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This study investigated how the use of virtual manipulatives in integer instruction impacts student achievement for integer addition and subtraction. Of particular interest was the influence of using virtual manipulatives on students' ability to create and translate among representations for integer computation. The research employed a quasi-experimental pretest-posttest design. Ninety-nine sixth-grade students participated over a four-week period. Six classes were randomly assigned to one of three treatment groups. Students increased in integer computation achievement and demonstrated facility with pictures and written representations. Students had more difficulty creating symbolic representations and making connections among this and other representational forms.

Many mathematics educators agree that the gateway to higher mathematics is the first year algebra course. One topic important to the study of algebra is integer arithmetic. Students must have the opportunity to engage in experiences that enable them to develop a deep understanding of these fundamental concepts. Although integers and integer arithmetic concepts are generally considered important topics in mathematics, there has been little research conducted in this area (Kilpatrick, Swafford, & Findell, 2001).

Some studies have examined students' pre-instructional knowledge (Mukhopadhyay, Resnick, & Schauble, 1990; Peled, Mukhopadhyay, & Resnick, 1989). These results indicate that students have some knowledge prior to instruction and often use their knowledge of whole number arithmetic to reason through problems involving negative integers. Other research has focused on appropriate instructional models for integer arithmetic. Even fewer studies have examined how models for integers can be used for instruction using technology such as virtual manipulatives.

The present study was designed to address the need for understanding how virtual manipulatives might support students' representational fluency in the addition and subtraction of integers. Specifically, we investigated how the use of virtual manipulatives, representing two different models for integers, impacts student achievement for integer addition and subtraction. Also of interest in this study was the influence of virtual manipulatives on students' ability to create and translate among representations for integer addition and subtraction.

LITERATURE REVIEW

Integers

Representational metaphors used to facilitate the learning of integer arithmetic include positively and negatively charged particles, debts and assets, and movements along a number line. Different models emphasize different attributes of number. The number line model focuses on the measurement aspect of number, or the notion of *how much*; debts and assets models focus on the quantity aspect of number, or *how many*. While these models may help students conceptualize integer values and, in some cases, integer addition, a common criticism is that they do not adequately represent other operations with integers, particularly subtraction (Kilpatrick, Swafford, & Findell, 2001). Other proposed models, including Janvier's (1985) hybrid model and a model based on algebraic geometry (Moses, Kamii, Swap, & Howard, 1989; Carson & Day, 1995) focus on quantity and directional aspects of number used in algebra.

While proponents of the various models advocate for one over another, few attempts have been made to compare the effectiveness of different models. Two studies comparing quantity and number line models found that students using quantity models outperformed those using number line models (Liebeck, 1990; Sherzer, 1973). One study comparing Janvier's hybrid model and a quantity model found that students using the hybrid model outperformed those using a quantity model (Janvier, 1985). The limited research does not provide adequate evidence to determine conclusively any difference in student achievement among the models.

Regardless of the model used, it is clear that integer arithmetic remains an area of difficulty for many students (Bruno et al., 1997; Peled, 1991). Many of these difficulties are presented when students work with symbolic representational forms for integer addition and subtraction. Lesh, Post, and Behr (1987) discuss five different representational forms that support concept development: manipulative models, pictures, written symbols, real-world situations, and oral language. Each form has the potential to highlight different features of a mathematical idea or relationship (NCTM, 2000); Translation among representational forms builds understanding (Hiebert, 1990). Research indicates that after instruction, students can work with story problems to solve integer addition and subtraction situations (Peled & Carraher, 2007). However, students are not as successful with symbolic-only forms. Some attribute this difficulty, in part, to the fact that students' whole number interpretations of addition and subtraction do not easily translate to these new situations (Kilpatrick, Swafford, & Findell, 2001; Moses, Kamii, Swap, & Howard, 1989). To make sense of integer computation, the learner needs to develop a different perception of number compared to that used in whole number arithmetic. In whole number arithmetic, the focus on number is mainly one of magnitude or quantity. However, when students expand their work with number to include negative numbers, they must also consider direction (Moses, Kamii, Swap, & Howard, 1989; Peled & Carraher, 2007).

Students must also expand their understanding of the meaning of operations. Vershcaffel, Greer, and De Corte (2007) categorize additive situations in whole number arithmetic as involving changes in quantities, combinations of two discrete sets, and comparison of two discrete sets. When the largest quantity is unknown, adding the two given quantities will produce the solution; when one of the smaller quantities is unknown, subtracting the smaller set from the larger set will produce the solution. However, this interpretation does not translate to operations with integers in which students can no longer think of addition as making bigger and subtraction as making smaller (Peled & Carraher, 2007). Subtraction presents particular difficulty. Vlassis (2008) points out that integer computation involves multiple and varied uses of the minus sign, including: a unary structural signifier (e.g., to indicate a value, -2, as opposed to 2); a binary operational signifier (e.g., to indicate subtracting in both arithmetic [take away or difference] and algebraic [adding the opposite] contexts; and a symmetric operational signifier (e.g., taking the opposite of a number). The need to expand one's interpretation of number, operations, and symbols adds to the complexity of integer study.

