5-2009

Symmetry Analysis of General Rank-3 Pfaffian Systems in Five Variables

Francesco Strazzullo
Utah State University

SYMMETRY ANALYSIS OF GENERAL RANK-3 PFAFFIAN SYSTEMS IN FIVE VARIABLES

by

Francesco Strazzullo

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematical Sciences

Approved:

Dr. Ian M. Anderson Major Professor	Dr. Mark Fels Committee Member
Dr. Charles Torre Committee Member	Dr. Zhi-Qian Wang Committee Member
Dr. Dariusz Wilczynski Committee Member	Dr. Byron R. Burnham Dean of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

2009

ABSTRACT

Symmetry Analysis of General Rank-3 Pfaffian Systems in Five Variables

by

Francesco Strazzullo, Doctor of Philosophy

Utah State University, 2009

Major Professor: Dr. Ian M. Anderson
Department: Mathematics and Statistics

In this dissertation we applied geometric methods to study underdetermined second order scalar ordinary differential equations (called general Monge equations), nonlinear involutive systems of two scalar partial differential equations in two independent variables and one unknown and non-Monge-Ampère Goursat parabolic scalar PDE in the plane. These particular kinds of differential equations are related to general rank-3 Pfaffian systems in five variables. Cartan studied these objects in his 1910 paper. In this work Cartan provided normal forms only for some general rank-3 Pfaffian systems with 14-, 7-, and 6-dimensional symmetry algebras.

In this dissertation we provided normal forms of all general rank-3 Pfaffian systems in five variables with a freely acting transverse 3-dimensional symmetry algebra. We applied our normal forms to

[i] sharpen Cartan's integration method of nonlinear involutive systems,

[ii] classify all general Monge equations with a freely acting transverse 3-dimensional symmetry algebra, of which many new examples are presented, and

[iii] provide a broad classification of non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane.

We developed a computer software, called FiveVariables, that classifies general rank-3 Pfaffian systems. FiveVariables runs in the environment DifferentialGeometry of Maple, version 11 and later.

(235 pages)

DEDICATION

I dedicate this dissertation to my wife, Lucia, who gave me love and strength during my doctorate and beyond, to our sons Alessandro and Marco, my joy and motivation, and to my family in Italy, that always supports me.

ACKNOWLEDGMENTS

I wish to express my gratitude to Dr. Anderson, who gave me the opportunity to work on this challenging project, and the means to complete it. He helped me to grow professionally and personally.

I greatly thank Dr. Fels for our instructive discussions and his observations about this project.

I am grateful to Dr. Thompson, Dr. Torre, Dr. Wang, and Dr. Wilczynski for their contributions in the final version of this dissertation. I am also grateful to Dr. Emanuel and Dr. Summers for proofreading my manuscript.

Many thanks go to the staff of the Mathematics and Statistics Department at USU. They helped me in so many ways.

I want to remember my undergraduate advisor, Dr. Rotondaro, who introduced me to the study of differential geometry and to the exploitation of the computer algebra systems in this matter.

I have been blessed with many friends, that I thank all. An especial mention goes to the Bahlers and the Runges.

This dissertation is the result of the dedication and work of all the teachers I had in these years, first of all, my parents, Graziella and Biagio.

Finally, to Whom Who is above all, I give thanks for everything I have.

Francesco

CONTENTS

Page

ABSTRACT . . . . **iii**

DEDICATION . . . . **iv**

ACKNOWLEDGMENTS . . . . **v**

LIST OF TABLES . . . . **ix**

LIST OF FIGURES . . . . **x**

1 INTRODUCTION **1**

2 PRELIMINARIES **10**
- 2.1 Preliminary definitions and properties 10
- 2.2 Pfaffian systems, their numerical invariants and basic properties 12
- 2.3 Rank-1 Pfaffian systems 19
- 2.4 Jet spaces, PDE, and Pfaffian systems 20
- 2.5 The jet space $J^1(\mathbb{R}^m, \mathbb{R}^n)$ 21
- 2.6 The jet space $J^k(\mathbb{R}, \mathbb{R})$ 23
- 2.7 Frobenius, Bryant, and special Goursat normal forms on a 5-manifold 26

3 RANK-2 AND -3 PFAFFIAN SYSTEMS ON A 5-MANIFOLD **28**
- 3.1 Antiderived systems 28
- 3.2 Characterization of the contact system $\mathrm{C}^1_{1,2}$ 36
- 3.3 GR_2D_5 Pfaffian systems 39
- 3.4 GR_3D_5 Pfaffian systems and their general Goursat normal form 45
- 3.5 Monge equations and GR_3D_5 Pfaffian systems 49
- 3.6 Cartan tensors and particular normal forms obtained by Cartan 60
- 3.7 Summary of non-Frobenius rank-2 and -3 Pfaffian systems in five variables . . 62

4 CARTAN INTEGRATION METHOD FOR NON-LINEAR INVOLUTIVE SYSTEMS OF PDE AND GENERAL GOURSAT EQUATIONS **64**
- 4.1 Involutive systems and GR_3D_5 Pfaffian systems 64
- 4.2 Goursat equations and GR_3D_5 Pfaffian systems 86
- 4.3 A special bijection 95
- 4.4 Lifting GR_3D_5 Pfaffian systems to a 6-manifold 98

5 NORMAL FORMS OF $\mathbf{GR_3D_5}$ PFAFFIAN SYSTEMS WITH 3-DIMENSIONAL TRANSVERSE AND FREE SYMMETRY ALGEBRA **102**
- 5.1 Gauge formalism 102
- 5.2 The list of the 3-dimensional real Lie algebras 109
- 5.3 $3A_1$: the abelian algebra 111
  - 5.3.1 Step 1 111
  - 5.3.2 Step 2 112
  - 5.3.3 Step 3 113
  - 5.3.4 Step 4 116
- 5.4 $A_1 \oplus A_2$: the decomposable solvable algebra 119
  - 5.4.1 Step 1 119

5.4.2 Step 2 . . . 120
5.4.3 Step 3 . . . 121
5.4.4 Step 4 . . . 124
5.5 $A_{3,1}$: the Heisenberg (nilpotent) algebra . . . 128
5.5.1 Step 1 . . . 128
5.5.2 Step 2 . . . 129
5.5.3 Step 3 . . . 130
5.5.4 Step 4 . . . 133
5.6 $A_{3,2}$: the solvable algebra not belonging to a family of algebras . . . 137
5.6.1 Step 1 . . . 137
5.6.2 Step 2 . . . 138
5.6.3 Step 3 . . . 139
5.6.4 Step 4 . . . 141
5.7 $A^{\epsilon}_{3,5}$: the one parameter family of solvable algebras with $\epsilon \neq 0$ . . . 145
5.7.1 Step 1 . . . 145
5.7.2 Step 2 . . . 146
5.7.3 Step 3 . . . 147
5.7.4 Step 4 . . . 149
5.8 $A^{\epsilon}_{3,7}$: the one parameter family of solvable algebras with $\epsilon \geq 0$ . . . 153
5.8.1 Step 1 . . . 153
5.8.2 Step 2 . . . 154
5.8.3 Step 3 . . . 155
5.8.4 Step 4 . . . 157
5.9 $A_{3,8}$: the special linear algebra . . . 161
5.9.1 Step 1 . . . 161
5.9.2 Step 2.1 . . . 163
5.9.3 Step 3.1 . . . 164
5.9.4 Step 4.1 . . . 166
5.9.5 Step 2.2 . . . 170
5.9.6 Step 3.2 . . . 171
5.9.7 Step 4.2 . . . 173
5.10 $A_{3,9}$: the special orthogonal algebra . . . 177
5.10.1 Step 1 . . . 177
5.10.2 Step 2 . . . 179
5.10.3 Step 3 . . . 179
5.10.4 Step 4 . . . 181

6 EXAMPLES . . . 186
6.1 $3A_1$ . . . 187
6.2 $A_1 \oplus A_2$ . . . 188
6.3 $A_{3,1}$ . . . 189
6.4 $A_{3,2}$ . . . 191
6.5 $A^{\epsilon}_{3,5}$ . . . 192
6.6 $A^{\epsilon}_{3,7}$ . . . 194
6.7 $A_{3,8,1}$ . . . 195
6.8 $A_{3,8,2}$ . . . 196
6.9 $A_{3,9}$ . . . 197
6.10 Summary of root-type lists . . . 198

