

Note that, by (1), \(T(i_1,j_{\alpha})/T(i_1,j_{\alpha^*+1}) = T(j_{\alpha^*+1},j_{\alpha}) = T(i_2,j_{\alpha})/T(i_2,j_{\alpha^*+1})\). Hence, the above inequality further reduces to

\[A(i_1,k_1)/T(i_1,j_{\alpha^*+1}) \geq A(i_2,k_2)/T(i_2,j_{\alpha^*+1}).\]
Divide both sides by \( T(j_{a+1}, 1) \) and then use (1) to get

\[
A(i_1, k_1)/T(i_1, 1) \geq A(i_2, k_2)/T(i_2, 1),
\]

as required. \((\text{End of Claim 1})\)

Since the total cost is \( \sum_{i \in N} \sum_{k=1}^{S(i)} C(i, k) = \gamma \sum_{i \in N} S(i) \) from our assumption, the optimal policy (that minimizes the total cost) is derived through choosing the \( S \in S \) that minimizes the total number of abatement steps \( \sum_{i \in N} S(i) \) subject to (15), i.e.,

\[
\min_{S \in S} \sum_{i \in N} S(i) \quad \text{subject to} \quad c \leq \sum_{i \in N} \sum_{k=1}^{S(i)} A(i, k)/T(i, 1).
\]

This problem is solved by consecutively advancing the abatement step of the trader \( i \) whose \( A(i, k)/T(i, 1) \) is maximum, which, by Claim 1, is satisfied by consecutively choosing the trader whose effective average cost is minimum. \((\text{According to Claim 1, this statement holds except for the very last iteration. However, in the very last iteration, only those traders who can satiate the abatement deficits have the maximum effective average cost by (9) because of the constant cost } \gamma. \text{ Hence, choice of a trader with the largest effective average cost is still valid in the very last iteration.})\) But this process is exactly the advancement algorithm, and hence the abatement policy \( S \in S \) resulting from the advancement algorithm minimizes the total cost \((\text{with } P_m \text{ doing nothing but distributing the abatements specifically in the order of } 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \text{ by (14))}.\) Hence, the advancement algorithm leads to an optimal policy \((S^*, P^*)\). \(\square\)
References