Technology in Integer Instruction

Technology can support students' developing understanding of integer concepts. Cognitive technology tools (Pea, 1985), like virtual manipulatives, can connect verbal (e.g., written directions and feedback) and nonverbal (e.g., visual/pictorial representations and dynamic objects) representations. The interconnections between verbal and nonverbal systems proposed in Dual Coding Theory (DCT, Paivio, 1991) increase the efficiency of the working memory. Logogens (i.e., language units, such as words, phrases or sentences), that process information sequentially, and imagens (i.e., representation units that activate different types of imagery), that process concepts in synchronous hierarchies, support the formation of representations of mathematical concepts by allowing the learner to manipulate objects, observe changes, and make connections.

A thorough review of the research on virtual manipulatives (see Moyer-Packenham, Westenskow, & Salkind, *manuscript under review*) produced only four studies specifically focusing on using virtual manipulatives to teach integer concepts (i.e., positive and negative integers, addition and subtraction of integers). Three of these studies were dissertations (Bolyard, 2006; Smith, 1995; Smith, 2006) and one was a journal publication (Moreno & Mayer, 1999). Overall, these studies found that students using virtual manipulatives made gains in achievement of integer concepts.

The limited research base on the teaching and learning of integer arithmetic using virtual manipulatives leaves room for further exploration. This study was designed to contribute to that knowledge. The following research questions guided the study:

1. How does the use of three different web-based virtual manipulatives for integer addition and subtraction impact students' achievement in computation? And, are there differences in achievement among the three treatment groups using each web-based virtual manipulative?
2. How does the use of web-based virtual manipulatives for integer addition and subtraction influence students' creation of and translation among representations during task solutions?

METHODOLOGY

This study employed a quasi-experimental pretest-posttest design. The research was conducted over a four-week period in two public middle schools in a single school district. Six classes were randomly assigned to one of three treatment groups: Virtual Integer Chips (VIC), Virtual Integer Chips with Context (VICC), and Virtual Number Line (VNL).

Participants and Site Description

Ninety-nine sixth-grade students (46% female; 54% male) participated in the study during regular mathematics class instruction. The student demographics included 73% White, 10% Asian/Pacific Islander, 7% Black, 5% Hispanic, 4% Other, and 1% American Indian/Alaskan Native. Each middle school in the district was equipped with three computer labs and four computers per classroom. However, the participating teachers reported little use of computers during mathematics instruction.

The classes were taught by four regular classroom teachers using an instructional unit designed by the researchers. The regular classroom teachers participated in a two-week summer institute taught by one of the researchers designed to provide professional development in the use of virtual and physical manipulatives for middle school mathematics. Choosing these teachers controlled for variability due to teacher expertise with the tools.

Instructional Materials and Procedures

The student participants used one of three different web-based virtual manipulatives: Virtual Integer Chips (VIC), Virtual Integer Chip with Context VICC), and Virtual Number Line (VNL). The VIC and VICC manipulatives were chosen as representative of a quantity model for integer addition and subtraction. The VNL manipulative was chosen as representative of a number line model. Each of these tools is presented electronically in the form of an "applet" or small, stand-alone application program.

The VNL manipulative was the Adding and Subtracting Integers applet, available from www.explorelearning.com and required a subscription. This applet presents the user with a number line, one red and one blue slider for representing integer values on the number line, and the symbolic sentence 0 + 0 = 0. A button at the top of the screen allows the user to choose either addition or subtraction. Users move the sliders left (to indicate a negative value) or right (to indicate a positive value) resulting in an arrow of appropriate length and direction to appear on the number line. Simultaneously, the values in the displayed symbolic sentence adjust according to the user's actions on the slider. Users also have the option of typing in a symbolic value in a box at the end of each slider. (See Figure 1).

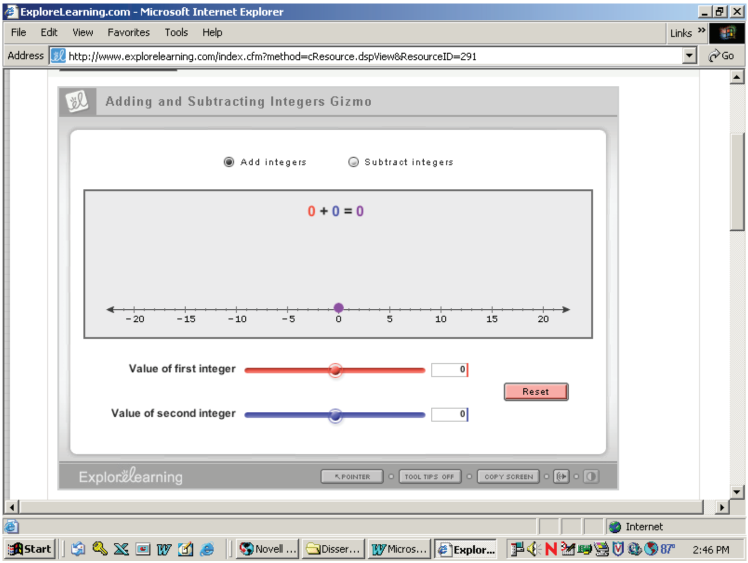

Figure 1. Virtual number line applet. Copyright © 1999-2003 ExploreLearning (http://explorelearning.com). All rights reserved. Used by permission.