7 DARBOUX INTEGRABLE HYPERBOLIC PDE IN THE PLANE 200
7.1 Basic definitions 200
7.2 Non-Monge-Ampère equations 203
7.3 Examples 207
7.4 Addendum: the exceptional Lie algebra $\mathfrak{g}_2$ 209

8 FIVEVARIABLES PACKAGE 215

REFERENCES 222

VITA 225

LIST OF TABLES

Table — Page

1.1 Symmetry normal forms. 8

1.2 General Monge normal forms. 9

2.1 Jet-related normal forms on a 5-manifold. 27

3.1 General Monge normal forms in Cartan 1910 [10]. 63

3.2 Normal forms of non-Frobenius rank-2 and -3 Pfaffian systems on a 5-manifold . . . 63

7.1 Non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane (1). 211

7.2 Non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane (2). 212

7.3 A Chevalley basis for $\mathfrak{g}_2$. 213

7.4 Lie brackets in $\mathfrak{g}_2$. 214

LIST OF FIGURES

Figure Page

1.1 Dissertation overview. 1

1.2 Example of the theory in Chapters 3 and 4. 5

CHAPTER 1

INTRODUCTION

The subject of this dissertation falls within the geometric study of differential equations. The dissertation consists of four parts. The first part (Chapters 2, 3, and 4) deals with the general theory of rank-2 and rank-3 Pfaffian systems on a 5-dimensional manifold and its relation to the problem of solving certain special types of partial differential equations. In the second part (Chapters 5 and 6), we carry out a detailed symmetry analysis of general rank-3 Pfaffian systems. The third part provides an application of our results to the theory of Darboux integrable hyperbolic second-order scalar PDE in the plane (Chapter 7). In the last part (Chapter 8), we describe how we implemented the equivalence method of Cartan for GR_3D_5 Pfaffian systems with a computer algebra system.

The diagram in Figure 1.1 summarizes the structure of this dissertation.

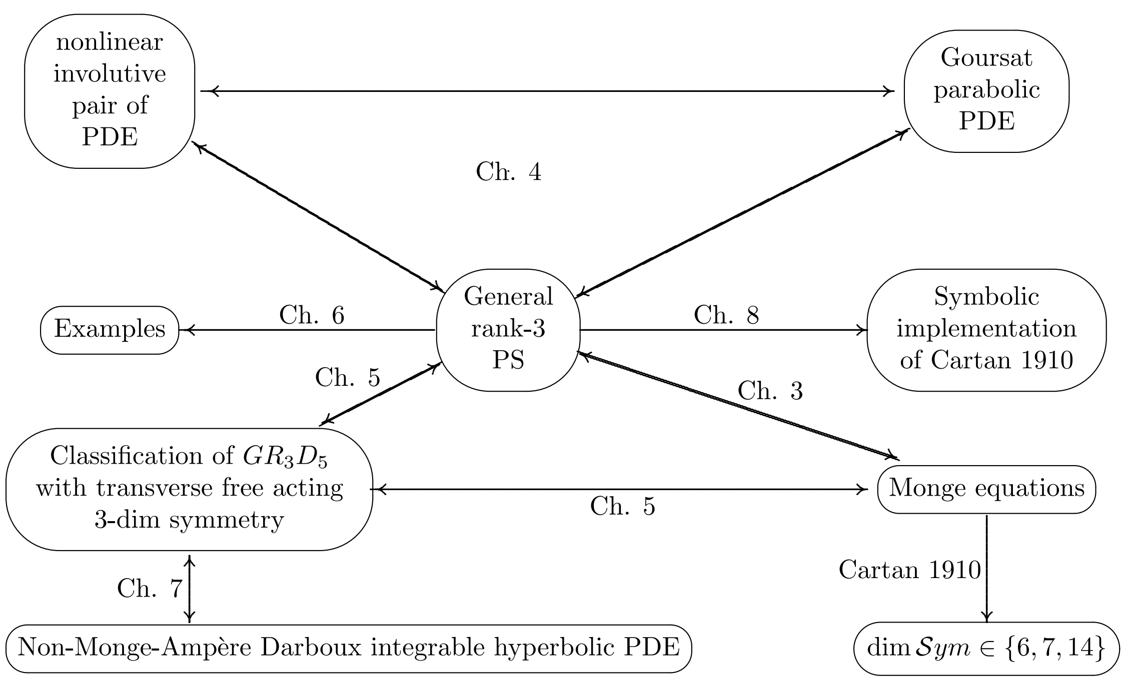

Fig. 1.1: Dissertation overview.

This dissertation started from the study of Èlie Cartan's famous *five variables* (or 1910) paper [10]. In this lengthy article, Cartan considers five main topics.

[A] The analysis of rank-2 and -3 Pfaffian systems on a 5-manifold (§II).

[B] The application of geometric methods to the integration of involutive systems of PDE and Goursat parabolic PDE in the plane (§III to §V).

[C] The application of the equivalence method to compute the fundamental invariant of general rank-3 (or GR_3D_5) Pfaffian systems in five variables (§VI).

[D] The distribution of GR_3D_5 Pfaffian systems into six main classes (§VII to §XI).

[E] The analysis of some special classes of GR_3D_5 Pfaffian systems (§XII to §XIV).

A recent exposition of these topics can be found in Stormark [38].

In Chapters 2 and 3, we provide a detailed account of topic **[A]**. We formalize the definition of a general rank-3 Pfaffian systems in five variables, as follows.

Definition 1. A rank-3 Pfaffian system I defined on a 5-manifold is said to be ***general*** and denoted GR_3D_5, if and only if the derived type of I is $[3, 2, 0]$ (see Definition 2.2.7). ■

Our main contribution to topic **[A]** is the following Theorem 1, which was only cited by Cartan as a consequence of Goursat's work [26].

Theorem 1. *A rank-3 Pfaffian system I defined on a 5-manifold is a GR_3D_5 Pfaffian system if and only if I is locally realized by an underdetermined second-order ordinary differential equation $Z' = H(X, Y, Z, Y', Y'')$ for which* $\dfrac{\partial^2 H}{\partial Y''^2} \neq 0$.

The equations of the type considered in Theorem 1 were called (general) second-order Monge equations by Gardner [18, page 148]. The proof of this theorem provides an algorithm, which we called Monge Algorithm 3.5.8, that is repeatedly used in Chapter 5.

We can say that topic **[B]** is the goal of the 1910 paper. Every system of two PDE in the plane gives rise to a rank-3 Pfaffian system in 6-variables, say I_2. Cartan proved that such systems are involutive if and only if they admit a unique Cauchy characteristic directional field. Consequently I_2 can be reduced to a rank-3 Pfaffian system in five variables. In Chapter 4 we summarize this result, obtaining the following.

Theorem 2. *Let $z = z(x, y)$ be a scalar function on the plane and let*

$$z_{xx} = R(x, y, z, z_x, z_y, z_{yy}), \quad z_{xy} = S(x, y, z, z_x, z_y, z_{yy}), \tag{1.1}$$

be an involutive system. Assume that I_2 is the Pfaffian system generated by (1.1) *and that C is a Cauchy characteristic of I_2. Then I_2 is reduced by C to a GR_3D_5 Pfaffian system if and only if* $\dfrac{\partial^2 S}{\partial {z_{yy}}^2} \neq 0$.

In Chapter 4, combining Theorems 1 and 2, we arrive at an integration method for nonlinear involutive systems, which was one of the goals in topic [**B**].

The simplest example we can use to illustrate the general theory developed in Chapters 2 through 4 is the following. In terms of the standard notation $(p, q, r, s, t) = (z_x, z_y, z_{xx}, z_{xy}, z_{yy})$, the system of scalar PDE in the plane

$$r = \frac{1}{3}t^3, \quad s = \frac{1}{2}t^2, \tag{1.2}$$

is of the kind treated in Theorem 2. The system (1.2) gives rise to the rank-3 Pfaffian system

$$I_2 = \begin{cases} dz - p\,dx - q\,dy, \\ dp - \dfrac{t^3}{3}\,dx - \dfrac{t^2}{2}\,dy, \\ dq - \dfrac{t^2}{2}\,dx - t\,dy, \end{cases} \tag{1.3}$$

on the 6-manifold M_6 with local coordinates (x, y, z, p, q, t). A Cauchy characteristic of (1.2) or I_2 is

$$C = \partial_x - t\,\partial_y + (p - qt)\,\partial_z - \frac{t^3}{6}\,\partial_p - \frac{t^2}{2}\,\partial_q. \tag{1.4}$$

A complete set of invariants for C is

$$x^1 = t, \quad x^2 = y + tx, \quad x^3 = p + \frac{t^3}{6}x, \quad x^4 = q + \frac{t^2}{x}, \quad x^5 = z + \frac{t^3}{6}x^2 + txq - xp. \tag{1.5}$$

Consequently, I_2 is reduced to the rank-3 Pfaffian system I

$$I = \begin{cases} dx^5 - x^4 dx^2, \\ dx^3 - \dfrac{{x^1}^2}{2}\,dx^2, \\ dx^4 - x^1\,dx^2, \end{cases} \tag{1.6}$$

on the quotient manifold M_5 with local coordinates $(x^1, \ldots, x^5)$. Cartan proved that I is realized by the Hilbert-Cartan equation

$$\frac{dZ}{dX} = \left(\frac{d^2Y}{dX^2}\right)^2. \tag{1.7}$$

Using the solutions of (1.7), we can construct the 2-dimensional integral manifold $\mathbf{s} : (X, \mu) \in \mathbb{R}^2 \to M_6$ of (1.2). Indeed, in Example 4.1.14, we see that the integral manifold of (1.2) is given by

$$x = \mu, \quad y = X\mu + F'', \quad z = -\frac{1}{6}\mu^2 X^3 + \left(F - \frac{1}{2}X^2 F''\right)\mu - \frac{1}{2}\left(G + XF''^2 - 2F'F''\right), \tag{1.8}$$

for every solution $Y = F(X)$ and $Z = G(X)$ of (1.7). For instance, from the solution

$$Y = \frac{1}{12}X^3, \quad Z = \frac{1}{12}X^3 + k, \quad k = \text{const.} \tag{1.9}$$

of (1.7) we obtain the solution

$$z = -\frac{y^3}{3(2x+1)} - \frac{k}{2} \tag{1.10}$$

of (1.2).

This integration method outlined by Cartan requires the realization of (1.6) as the rank-3 Pfaffian system in five variables generated by (1.7). Although such a realization always exists, its explicit computation is quite complicated. In Chapter 4 we apply this integration method to a family of nonlinear involutive systems for which (1.2) is particular (see Theorem 4.1.13).

It turns out a similar procedure can be applied to Goursat parabolic PDE in the plane which are non-Monge-Ampère equations (or simply general Goursat equations). Indeed, following Cartan's topic **[B]** in Chapter 4, we show that there is a bijective mapping between general Goursat equations and nonlinear involutive systems (see Section 4.3). For instance, the general Goursat equation

$$32s^3 - 12t^2s^2 + 9r^2 - 36rts + 12rt^3 = 0, \tag{1.11}$$

is associated to (1.2) (see Example 4.3.2). We display this example in the diagram of Figure 1.2 (page 5).

While developing topic **[C]**, Cartan proved that all GR_3D_5 Pfaffian systems have a finite dimensional symmetry algebra, namely of dimensions 14 or less than 8. Cartan then provided normal forms for those GR_3D_5 Pfaffian systems with symmetry algebra of dimension 14 or 7, and some normal forms for those with 6-dimensional symmetry algebra. No normal forms were given for GR_3D_5 Pfaffian systems whose symmetry algebra is 5-dimensional or smaller. In Chapter 5 we provide normal forms for all the GR_3D_5 Pfaffian systems with a transverse free-acting 3-dimensional symmetry algebra. We summarize the results of Chapters 5 in the following theorems and tables, where a

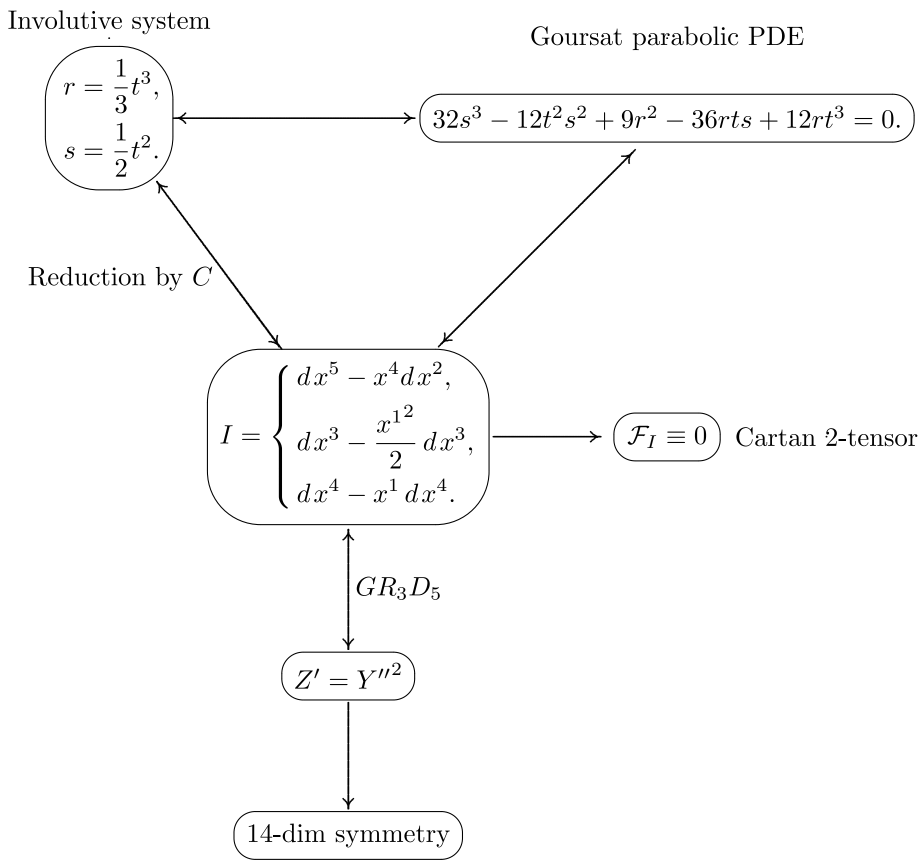

Fig. 1.2: Example of the theory in Chapters 3 and 4.

reference to the proof of each entry is provided.

Theorem 3 (GR_3D_5 symmetry normal forms)**.** *Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Then about each point of M there are local coordinates (a, b, c, u, v) such that I and Γ can be expressed in one of the normal forms of Table 1.1 (page 8). There, $[\mathfrak{g}]$ denotes the algebraic type of Γ according to [36], $F = F(u, v)$ is a differentiable function and $K = K(u, v, F)$ is subject to the constraint $D_{vv}[K] \neq 0$.*

The Monge Algorithm 3.5.8 that we elaborated together with Theorem 3 provide the following.

Theorem 4 (General Monge normal forms)**.** *Let*

$$Z_1 = H\left(X, Y, Z, Y_1, Y_2\right), \qquad \frac{\partial^2 H}{\partial {Y_2}^2} \neq 0, \tag{1.12}$$

be a general second order Monge equation on the 5-manifold N, with local coordinates (X, Y, Z, Y_1, Y_2). Assume that (1.12) *admits a 3-dimensional symmetry group G which acts freely and transversely on N and denote by Γ a set of infinitesimal generators of the action of G on N. Then* (1.12) *and Γ can be expressed as in Table 1.2 (page 9). There, $[\mathfrak{g}]$ is the algebraic type of Γ according to [36] and h is a differentiable function depending only on two arguments, which is non-zero for the algebraic type $A_{3,9}$ while for all the other algebraic types h is such that $D^2_{[2]}h \neq 0$.*

As we already mentioned, obtaining the general Monge normal form of a GR_3D_5 Pfaffian system is not easy task. The assumptions about the symmetry algebra in Theorem 4 make this task more accessible. These results combined with that of Chapter 4 sharpen Cartan's integration method for nonlinear involutive systems. To illustrate this point, we give the following theorem.