The VIC manipulatives were the Color Chips Addition and Color Chips Subtraction applets, available at http://matti.usu.edu/nlvm/nav/ and free on the Internet. These applets present the user with a problem statement in symbolic form (i.e., -2 + 3), a color chip bank showing positive (black) and negative (red) chips, and a workspace. The applet instructs the user to drag chips into the workspace to model the problem presented and then click "Continue." In the Color Chips Addition applet, the user drags pairs of positive and negative chips together to create zero pairs. As the zero pairs are created, the chips disappear from the screen. The remaining chips represent the simplified value of the original problem statement. The user then types in an answer (See figure 2).

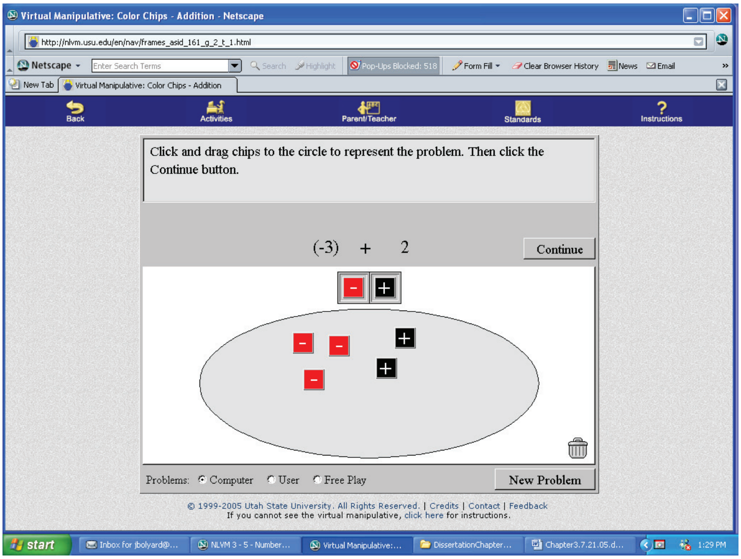

Figure 2. Virtual integer chips applet. The National Library of Virtual Manipulatives (http://matti.usu.edu). © 2003 Utah State University. All rights reserved. Used with permission.

In the Color Chips Subtraction applet, the user models the minuend of the problem using chips, then clicks "Continue." Next, the user drags zero pairs (combinations of one positive and one negative chip) to enable him or her to remove the designated value of positive or negative chips. The user then enters the answer. In both applets, the program controls actions and provides feedback. For example, the program prompts the user to re-examine his or her response with suggestions such as "Your chips don't represent the problem shown" or "Check the color of the chips and try again." The VICC manipulative is a modified version of the VIC manipulative. However, the VICC applet presents the initial problem using the context of debts and assets. After the initial introduction of the problem, the program continues in the same manner as the VIC manipulative.

The researchers created the following instructional materials for teachers: a detailed instructional plan, integer task cards, technology guides, student recording sheets, and student practice sheets. Instructional activities using the VNL model used the metaphor of walking along a line a specified number of spaces in the direction indicated. Instructional activities using the VIC and VICC models employed the metaphor of debts and assets. These metaphors were used during all classroom discourse as well as on all instructional handouts.

During instruction, introductory activities focused on establishing the respective metaphor using related physical and pictorial models. Students also participated in extended computer lab sessions (30 to 40 minutes) in which they used the virtual manipulatives to explore integer addition and subtraction. Following each computer session, students engaged in follow-up discussions focused on articulating their observations. Instruction took place during four, 90 minute alternating day periods over two weeks to accommodate the middle school block scheduling format used in the school.

DATA SOURCES AND ANALYSIS

Data were collected and analyzed from three sources: integer addition and subtraction pretests and posttests and student interviews. Prior to instruction, students took a 24-item paper and pencil pretest to assess students' prior knowledge of integer addition and subtraction. The test was researcher created and contained 12 addition and 12 subtraction items, all presented in symbolic form (ex. $2 - {}^{-}3$). The problems were similar to those explored during the instructional sessions and were drawn from district and state assessment standards. Following treatment, students were given a 24-item paper and pencil posttest. The posttest was of the same format as the pretest but with different items in order to reduce the threat of test reactivity. For each assessment, five individuals identified as expert reviewers conducted an analysis of the test items, examining 1) how appropriately each measured student achievement in the concept and 2) level of difficulty. During analysis, the addition and subtraction sections of the pre- and posttest were examined separately. Each item was worth one point for a total of 12 possible points for each section.

Following the treatment, one researcher conducted task-based interviews (Goldin, 1997) with three randomly selected students from each treatment group (n=9). Each interview was audiotaped and transcribed to maintain a written record of students' responses. During the interviews, students solved six problems (three addition and three subtraction) presented in symbols, pictures, or written/metaphorical (story) form. For each item, participants were asked to 1) simplify given problems, 2) represent the problems in two additional forms (other than the one initially used), and 3) explain their thinking and solution process. Student work on the tasks was collected for analysis. During analysis, a paired samples *t* test was performed using pre- and posttest scores for the three virtual treatment groups to determine if there was an overall difference in achievement after using the virtual manipulatives during instruction. To determine if there were differences in student achievement, an Analysis of Variance (ANOVA) was performed on the posttest measures of the three treatment groups.