Theorem 5. *Let I be the reduction by the Cauchy characteristic of the Pfaffian system generated by a nonlinear involutive system*

$$r = R\,(x, y, z, p, q, t)\,, \quad s = S\,(x, y, z, p, q, t)\,. \tag{1.13}$$

Assume that the Heisenberg algebra is a symmetry algebra of I, acting transversely and freely. Then the integral manifolds of (1.13) *are expressed in terms of two independent variables, $\hat{x}$ and X, one generic function $Y = f(X)$, its derivatives Y' and Y'', and a function $Z = G(X)$ such that $Z' = Y + h(Y', Y'')$.*

One future work is the implementation in a computer algebra system of this theory and our algorithm, in order to create a solver for involutive systems.

Another conclusion obtained in topic **[C]** is the construction of the Cartan 2-tensor $\mathcal{F}_I$ associated to every GR_3D_5 Pfaffian system I. This is a homogeneous fourth degree polynomial in 2-variables. GR_3D_5 Pfaffian systems can be classified according to the root type of $\mathcal{F}_I$, as Cartan did in topic **[D]**. In Chapter 6 we provide many examples of inequivalent GR_3D_5 Pfaffian systems for each root type, which are new to the literature.

In Chapter 7, Theorem 7.2.2, we provide another application of our symmetry analysis of GR_3D_5 Pfaffian systems (Theorem 3). This is based upon the recent work of Anderson, Fels, and Vassiliou [3], which allows a broad classification of all non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane. For instance, we derive the following

Theorem 6 (Heisenberg Darboux integrability)**.** *Let*

$$F(x, y, z, p, q, r, s, t) = 0 \tag{1.14}$$

be a non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane with Vessiot algebra the Heisenberg algebra. Then (1.14) *is the quotient of two copies of*

$$Z' = Y + h(Y', Y'')$$

by the diagonal action of

$$\Gamma = \langle \partial_Z,\, \partial_X,\, \partial_Y + X\,\partial_Z \rangle.$$

Conversely, every such quotient gives rise to a non-Monge-Ampère Darboux integrable hyperbolic PDE (1.14) *with Heisenberg Vessiot algebra.*

In topic **[C]** Cartan actually constructed two invariants of a GR_3D_5 Pfaffian system I. One is $\mathcal{F}_I(x_1, x_2)$, and the other is a homogeneous fourth degree polynomial in 3-variables $\mathcal{G}_I$ such that $\mathcal{G}_I(x_1, x_2, 0) = \mathcal{F}_I$. The computation of the Cartan tensors is very complicated; Cartan himself could provide only few explicit expressions. In Chapter 8 we summarize the basic steps of topic **[C]**. With an important contribution of Hsiao [29], we implemented the equivalence method for GR_3D_5 Pfaffian systems and nonlinear involutive systems. We briefly describe the program *FiveVariables* that we developed in order to compute the Cartan tensors and the root type of $\mathcal{F}_I$. This program works under the DifferentialGeometry package of Maple (version 11 and later).

Table 1.1: Symmetry normal forms. $D_{vv}[K] \neq 0$.

$[\mathfrak{g}]$	Γ	I; K.	Prop.
$3A_1$	$-\partial_a,\ -\partial_b,\ -\partial_c.$	$db - u\,da,\ dc - v\,da - F\,du,$ $da + F_v\,du; \quad K = F.$	5.3.2
$A_1 \oplus A_2$	$-e^c\,\partial_a,\ -\partial_c,\ -\partial_b.$	$da - a\,dc + F\,du + dv,\ db - u\,dc,$ $dc + F_v\,du; \quad K = F.$	5.4.2
$A_{3,1}$	$-\partial_b,\ -\partial_a - c\,\partial_b,\ -\partial_c.$	$db - a\,dc + F\,du,\ dc - u\,da - dv,$ $da + F_v\,du; \quad K = F.$	5.5.2
$A_{3,2}$	$-\,e^c\,\partial_b,\ -e^c\,\partial_a - ce^c\,\partial_b,$ $-\,\partial_c.$	$db - (a+b)\ dc + F\,du + dv,$ $da - a\,dc + du,\ dc + F_v\,du; \quad K = F.$	5.6.2
$A^{\epsilon}_{3,5}$ $\epsilon \neq 0$	$-e^c\,\partial_a,\ -e^{\epsilon c}\,\partial_b,\ -\partial_c.$	$da - a\,dc + du,\ db - \epsilon b\,dc + F\,du + dv,$ $\epsilon\,dc + F_v\,du; \quad K = F.$	5.7.2
$A^{\epsilon}_{3,7}$ $\epsilon \geq 0$	$A\,\partial_a + B\,\partial_b,\ A\,\partial_b - B\,\partial_a,$ $-\,\partial_c.$ $A = \dfrac{e^{\epsilon c}\,(\epsilon \sin c - \cos c)}{1+\epsilon^2},$ $B = \dfrac{e^{\epsilon c}\,(\epsilon \cos c + \sin c)}{1+\epsilon^2}.$	$da - \epsilon\,db + \left(\epsilon^2 b - b - 2\epsilon a\right)\ dc + F\,du + dv,$ $\epsilon\,da + db + \left(a - \epsilon^2 a - 2\epsilon b\right)\ dc + du,$ $dc - \dfrac{F_v}{\epsilon^2+1}\left((F-\epsilon)\,du + dv\right);$ $K_v = (F - \epsilon)\,F_v - F_u.$	5.8.2
$A_{3,8,1}$	$-\,\partial_a - 2b\,\partial_b + 2c\,\partial_c,$ $c\,\partial_a + (1+2cb)\ \partial_b - c^2\,\partial_c,$ $\partial_c.$	$da - b\,dc + F\,du + dv,$ $e^{-2a}\,db - b^2 e^{-2a}\,dc + du,$ $e^{2a}\,dc - \left(\dfrac{1}{2} - FF_v\right)\ du - F_v\,dv;$ $K_v = FF_v - v - F_u.$	5.9.2
$A_{3,8,2}$	$-\,\partial_a - 2b\,\partial_b + 2c\,\partial_c,$ $c\,\partial_a + (1+2cb)\ \partial_b - c^2\,\partial_c,$ $\partial_c.$	$e^{-2a}\,db - b^2 e^{-2a}\,dc + F\,du + dv,$ $e^{2a}\,dc + du,\ da - b\,dc - \dfrac{1}{2}F_v\,du;$ $K = F + v^2.$	5.9.5
$A_{3,9}$	$A_c\,\partial_a + B_c\,\partial_b + C_c\,\partial_c,$ $A\,\partial_a + B\,\partial_b + C\,\partial_c,\ -\partial_c.$ $A = \dfrac{-\sin c}{\cos b},\ B = -\cos c,$ $C = A \sin b.$	$da - \sin b\,dc + du,$ $\cos a\,db + \cos b \sin a\,dc + F\,du + dv,$ $\cos b \cos a\,dc - \sin a\,db + FF_v\,du + F_v\,dv;$ $K_v = FF_v + v - F_u.$	5.10.2

Table 1.2: General Monge normal forms. $Z_1 = H$ and $H_{Y_2Y_2} \neq 0$.

$[\mathfrak{g}]$	Γ	H	Prop.
$3A_1$	$\partial_X,\, \partial_Y,\, \partial_Z.$	$h(Y_1, Y_2)$	5.3.3
$A_1 \oplus A_2$	$\partial_Z,\, Z\,\partial_Z - X\,\partial_X,\, \partial_Y.$	$X^{-2}\,h\big(XY_1, X^2Y_2\big)$	5.4.3
$A_{3,1}$	$\partial_Z,\, \partial_X,\, \partial_Y + X\,\partial_Z.$	$Y + h(Y_1, Y_2)$	5.5.3
$A_{3,2}$	$\partial_Z,\, \ln X\,\partial_Z + X\,\partial_Y,$ $Z\,\partial_Z - X\,\partial_X.$	$X^{-2}Y + X^{-2}\,h\big(Y - XY_1, X^2Y_2\big)$	5.6.3
$A^{\epsilon}_{3,5}$ $\epsilon \neq 0$	$X^{1/\epsilon}\,\partial_Y,\, \partial_Z,$ $\epsilon\,(Z\,\partial_Z - X\,\partial_X)\,.$	$\frac{1}{\epsilon}X^{-2}\,h\big(\epsilon XY_1 - Y, \epsilon^2X^2Y_2 + \epsilon^2XY_1 - Y\big)$	5.7.3
$A^{\epsilon}_{3,7}$ $\epsilon \geq 0$	$e^{\epsilon X}\sin X\,\partial_Y,$ $-\,e^{\epsilon X}\cos X\,\partial_Y,$ $-\,\partial_X.$	$h\big(Z, Y_2 - 2\epsilon Y_1 + \epsilon^2 Y + Y\big)$	5.8.3
$A_{3,8,1}$	$2X\,\partial_X - 2\partial_Y,$ $2X\,\partial_Y - X^2\,\partial_X,$ $\partial_X.$	$e^Y\,h\big(Z, Y_2e^{-2Y} - \frac{1}{2}{Y_1}^2e^{-2Y}\big)$	5.9.3
$A_{3,8,2}$	$2X\,\partial_X - 2Z\,\partial_Z,$ $(1 + 2XZ)\,\partial_Z - X^2\,\partial_X,$ $\partial_X.$	$Z^2 + {Y_1}^2\,h\left(Y, \dfrac{Y_2 - 2Y_1Z}{{Y_1}^2}\right)$	5.9.6
$A_{3,9}$	$\dfrac{\cos X}{\sqrt{1 - {Y_1}^2}}\,(Y_1\,\partial_X + \partial_Y)\,,$ $\dfrac{-\sin X}{\sqrt{1 - {Y_1}^2}}\,(Y_1\,\partial_X + \partial_Y)\,,$ $\partial_X.$	$\sqrt{1 - {Y_1}^2 + \dfrac{{Y_2}^2}{1 - {Y_1}^2}}\,h\left(Z, Y + \arctan\dfrac{Y_2}{1 - {Y_1}^2}\right)$	5.10.3

CHAPTER 2

PRELIMINARIES

In this Chapter we report the basic notations and definitions used throughout this dissertation.

We will assume familiar to the reader some basic concepts of differential geometry, like those treated in the first chapters of [40], [6], and [37, v. I]. The first two chapters of [7] are our guiding light and [30] has been a useful source.

2.1 Preliminary definitions and properties

For the most part of this section we will refer to Gardner [16], [17], and [18], for definitions and proofs. Our goal is to share with the reader our notation and those properties that we give for granted.

We will be consistent with the following notation. M or M_m denote a differentiable m-dimensional manifold (or m-manifold); $C^\infty(M)$ is the ring of real smooth functions on M; $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on M; $\Omega(M)$ is the algebra of differential forms on M, with $\Omega^k(M)$ denoting the module of k-forms on M; TM is the tangent bundle; T^*M is the cotangent bundle and $\Lambda^k(M) = \Lambda^k(T^*M)$ is the bundle of k-covectors.

Definition 2.1.1. The ***hook operator*** (or interior product or contraction) is the map

$$\lrcorner : \mathfrak{X}(M) \times \Omega^{k+1}(M) \to \Omega^k(M)$$

defined by

$$[X \lrcorner \omega]\,(X_1, \ldots, X_k) = \omega\,(X, X_1, \ldots, X_k)\,.$$

In particular, if ω is a 1-form on M then $X \lrcorner \omega = \omega(X) \in \Omega^0(M) = C^\infty(M)$.

The ***annihilator*** of a vector field X is defined as $X^\perp = \{\omega \in \Omega^1(M) \mid X \lrcorner \omega = 0\}$. ■

Proposition 2.1.2. Let M be a manifold.

[i] If $\omega \in \Omega(M)$ and $X_i \in \mathfrak{X}(M)$, then $X_1 \lrcorner (X_2 \lrcorner \omega) = -X_2 \lrcorner (X_1 \lrcorner \omega)$.

[ii] The hook operator is an ***antiderivation***, that is, if $\omega \in \Omega^k(M)$, $\eta \in \Omega^j(M)$ and $X \in \mathfrak{X}(M)$, then

$$X \lrcorner (\omega \wedge \eta) = (X \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (X \lrcorner \eta)$$

Remark 2.1.3. Let $X \in \mathfrak{X}(M)$ and consider the (local) flow $\mathrm{Fl}^X = \{\mathrm{Fl}^X_t\}_{t \in (-\epsilon, \epsilon)}$ of X. Then around every point $p \in M$ the function Fl^X_t is a local diffeomorphism (see [37, v. I, pages 238-318], [6, page 27, 130], and [40, page 70]).

Definition 2.1.4. The ***Lie derivative*** is the map

$$\mathcal{L} : \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^k(M)$$

defined by

$$\mathcal{L}(X, \omega)_p = [\mathcal{L}_X \omega]_p = \lim_{h \to 0} \frac{\mathrm{Fl}^{X\,*}_h (\omega_{\mathrm{Fl}^X_{-h}(p)}) - \omega_p}{h}.$$

■

Proposition 2.1.5. The following are some properties of the Lie derivative.

[i] $\mathcal{L}_{fX}\omega = f\,\mathcal{L}_X \omega + df \wedge (X \lrcorner \omega)$.

[ii] $\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X \omega) \wedge \eta + \omega \wedge (\mathcal{L}_X \eta)$.

[iii] $\mathcal{L}_X(Y \lrcorner \omega) = Y \lrcorner \mathcal{L}_X \omega + (\mathcal{L}_X Y) \lrcorner \omega$.

[iv] $\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X(\mathcal{L}_Y \omega) - \mathcal{L}_Y(\mathcal{L}_X \omega)$.

Here $X, Y \in \mathfrak{X}(M)$, $\omega, \eta \in \Omega(M)$ and $f, g \in C^\infty(M)$.

Definition 2.1.6. The exterior derivative on M is the operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ defined by

$$\begin{aligned} d\omega\,(X_1, \ldots, X_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i+1} X_i \left(\omega\left(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}\right)\right) \\ & + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}\right), \end{aligned} \tag{2.1}$$

where $\omega \in \Omega^k(M)$ and $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$. In particular, if ω is a 1-form, then (2.1) becomes $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$. ■

Proposition 2.1.7. The following are some properties of the exterior derivative on M.

[i] If $f : M \to N$ is smooth and $\omega \in \Omega(N)$ then $d_M(f^*\omega) = f^* d_N \omega$.

[ii] Let $\omega \in \Omega(M)$ and $X \in \mathfrak{X}(M)$ then the following ***Cartan's formula*** holds

$$\mathcal{L}_X \omega = d(X \lrcorner\, \omega) + X \lrcorner\, d\omega. \tag{2.2}$$

2.2 Pfaffian systems, their numerical invariants and basic properties

In this section we define Pfaffian systems, their numerical invariants and some of their basic properties.

Definition 2.2.1. Let M be an m-manifold. $\mathcal{I} \subset \Omega(M)$ is an ***exterior differential systems (EDS)*** on M if

[i] $\mathcal{I}$ is an algebraic ideal of $\Omega(M)$, that is,

[i.a] $f \in C^\infty(M), \alpha \in \mathcal{I} \Rightarrow f\,\alpha \in \mathcal{I}$,

[i.b] $\alpha^1, \ldots, \alpha^s \in \mathcal{I} \Rightarrow \sum_{i=1}^s \alpha^i \in \mathcal{I}$,

[i.c] $\alpha \in \mathcal{I}, \beta \in \Omega(M) \Rightarrow \alpha \wedge \beta \in \mathcal{I}$ (and thus by **[i.a]** we have $\beta \wedge \alpha \in \mathcal{I}$);

[ii] $\mathcal{I}$ is differentially closed, $d\mathcal{I} \subset \mathcal{I}$, that is, if $\alpha \in \mathcal{I}$ then $d\alpha \in \mathcal{I}$.