Qualitative analysis procedures examined students' abilities to flexibly apply knowledge of integer addition and subtraction to various representations of the concept and make connections among representations. Interview transcripts were coded using conventional content analysis (Hsieh & Shannon, 2005). Codes were then categorized to compare the data, identify patterns, and determine emerging themes and categories with respect to students' level of conceptual understanding (Glesne, 1999; Maxwell, 1996). Two major themes used in this analysis were flexibility in using different representational forms and the ability to make connections among representational forms.

RESULTS

The results below are presented based on our two major research questions. To answer our first research question (How does the use of three different web-based virtual manipulatives for integer addition and subtraction impact students' achievement in computation? And, are there differences in achievement among the three treatment groups using each web-based virtual manipulative?), we used paired t tests to compare pretest and posttest scores on the integer addition and integer subtraction tests for each treatment group. The size of each significant effect was assessed using Cohen's d, with .20, .50, and .80 representing the lower limit of small, medium, and large effect sizes, respectively. A summary of the means and standard deviations for the pre- and posttests for each treatment group are presented in Table 1. There were significant differences for all three groups for the addition (VIC, $t[34] = 4.46$, $p < .01$, $d = .75$, VICC, $t[36] = 4.44$, $p < .01$, $d = .73$, and VNL, $t[26] = 3.26$, $p < .01$, $d = .63$) and subtraction (VIC, $t[34] = 8.05$, $p < .01$, $d = 1.36$, VICC, $t[36] = 10.23$, $p < .01$, $d = 1.68$, and VNL, $t[26] = 9.53$, $p < .01$, $d = 1.83$) portions of the pretest and posttest scores. Thus, all treatment groups showed a significant increase in student achievement.

Table 1
Pre and Posttest Means and Standard Deviations by Treatment Group

Integer Addition		
Treatment Group	Pretest	Posttest
Virtual Integer Chips (VIC) *n* = 35	*M* 71.90 *SD* 30.79	*M* 93.57 *SD* 11.63
Virtual Integer Chips with Context (VICC) *n* = 37	*M* 70.86 *SD* 30.21	*M* 95.05 *SD* 9.71
Virtual Number Line (VNL) *n* = 27	*M* 69.14 *SD* 28.76	*M* 89.51 *SD* 19.69
Integer Subtraction		
Treatment Group	Pretest	Posttest
Virtual Integer Chips (VIC) *n* = 35	*M* 45.95 *SD* 22.90	*M* 81.90 *SD* 19.23
Virtual Integer Chips with Context VICC) *n* = 37	*M* 47.75 *SD* 18.39	*M* 86.04 *SD* 19.94
Virtual Number Line (VNL) *n* = 27	*M* 40.43 *SD* 24.31	*M* 86.73 *SD* 15.20

The next part of the analysis for our first research question examined differences for each web-based virtual manipulative. An analysis of variance (ANOVA) was performed on the integer addition and subtraction pretest scores to determine if there were pre-existing differences among the three treatment groups. There were no significant differences among the groups for addition, $F(2, 96) = .07$, *ns*, or subtraction, $F(2, 96) = .92$, *ns* pretests. The ANOVA indicated no significant differences in posttest addition scores among the three treatment groups: VIC ($M = 93.57$, SD=11.63), VICC ($M = 95.05$, $SD = 9.71$) or VNL ($M = 89.51$, $SD = 19.69$), $F(2, 96) = 1.32$, *ns*. Similarly, there were no significant differences among the groups for subtraction: VIC ($M = 81.90$, SD=19.23), VICC ($M = 86.04$, $SD = 19.94$), and VNL ($M = 86.73$, $SD = 15.20$), $F(2, 96) = .66$, *ns*.

Analysis of Students' Work with Representational Forms for Integer Computation

To answer our second research question (How does the use of web-based virtual manipulatives for integer addition and subtraction influence students' creation of and translation among representations during task solutions?), we analyzed student work on the interview tasks. These interviews revealed that students were generally able to work flexibly within the written/metaphorical and picture representational forms for integer addition and subtraction. Students had more difficulty working with the symbolic representation for integer addition and subtraction. Table 2 summarizes the number of correct responses for each type of interview task.

Table 2

Number of Correct Responses on Interview Tasks by Representational Form (N = 9)

Presentation Mode	Correct Responses by Student Response Form Types		
	Addition Items		
	Written/Metaphorical Response	Picture Response	Symbols Response
Item 1: $^{-}7 + 10$ (presented in written/metaphorical)	9	9	9
Item 2: $^{-}3 + 2$ (presented in pictures)	9	9	7
Item 3: $^{-}4 + ^{-}5$ (presented in symbols)	9	9	9
	Subtraction Items		
	Written/Metaphorical Response	Picture Response	Symbols Response
Item 4: $3 - 5$ (presented in pictures)	8	8	6
Item 5: $^{-}9 - ^{-}3$ (presented in written/metaphorical)	9	9	6
Item 6: $^{-}4 - 5$ (presented in symbols)	4	4	4

Facility with Representational Forms: Pictures and Written/Metaphorical

Students in this study were successful in evaluating integer addition and subtraction items presented in written/metaphorical and picture representational forms. Further, the students were generally successful in translating items to written/metaphorical and picture forms.