We denote the set of (homogeneous) k-forms in $\mathcal{I}$ by $\mathcal{I}^k = \mathcal{I} \cap \Omega^k(M)$. *We assume that* $\mathcal{I}^0 = 0$. A ***solution*** (or integral manifold) of $\mathcal{I}$ is an immersion $\phi : N \to M$ such that $\phi^*\mathcal{I} = 0$. ■

We will deal with the following special kind of EDS.

Definition 2.2.2. A ***Pfaffian system of rank-s*** on an m-manifold M is an EDS $\mathcal{I}$ on M algebraically generated, in a neighborhood of each point of M, by a set of s linearly independent 1-forms $I = \{\theta^1, \ldots, \theta^s\}$ and the set of their exterior derivatives $dI = \{d\theta^1, \ldots, d\theta^s\}$. A minimal set of (1-forms) generators I is called a ***basis*** of the Pfaffian system $\mathcal{I}$. We shall often identify a Pfaffian system $\mathcal{I}$ with one of its bases $I \subset \Omega^1(M)$. The rank of a Pfaffian system I is denoted by $\dim I$, while the integer $p = m - s = \operatorname{codim} I$ is called the corank of I. A Pfaffian system K such that $K \subset I$ is called a ***subsystem*** of I. ■

Given a rank-s Pfaffian system $I = \left\{\theta^1, \ldots, \theta^s\right\}$ on M_m, we can complete the basis I to a local coframe $\theta^1, \ldots, \theta^s, \theta^{s+1}, \ldots, \theta^m$ on M. For each $\theta^i \in I$, the exterior derivative $d\theta^i$ is a 2-form on

M and therefore $d\theta^i$ is uniquely expressed as a linear combination of 2-fold wedge products of the 1-forms $\theta^1, \ldots, \theta^m$, that is,

$$d\theta^i = A^i_{jk}\theta^j \wedge \theta^k, \quad j, k \in \{1, \ldots, m\}. \tag{2.3}$$

Here we have $A^i_{jk} \in C^\infty(M)$ and we use the Einstein's notation for summation. The expressions (2.3) for $i = 1, \ldots, s$ are called the ***structure equations*** of I with respect to the local coframe $\theta^1, \ldots, \theta^m$.

Definition 2.2.3. Let $\mathcal{I}$ be a rank-s Pfaffian system on M, $I = \{\theta^1, \ldots, \theta^s\}$ a basis of $\mathcal{I}$, $\theta^1, \ldots, \theta^s$, $\theta^{s+1}, \ldots, \theta^m$ a local coframe on M, and $\omega^1, \omega^2 \in \Omega(M)$. We define $\omega^1 \equiv \omega^2 \mod I$ if and only if $\omega^1 - \omega^2$ is in the algebraic ideal generated by I.

One can prove that $\omega^1 \equiv \omega^2 \mod \{\theta^1, \ldots, \theta^s\}$ if and only if $\theta^1 \wedge \ldots \wedge \theta^s \wedge (\omega^1 - \omega^2) = 0$. We suggest [7, Chapters 1 and 2] and [30, Chapter 1 and Appendix B] for more details.

Remark 2.2.4. Let $\mathcal{I}$ be a rank-s Pfaffian system on M, $I = \{\theta^1, \ldots, \theta^s\}$ a basis of $\mathcal{I}$, and $\theta^1, \ldots, \theta^s, \theta^{s+1}, \ldots, \theta^m$ a local coframe on M. With the previous definition we can write the structure equations (2.3) as

$$d\theta^i \equiv A^i_{jk}\theta^j \wedge \theta^k \mod I, \quad j, k \in \{s+1, \ldots, m\}, \quad i = 1, \ldots, s.$$

Consider $\theta = \lambda_i\theta^i \in I$, then $d\theta \equiv 0 \mod I$ if and only if $\theta^1 \wedge \ldots \wedge \theta^s \wedge d\theta = 0$. Consequently $d\theta \equiv 0 \mod I$ if and only if about every point of M we have $\lambda_i A^i_{jk} = 0$. *We assume that this linear system in the variables λ_i has constant rank on M.*

Definition 2.2.5. Let I be a Pfaffian system.

[i] The ***derived system*** of I is the Pfaffian system (with basis) $I' = \{\theta \in I \mid d\theta \equiv 0 \mod I\}$ (see Remark 2.2.4).

[ii] The ***derived series*** of I is the flag of Pfaffian systems $I^{(0)} \supseteq \ldots \supseteq I^{(m)}$ recursively defined by $I^{(0)} = I$, $I^{(i+1)} = \left(I^{(i)}\right)' = \{\theta \in I^{(i)} \mid d\theta \equiv 0 \mod I^{(i)}\}$ for $i \geq 0$.

[iii] The ***derived length*** of I is the smallest integer N such that $I^{(N+1)} = I^{(N)}$. ■

Definition 2.2.6. Let I be a Pfaffian system on a manifold M.

[i] The space of ***Cauchy characteristics*** of I is the distribution

$$\mathrm{Cau}(I) = \{X \in \mathfrak{X}(M) \mid X \lrcorner\, \theta = 0, X \lrcorner\, d\theta \in I, \forall \theta \in I\}.$$

[ii] The ***Cartan system*** of I is the Pfaffian system $\mathcal{CS}(I)$ with basis

$$(\mathrm{Cau}(I))^{\perp} = \left\{\alpha \in \Omega^1(M) \mid \mathrm{Cau}(I) \lrcorner\, \alpha = 0\right\}.$$

[iii] The ***class*** of I is the integer $\mathrm{class}(I) = \dim \mathcal{CS}(I) = \operatorname{codim} \mathrm{Cau}(I)$.

[iv] The ***Engel-rank*** of I is the smallest integer $\mathrm{Eng}(I) = r \geq 0$ such that $(d\theta)^{[r+1]} \equiv 0 \bmod I$ for all $\theta \in I$. Here we use the power notation for the wedge product, that is, for any form α we define $\alpha^{[1]} = \alpha$ and $\alpha^{[r+1]} = \alpha^{[r]} \wedge \alpha$.

[v] The ***Cartan-rank*** of I is the smallest number $\mathrm{Car}(I) = v$ of linearly independent 1-forms $\pi^1, \ldots, \pi^v \in \Omega^1(M)/I$ such that $d\theta \wedge \pi^1 \wedge \ldots \wedge \pi^v \equiv 0 \bmod I$ for all $\theta \in I$. ■

We can now list the fundamental numerical invariants of a Pfaffian system.

Definition 2.2.7. Let I be a Pfaffian system with derived length N. The following numerical sequences of length $N+1$ are the ***fundamental invariants of*** I.

[i] The ***derived type*** is $\mathbf{DT}(I) = [\dim I, \dim I^{(1)}, \ldots, \dim I^{(N)}]$.

[ii] The ***Cauchy type*** is $\mathbf{Cau}(I) = [\dim \mathrm{Cau}(I), \dim \mathrm{Cau}\big(I^{(1)}\big), \ldots, \dim \mathrm{Cau}\big(I^{(N)}\big)]$.

[iii] The ***Engel type*** is $\mathbf{Eng}(I) = \big[\mathrm{Eng}(I), \mathrm{Eng}\big(I^{(1)}\big), \ldots, \mathrm{Eng}\big(I^{(N)}\big)\big]$.

[iv] The ***Cartan type*** is $\mathbf{Car}(I) = \big[\mathrm{Car}(I), \mathrm{Car}\big(I^{(1)}\big), \ldots, \mathrm{Car}\big(I^{(N)}\big)\big]$. ■

We will consider only Pfaffian systems whose ***numerical invariants are constant*** on M.

Now we turn to the properties of these invariants. First let's notice that the Cauchy characteristics, the Engel-rank and the Cartan-rank of a Pfaffian system I can be determined simply in a basis.

Proposition 2.2.8. Let I be a rank-s Pfaffian system with $I = \{\theta^1, \ldots, \theta^s\}$ and $I' = \{\theta^1, \ldots, \theta^{s_1}\}$ $(s_1 \leq s)$. Then the following properties hold.