Items given in picture form. All students were able to correctly evaluate the addition item (representing –3 + 2) using the picture model; all students but one correctly evaluated the subtraction problem (representing 3 – 5) presented in pictures. On the addition item, most students described using the picture to arrive at their answer giving responses such as, "there were three negatives and two positives then we would cancel that out and there would be one negative left," (VIC student) or "the ending position was at negative one" (VNL student). Although most students were able to correctly complete the picture on the subtraction item, they showed evidence of struggling with this item and relied on other representations to help them create and verify their responses. When asked to create a story representing the picture, students personalized the stories by inserting themselves into the narrative. For example, a VICs student said, "I owed my sister three dollars and then my mom gave me two dollars." A VNL student related his story to football: "You gain three yards then lose five yards."

Facility with Representational Forms: Symbolic

Students demonstrated less facility working with symbolic representational forms. When students were given an addition item, some could explain that they knew to use a negative seven and a positive ten to write the statement, $^{-}7 + 10$, by stating: "You move seven steps left on a number line. When you go left those are the negatives" (VNL student); and "Um, debt, because a debt is something that you take away" (VICC student). However, after writing $^{-}7 + 10$, one VICC student explained, "sometimes when I do these I do, like, ten minus seven." Asked why she thought of it as subtraction even though she had written it as an addition statement, the student responded uncertainly, "Because this (referring to the $^{-}7$) is a negative number?" The student used whole number reasoning to simplify the problem. Another VICC student explained his decision to use addition in his mathematical sentence by stating, "I thought it must be adding because most of the time when it talks about this and it doesn't say to subtract it usually means add." These examples illustrate, that while students were able to connect the words of the story problem to the values of the addends used in their mathematical sentences, they were not as confident in describing the connection between the problem statement and their use of the addition operation signifier (+).

Work on the addition item given in pictures ($^{-}3 + 2$) further demonstrated students' incomplete understanding of the symbolic form of representing integer addition. While most students were able to produce an accurate symbolic statement, this task presented difficulties for some students. One student in the VIC group wrote the statement as subtraction ($^{-}3 - {+2}$) rather than addition and explained: "Um, there was negative three chips and positive two chips so I figured it would be a subtraction problem because there were more of one. So, I, my final answer was -1 because negative three minus positive 2 was negative 1." A VNL student, who evaluated the picture representation correctly, wrote a symbolic statement of $^{-}3 + {}^{-}2$. The student indicated that she looked at the arrow indicating three steps left to determine $^{-}3$ and the arrow indicating two steps right to determine $^{-}2$. These students did not appear to notice any inconsistencies between the result of their initial evaluation of the given problem statement and the symbolic statement they created in the second task.

The symbolic form presented even greater challenges for students in subtraction contexts. For example, on the first item (3 – 5), presented in pictures, six students correctly produced a symbolic statement of $3 - 5$. However, three students had difficulties correctly translating their work from pictures to symbols. Two VIC students drew a picture in which they added five negatives rather than subtracting or removing five positives (by adding zero pairs). One student used a story about owing her friends $5 to help her complete her statement. These students recognized that removing five positive tiles from the picture was equivalent to adding five negative tiles. However, this connection failed to translate to symbols. Both students created problem statements of $3 - ({}^{-}5)$ rather than $3 + ({}^{-}5)$ as they had represented in their pictures. In this scenario, the minus sign serves a binary function (Vlassis, 2008). However, in the symbolic form, these students included two minus signs: the first serving as a binary sign to indicate subtraction and the second serving as a unary sign attached to the numeral 5.

On the second subtraction item ($^{-}9$ - $^{-}3$), presented in written/metaphorical form, students correctly represented their interpretation of the story problem using a picture. Similar to their work in addition, students' comments illustrated that they were making connections between the words of the story and the integer values they represent, i.e., "if you owe your brother $9 that means it's a debt, so that would be negative nine" (VIC student). Students used both negative three and positive three in their mathematical sentences. Students had more difficulty choosing a correct operation to complete their symbolic statement for ($^{-}9$ - $^{-}3$). Six students created correct statements. These students were able to use the words of the story problem to articulate their reasoning for the chosen operation. For example, one VIC student explained her use of subtraction: "it says 'erases' and, so, that's like taking away." The student made connections between the idea of erasing a \$3 debt and the symbolic notation $- {}^{-}3$. Students creating ${}^{-}9 - {}^{-}3$ represented their interpretation of the story with the use of two minus signs in their symbolic statement: the first as a binary sign indicating "take away" and the second as a unary sign attached to the numeral 3. Students who produced the addition statement ${}^{-}9 + 3$ to represent the problem also used the context of the words to make connections to their choice of operation. One VIC student describing his reasoning for using addition explained that if someone takes away a debt, "it's positive." A VICC student who wrote ${}^{-}9 + 3$ was able to see that either an addition or a subtraction statement was accurate after rereading the problem concluding, "I guess I could also have it be ${}^{-}9 - {}^{-}3$." The symbolic statement produced by these six students related to their picture interpretation. In other words, students who wrote an addition statement drew a picture indicating addition; those who wrote a subtraction statement drew a picture that illustrated taking away or subtracting values. The remaining three students (one in each group) created an incorrect statement (${}^{-}9 - 3$) and did not appear to make connections between their work in pictures and their work in symbols. Each of these students created a picture representing the addition problem, ${}^{-}9 + 3$, interpreting the situation using a binary function of adding the opposite.

The final subtraction item presented the students with the statement ${}^{-}4 - 5$. By far, this item gave students the greatest difficulty with fewer than half of the students able to respond correctly using written/metaphorical, pictorial, or symbolic forms. Four students (one VICC, one VIC, and two VNL) correctly interpreted the symbolic statement as ${}^{-}9$ and completed the remaining two tasks correctly. The other five students were not able to accurately interpret the given problem statement. One VIC student initially crossed out the negative on the first addend and changed subtraction sign to an addition sign, resulting in $4 + 5$. As the student created his picture and story representations, he expressed uncertainty in his initial answer: "It's one or nine. I know it's like one or nine, but I'm thinking that it's probably nine . . . ." The remaining four students misinterpreted the statement given (${}^{-}4 - 5$) as ${}^{-}4 + 5$, reported an answer of one, and then continued to create pictures and stories based on this interpretation of the problem. While some students did not question their initial interpretation, it appeared that others did continue to reason through the problem and attempt to make connections as they worked with the other representations. For example, one VIC student who had created a pictorial representations for ${}^{-}4 + 5$, began to create a correct story for the original problem, ${}^{-}4 - 5$. However, upon realizing that her scenario would not result in an answer of positive one, she then changed her story to match the sentence $^{-}4 + 5$. When in doubt, students forced the other representations to support the symbolic statement.

DISCUSSION

This research study investigated the impact of using virtual manipulatives to teach integer addition and subtraction on students' computation skills and use of representations. The four major findings of this study presented below will be discussed in detail in the sections that follow.

Finding 1: Students demonstrated significant gains in computation achievement after using three different virtual manipulative applets during instruction of integer addition and integer subtraction concepts.

Finding 2: There were no statistically significant differences in achievement among students using three different virtual manipulative applets designed for integer instruction.

Finding 3: Students successfully translated between pictorial and written/metaphorical representational forms for integer addition and subtraction.

Finding 4: Students demonstrated less facility in creating an accurate symbolic representation and connecting this with other representations (i.e., pictorial and written/metaphorical), particularly for subtraction.

Discussion of Student Achievement

Our first two findings showed significant gains for all treatment groups after using three different virtual manipulatives, and no significant differences among the three virtual manipulatives applets. Cohen's d values calculated on the integer pretest-to-posttest mean differences for all three treatment groups met the standard of a medium effect size for addition items (smallest d = .63) and the standard of a large effect size for subtraction items (smallest d = 1.36). The fact that students made larger gains in subtraction than addition is not surprising. Analysis of the pretest scores indicated that students had some prior knowledge of integer addition but less prior knowledge of integer subtraction.

Analysis of achievement by treatment group for the three virtual manipulatives applets indicated no significant differences. While these three virtual manipulatives shared several key features (dynamic linked representations, interactivity, multiple representations, and immediate feedback), there were some features (type of user input required, degree of guidance provided by the applet, and problem presentation) that were unique to each virtual manipulative. One conclusion that might be drawn from this is that the features shared by these three applets had a larger impact on students' learning of integer computation than those that were unique to any one specific tool.

Discussion of Students' Facility with Different Representational Forms

Although the results of the pre- and post-testing in each of the three different virtual manipulatives integer groups demonstrate that the different applets had a positive influence on students' abilities to perform integer computation, we believe that a more important result of this teaching experiment is the differences in the ways students translated among different representations for integer addition and subtraction problems. Specifically, students had much more difficulty creating representations and making connections when problems were presented in symbols or when a symbolic representation was requested, particularly for subtraction.

Students' work on the interview tasks showed they were making connections between written/metaphorical and pictorial representations of integer values. For example, students expressed thinking that demonstrated connections between phrases such as "debt" and "walking left" to negative integer values and "assets" and "walking right" to positive integer values. Students also used appropriate images (positive and negative tiles or left and right arrows) to represent words indicating positive and negative integer values in their pictorial models. There was also evidence that students were able to make personal connections to the metaphors based on the frequency with which students created personalized stories. English (1997) describes such analogies and metaphors as "illuminating devices" which help learners take concrete experiences and build them into mental models for abstract ideas.