[i] $\mathrm{Cau}(I) = \{X \in \mathfrak{X}(M) \mid X \lrcorner\, \theta^i = 0, X \lrcorner\, d\theta^j \in I,\ i = 1, \ldots, s,\ j = 1, \ldots, s_1\}$. In particular $\mathrm{Cau}(I) \subseteq \mathrm{Cau}(I')$.

[ii] $\mathrm{Eng}(I) = r$ if and only if r is the smallest integer such that $\left(d\theta^i\right)^{[r+1]} \equiv 0 \bmod I$ for $i = s_1 + 1, \ldots, s$.

[iii] $\mathrm{Car}(I) = v$ if and only if there are v linearly independent 1-forms $\pi^1, \ldots,\ \pi^v \in \Omega^1(M)/I$ such that $d\theta^j \wedge \pi^1 \wedge \ldots \wedge \pi^v \equiv 0 \bmod I$ for $j = 1, \ldots, s_1$.

A rank-s Pfaffian system I on an m-manifold M such that $I' = I$ is called ***Frobenius or completely integrable***. Trivially, if $s \geq m-1$ then I is completely integrable. A Frobenius system I has derived length $N = 0$, and hence all the numerical invariants of I have only one entry, namely,

$$\mathbf{DT}(I) = [s], \quad \mathbf{Cau}(I) = [m-s], \quad \mathbf{Eng}(I) = \mathbf{Car}(I) = [0]. \tag{2.4}$$

We shall encounter many normal forms of Pfaffian systems in this dissertation. The first is that of a completely integrable Pfaffian system, given in the following theorem.

Theorem 2.2.9 (Frobenius). *Let I be a rank-s Frobenius Pfaffian system on an m-manifold M. Then about every point of M there exists a coordinate system $\left(x^1, \ldots, x^s, \ldots, x^m\right)$ such that*

$$I = \left\{dx^1, \ldots, dx^s\right\}. \tag{2.5}$$

In particular, all rank-s Frobenius systems are locally equivalent.

From [16, page 515] we recall the following.

Remark 2.2.10. Let I be a rank-s Pfaffian system on an m-manifold M such that $\dim \mathrm{Cau}(I) = m-c$. The Cartan system $\mathcal{CS}(I)$ is the *smallest completely integrable Pfaffian system* $\{dx^1, \ldots, dx^c\}$ *containing* I such that locally the forms in I can be expressed in terms of functions of $x^1, \ldots, x^c$ and the 1-forms $dx^1, \ldots, dx^c$. ■

Example 2.2.11. Consider the rank-2 Pfaffian system $H = \{du - u'\,dt, dv - v'\,dt\}$ on a 5-manifold. The smallest completely integrable Pfaffian system containing H is the rank-3 Pfaffian system $L = \{du,\ dv,\ dt\}$. But L has two Cauchy characteristics, namely $\partial_{u'}$ and $\partial_{v'}$. Thus L can not be the

Cartan system of H, which is $\mathcal{CS}(H) = \{du',\, dv', du,\, dv,\, dt\}$ (see also Example 3.1.2 and Remark 3.1.6). ■

Before proceeding to specific examples, let's cite from [16, pages 516-518] and [7, page 46] some of the relations between the numerical invariants of a Pfaffian system.

Proposition 2.2.12. Let I be a rank-s Pfaffian system on a m-manifold. Assume that **[i]** the derived length of I is N, **[ii]** $\dim \mathrm{Cau}(I) = m - c$ and **[iii]** $p = c - s$. Then the following inequalities hold.

$$\mathrm{Eng}(I) \le \mathrm{Car}(I) \le 2\,\mathrm{Eng}(I). \tag{2.6}$$

$$\mathrm{Car}\Big(I^{(i+1)}\Big) \le \mathrm{Car}\Big(I^{(i)}\Big), \quad i = 0, \ldots, N-1. \tag{2.7}$$

$$2\,\mathrm{Eng}(I) \le p \le \mathrm{Eng}(I)\,(1 + \dim I - \dim I')\,. \tag{2.8}$$

$$\dim I' \ge s - \frac{p\,(p-1)}{2}. \tag{2.9}$$

We will be concerned with rank-2 and -3 Pfaffian systems on a 5-manifold. Therefore we give here the following application of (2.9).

Example 2.2.13. Let's consider a rank-s Pfaffian system I on 5-manifold M, with $s < 5 - 1 = 4$. Assume $\dim \mathrm{Cau}(I) = 0$, that is I has no Cauchy characteristics. Therefore, with the notation of Proposition 2.2.12, we have $c = m = 5$ and thus $p = 5 - s$.

[**Rank-**1] We have $s = 1$ and $p = 4$. From (2.9) one obtains $\dim I' \ge 1 - 6 = -5$, so that there is no constraint on $\dim I'$. *There are two derived types*, namely,

$$[1], \quad [1,0]. \tag{2.10}$$

[**Rank-**2] Here $s = 2$ and $p = 3$. From (2.9) one obtains $\dim I' \ge 2 - 3 = -1$ and, as before, there is no constraint for $\dim I'$. In this case *there are four derived types*, namely,

$$[2], \quad [2,1], \quad [2,1,0], \quad [2,0]. \tag{2.11}$$

[**Rank-**3] We have $s = 3$ and $p = 2$. This time (2.9) gives $\dim I' \geq 3 - 1 = 2$. Consequently, there exist two linearly independent 1-forms ω^1, ω^2 in I such that $\{\omega^1, \omega^2\} \subset I'$. We can conclude saying that in this case *there are five derived types*, namely,

$$[3], \quad [3,2], \quad [3,2,1], \quad [3,2,1,0], \quad [3,2,0]. \tag{2.12}$$

In particular, such Pfaffian systems can not have derived type $[3, 1, \ldots]$ nor $[3, 0]$. ■

We conclude this section with a tool that will be frequently used in the next chapters, that is, the quotient by a vector field.

Definition 2.2.14. Let M be an m-manifold. Assume $X \in \mathfrak{X}(M)$ to be non-singular and denote by Fl_t^X the local flow of X (see Remark 2.1.3). The relation $\sim^X$ on M defined by

$$p_1 \sim^X p_2 \text{ if and only if there exists } t \in (-\epsilon, \epsilon) \text{ such that } p_2 = \mathrm{Fl}_t^X(p_1),$$

is an equivalence relation. The equivalence classes $[p]_{\sim^X}$ of $\sim^X$ are called ***orbits*** of the flow of X and their set is denoted by $N = M/\sim^X$. Then on N there is a uniquely defined $(m-1)$-manifold structure such that the natural projection $\mathbf{q} : p \in M \to [p]_{\sim^X} \in N$ is a submersion (see [34, Theorem 3.18]). With this structure, N is called the ***quotient manifold*** of M by (the orbits of) X. ■

Theorem 2.2.15. *Let I be a rank-s Pfaffian system on an m-manifold M and $C \in \mathrm{Cau}(I)$. Then there is a unique rank-s Pfaffian system $\bar{I}$ on the quotient manifold N such that $\mathbf{q}^*\bar{I} = I$ (see [7, Theorem 2.2 and Corollary 2.3]).*

The Pfaffian system $\bar{I}$ of Theorem 2.2.15 is called the ***reduction*** of I by a Cauchy characteristic. Some arguments of the proof of Theorem 2.2.15 are reported in the following remark (see [7, page 31]).

Remark 2.2.16. Let I be a rank-s Pfaffian system on an m-manifold M and $C \in \mathrm{Cau}(I)$. Then on M there exists a local coframe $\theta^1, \ldots, \theta^s$, $\pi^1 \ldots, \pi^{m-s-1}$, η such that $I = \{\theta^1, \ldots, \theta^s\}$, $C \lrcorner\, \theta^i = C \lrcorner\, \pi^j = 0$, and $C \lrcorner\, \eta = 1$. Moreover, this local coframe can be chosen so that the structure equations of I are

$$d\theta^i \equiv \frac{1}{2} a^i_{jk} \pi^j \wedge \pi^k \mod I, \quad i = 1, \ldots, s, \tag{2.13}$$

where the functions a^i_{jk} are invariants of the quotient $\mathbf{q}$ by C.