In contrast to their facility with written and pictorial representational forms, students in this study had noticeably more difficulty working with the symbolic form. On addition tasks, students showed evidence of making connections among the words in the story and the images in the pictures to the values of the integers used in their mathematical sentences. However, they were not confident in describing the connection between the problem situation and their use of the addition operation in their symbolic sentences.

Students' difficulties with the symbolic form were evident across all three subtraction interview items. Several students successfully translated the subtraction items presented in pictures and words into related addition situations as they worked in these forms (i.e., they equated removing five positive tiles from the picture with adding five negative tiles to the picture). However, students made errors on their symbolic statements for these items because they failed to translate their understanding of this relationship (subtracting a number is equivalent to adding the opposite) to the appropriate symbols of the mathematical sentence. They appeared to treat the subtraction sign as a placeholder and did not fully understand its role and purpose as binary sign, indicating algebraic subtraction (Vlassis, 2008). Interestingly, once students misinterpreted the original symbolic problem, they proceeded to create pictures and stories to match this interpretation. Rarely were students able to overcome their initial error.

Many mathematics educators argue that students over-learn “take away” as an interpretation of subtraction in whole number arithmetic (Moses, Kamii, Swap, & Howard, 1989). This interpretation does not always translate to the modeling and counting up procedures many children naturally use to solve these problems (Baroody, 1984; Fuson, 1984). When students expand into arithmetic situations with negative integers, the complexity increases and “take away” does not adequately model subtraction with positive and negative integers. Thus, a more flexible interpretation of subtraction (i.e., one that includes comparison, difference, and other contexts) that allows for both quantity and direction features of integers to be made explicit is needed (Moses, Kamii, Swap, & Howard, 1989).

Confusion over the subtraction operator in integer arithmetic is common. Mathematics educators note the multiple and varied uses of the minus sign as one of the main obstacles students face in working with integers (Hativa & Cohen, 1995; Vlassis, 2008). Peled et al. (1989) found that students often disregard or (inappropriately) relocate the minus signs in problems in order to better accommodate their understanding. Students in this study demonstrated these errors. Ashlock (1994) points out that instructional experiences must “relate the model to a number sentence, numbers to their numerals, and operations to the sign for the operation.” (p. 246). It appears that while the students’ work with the virtual manipulatives was successful in relating models to number values in number sentences, it was not completely successful in relating the model to the operation used in the number sentences.

The structure of the virtual manipulative applets used in this study began with the symbolic statement and then created pictorial models from that statement. Therefore, students were only required to attend to the values of the integers in the statement (to produce the correct number of tiles or to move the slider in the correct direction). The applet provided the necessary structure to ensure that the pictorial model complied with the specific operation symbol in the statement. As a result, when creating a corresponding symbolic sentence, some students appeared to disregard the role of the "–" sign as an operation indicator and ignored its effects on the value of the subtrahend. These students would have benefited from tasks targeted on making that distinction explicit.

Limitations

Several limitations to the present study should be noted. First, the participants in this study were enrolled in what was labeled as an above-grade level course in the districts' curriculum. Therefore, the results may not be representative and generalizeable to all sixth-grade students. Second, the time of instruction could have been longer to allow students adequate opportunities to explore and practice the concepts of integer addition and subtraction. Third, because the researchers were not the instructors, there may have been inconsistencies in the students' instructional experiences due to individual teacher biases and practices. Finally, most of the students had never used any virtual manipulative tool prior to this study. Therefore, the novelty of these tools might have contributed to the positive results in computation.

CONCLUSIONS AND IMPLICATIONS FOR FURTHER RESEARCH

Although students in this study made positive gains in achievement on integer addition and subtraction computation, the students had not yet developed a complete understanding of integer addition and subtraction concepts. Similar to other research (e.g., Peled & Carraher, 2007), after working with the virtual manipulatives, students demonstrated a degree of flexibility in working with and among some representations for integers. However, they still showed some difficulties in interpreting and representing situations in symbolic form, particularly for subtraction items. Students need targeted experiences to help them build connections among symbolic and other representational forms of integer subtraction (Kilpartick, et al. 2001). Designers of future virtual manipulatives for integer instruction should consider adding features to the applet that would focus the student's attention on the purpose of the operation sign allowing them to make distinctions between, for example, the dual roles of the minus sign through their work with the virtual manipulatives.

Building on these results, further study should compare virtual manipulatives representing other models for integers to determine their effects on learning. For example, studies could examine the effectiveness of virtual manipulatives created to represent Janvier's hybrid model (Janvier, 1983) or an integer model based on algebraic geometry (Carson & Day, 1995). Finally, further research should examine the effect of virtual manipulatives designed specifically to help students make connections among interpretations of integer addition and subtraction situations and how these are represented in symbols. The process of working in different representational forms and reconciling inconsistencies among them can facilitate the development of a more complete and flexible understanding of integers and integer operations. The world of virtual manipulatives offers an excellent forum in which to design and study effective approaches to providing such experiences.

References

Ashlock, R. B.(1994). *Error patterns in computation: Using error patterns to improve instruction* (8th ed.). Upper Saddle River, NJ: Merrill Prentice Hall.

Baroody, A. J. (1984). Children's difficulties in subtraction: Some causes and questions. *Journal for Research in Mathematics Education, 15,* 203-213.

Bolyard, J. J. (2006). *A comparison of the impact of two virtual manipulatives on student achievement and conceptual understanding of integer addition and subtraction*. (Doctoral dissertation, George Mason University, 2006). Dissertation Abstracts International, 66 (11), 3960A.

Bruno, A., Espinel, M. C., & Martinon, A. (1997). Prospective teachers solve additive problems with negative numbers. *Focus on Learning Problems in Mathematics, 19*(4), 36-55.

Carson, C. L., & Day, J. (1995). *Annual report on promising practices: How the algebra project eliminates the "game of signs" with negative numbers.* San Francisco: Far West Lab for Educational Research and Development. (Eric Document Reproduction Service No. ED 394 828)

English, L. D. (Ed.). (1997). Analgoies, metaphors, and images: Vehicles for mathematical reasoning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images.* Mahwah, NJ: Lawrence Erlbaum Associates.

Fuson, K. (1984). More complexities in subtraction. *Journal for Research in Mathematics Education, 15,* 214-225.

Glesne, C. (1999). *Becoming qualitative researchers: An introduction* (2nd ed). New York: Addison Wesley Longman.

Goldin, G. A. (1997). Observing mathematical problem solving through task-based interview. *Journal for Research in Mathematics Education Monographs, 9*, 40-62.

Hativa, N. & Cohen, D. (1995). Self learning of negative number concepts by lower division elementary students through solving computer-provided numerical problems. *Educational Studies in Mathematics, 28*, 401-431.

Hsieh, H., & Shannon, S.E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15(9),*1277-1288. doi: 10.1177/1049732305276687

Janvier, C. (1985). Comparison of models aimed at teaching signed integers. In L. Streefland (Ed.), *Proceedings of the 9h International Conference for the Psychology of Mathematics Education* (vol. 1, pp. 135-140). International Group for the Psychology of Mathematics Education. (ERIC Document Reproduction Service No. ED 411 130)

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up. Helping children learn mathematics.* Washington, DC: National Academy Press.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (33-40). Hillsdale, NJ: Erlbaum.

Liebeck, P. (1990). Scores and forfeits: An intuitive model for integer arithmetic. *Educational Studies in Mathematics, 21*, 221-239.

Maxwell, J. A. (1996). *Qualitative research design: An interactive approach.* Thousand Oaks, CA: Sage.

Moreno, R. & Mayer, R. E. (1999). Multimedia-supported metaphors for meaning making in mathematics. *Cognition and Instruction, 17*, 215-248.

Moses, R. P., Kamii, M., Swap, S. M., & Howard, J. (1989). The Algebra Project: Organizing in the spirit of Ella. *Harvard Educational Review, 59*, 423-443.

Moyer-Packenham, P. S., Westenskow, A., & Salkind, G. (April, 2012). *A review of the research on virtual manipulatives in mathematics instruction.* Paper accepted for presentation at the annual meeting of the America Educational Research Association, Vancouver.

Mukhopadhyay, S., Resnick, L. D., & Schauble, L. (1990). Social sense-making in mathematics; Children's ideas of negative numbers. In G. Booker, P. Cobb, T. N. deMendicuti (Eds.), *Proceedings of the 14th International Conference for the Psychology of Mathematics* Education (vol. 3, pp. 281-288). International Group for the Psychology of Mathematics Education. (ERIC Document Reproduction Service No. ED 411 139)

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Peled, I. (1991). Levels of knowledge about signed numbers: Effect of age and ability. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (vol. 3, pp. 145-152). International Group for the Psychology of Mathematics Education. (ERIC Document Reproduction Service No. ED 413164)

Peled, I. & Carraher, D.W. (2007). Signed numbers and algebraic thinking. In J. Kaput. D. Carraher, & M. Blanton (Eds.), Algebra in the Early Grades. Mahwah, NJ, Erlbaum, pp. 303-327 (now Taylor & Francis).

Peled, I., Mukhopadhyay, S., & Resnick, L. B. (1989). Formal and informal sources of mental models for negative numbers. In G. Vergnaud, J. Rogalski, & M. Artique (Eds.), *Proceedings of the 13h International Conference for the Psychology of Mathematics Education* (vol. 3, pp. 106-110). International Group for the Psychology of Mathematics Education. (ERIC Document Reproduction Service No. 411 142)

Sherzer, L. (1973). Effects of different methods of integer addition instruction on elementary school students of different grade and aptitude levels. (Doctoral dissertation, University of Miami, 1973). *Dissertation Abstracts International, 34*(05), 2465A. (UMI No. 7325919)

Smith, J. P. (1995). The effects of a computer microworld on middle school students' use and understanding of integers. *Dissertation Abstracts International, 56*(09), 3492A.

Smith, L. (2006). *The impact of virtual and concrete manipulatives on algebraic understanding.* Unpublished doctoral dissertation, George Mason University, Fairfax, Virginia.

Vershcaffel, L., Greer, B., & De Corte, E. (2007) Whole number concepts and operations. In F. K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Charlotte, NC: Information Age.

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology, 21,* 555-570.