By the rectifying Theorem [40, Proposition 1.53], there are local coordinates $(X^1, \ldots, X^{m-1}, X^m)$ on M such that $C = \partial_{X^m}$. Take local coordinates $(x^1, \ldots, x^{m-1})$ on the quotient manifold N, then the projection by C, $\mathbf{q} : M \to N$, is locally defined by

$$x^i = X^i, \quad i = 1, \ldots, m-1.$$

In these terms, we can say that there is a local coframe on M satisfying the structure equations (2.13), for which the functions a^i_{jk} do not depend upon X^m and the 1-form dX^m does not appear in any of the 1-forms $\theta^1, \ldots, \theta^s$, $\pi^1 \ldots, \pi^{m-s-1}$. Consequently, writing $\mathbf{q}^*\bar{\omega} = \omega$, one easily checks that the reduction $\bar{I} = \{\bar{\theta}^1, \ldots, \bar{\theta}^s\}$ of I by C satisfies the structure equations

$$d\bar{\theta}^i \equiv \frac{1}{2}\, \bar{a}^i_{jk}\, \bar{\pi}^j \wedge \bar{\pi}^k \mod I, \quad i = 1, \ldots, s. \tag{2.14}$$

Here $\bar{a}^i_{jk} = \bar{a}^i_{jk}(x^1, \ldots, x^{m-1}) = a^i_{jk}(X^1, \ldots, X^{m-1})$.

This construction can be carried on for each $C_i \in \mathrm{Cau}(I)$ (which we assume non-singular). Assume $\dim \mathrm{Cau}(I) = m - n$ to be constant on M. Then the rank-s Pfaffian system I on the m-manifold M reduces to a uniquely defined rank-s Pfaffian system $\bar{I}$ on the n-manifold N. ■

Corollary 2.2.17. *Let I be a Pfaffian system and $\bar{I}$ the reduction of I by some Cauchy characteristics of I. Then $\mathbf{DT}(\bar{I}) = \mathbf{DT}(I)$.*

Finally, combining Remark 2.2.10 and Corollary 2.2.17, we can state the following.

Corollary 2.2.18. *Let I be a rank-s Pfaffian system on an m-manifold M such that $\dim \mathrm{Cau}(I) = m - n < m$. Then there exists a submanifold $\mathbf{s} : N \to M$ such that $\dim N = n$ and $\mathbf{s}^*\mathcal{CS}(I)$ is a coframe on N. Moreover, $\bar{I} = \mathbf{s}^* I$ is the reduction of I by $\mathrm{Cau}(I)$ and $\mathrm{Cau}(\bar{I}) = 0$.*

It is customary to identify I and $\bar{I}$, or in other words to consider Pfaffian systems without Cauchy characteristics.

We can now move on to more generic examples, always providing specific examples concerning 5-dimensional manifold.

2.3 Rank-1 Pfaffian systems

In this section we want to show how the problem of finding the normal form of a rank-1 Pfaffian system is solved. This is known as the Pfaff problem. If $I = \{\alpha\}$ is a rank-1 Pfaffian system, then, in view of Proposition 2.2.8, the definitions of Engel and Cartan ranks of I can be expressed as follows.

[**Engel-rank**] $\mathrm{Eng}(\alpha) = \mathrm{Eng}(I) = r$ if and only if $(d\alpha)^{[r+1]} \wedge \alpha = 0$ and $(d\alpha)^{[r]} \wedge \alpha \neq 0$.

[**Cartan-rank**] $\mathrm{Car}(\alpha) = \mathrm{Car}(I) = v$ if and only if v is the smallest number of linearly independent 1-forms $\pi^1, \ldots, \pi^v$ such that $\pi^1 \wedge \ldots \wedge \pi^v \wedge \alpha \neq 0$ and $d\alpha \wedge \pi^1 \wedge \ldots \wedge \pi^v \wedge \alpha = 0$.

Remember that the numerical invariants are assumed constant.

Theorem 2.3.1 (Pfaff normal form)**.** *Let α be a 1-form on an m-manifold M such that* $\mathrm{Eng}(\alpha) = r$*. Then there are local coordinates $\left(x^1, \ldots, x^r, z, p_1, \ldots, p_r, w^1, ..w^{m-(2r+1)}\right)$ on M and a non-zero smooth function $a \in C^\infty(M)$ such that*

$$a\,\alpha = d\,z - \sum_{i=1}^{r} p_i\, d\,x^i \tag{2.15}$$

When the local expression of α is (2.15), then α is said to be in ***Pfaff normal form*** (see [7, page 38]).

Corollary 2.3.2. *Let I be a rank-1 Pfaffian system such that* $\mathrm{Eng}(I) = r$*. Then I admits the* ***Pfaff normal form*** $I = \{d\,z - \sum_{i=1}^{r} p_i\, d\,x^i\}$.

A ***contact form*** is a 1-form α on an odd dimensional m-manifold M, with $m = 2r + 1 \geq 3$, such that α has maximal Engel-rank, that is, $\mathrm{Eng}(\alpha) = r$. A manifold with a contact form is said to have a ***contact structure***. Here we list some lower dimensional examples of contact structures.

[$\dim M = 3$] $r = 1$, then $d\alpha \wedge \alpha \neq 0$ and there are coordinates (x, u, u_1) such that $a\,\alpha = d\,u - u_1\, d\,x$.

[$\dim M = 5$] $r = 2$, then $d\alpha \wedge d\alpha \wedge \alpha \neq 0$ and there are coordinates (x, y, z, p, q) such that $a\,\alpha = d\,z - p\,d\,x - q\,d\,y$.

[$\dim M = 7$] $n = 3$, then $d\alpha \wedge d\alpha \wedge \alpha \neq 0$ and there are coordinates $(x^1, x^2, x^3, z, p_1, p_2, p_3)$ such that $a\,\alpha = d\,z - p_1\, d\,x^1 - p_2\, d\,x^2 - p_3\, d\,x^3$.

2.4 Jet spaces, PDE, and Pfaffian systems

For this section, we cite the nice and short introduction given by Gardner [18, §3] and the one by Ivey [30], to which we refer for details.

Let $m, k > 0$ be integers, we say that a k-tuple $\mathrm{J} = (j_1, \ldots, j_k)$ such that $1 \leq j_1 < \ldots < j_k \leq m$ is a multi-index of order at most k on m-indexes. We denote by $\mathcal{A}^m_k$ the set of such multi-indexes. One can compute $|\mathcal{A}^m_k| = \binom{m}{k} = \dfrac{m!}{k!\,(m-k)!}$.

Definition 2.4.1. The ***space of k-jets with m-dimensional source and n-dimensional target*** over the real numbers is the differential manifold $J^k(\mathbb{R}^m, \mathbb{R}^n)$ with the following properties:

[i] $\dim J^k(\mathbb{R}^m, \mathbb{R}^n) = m + n\binom{m+k}{k}$.

[ii] There exist ***natural coordinates*** $(\mathbf{x}, \mathbf{z}, \mathbf{p})$ on $J^k(\mathbb{R}^m, \mathbb{R}^n)$, such that $\mathbf{x} = (x^1, \ldots, x^m)$, $\mathbf{z} = (z^1, \ldots, z^n)$, and $\mathbf{p} = (p^1_1, \ldots, p^1_m, \ldots, p^n_1, \ldots, p^n_m, p^1_{11}, \ldots, p^1_{1m}, \ldots, p^n_{\underbrace{m\ldots m}_{k-\text{times}}})$.

[iii] Let $F : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth function with $z^i = F^i(\mathbf{x})$. Let $\mathrm{J} = (j_1, \ldots, j_s) \in \mathcal{A}^m_k$ be any multi-index of order at most k on m-indices, so that $1 \leq s \leq k$ and $1 \leq j_1 \leq j_s \leq m$. Then the mapping $p^i_{j_1\ldots j_s} = \dfrac{\partial^s F^i}{\partial x^{j_1} \ldots \partial x^{j_s}}$ defines a smooth function $\mathrm{j}^s(F) : \mathbb{R}^m \to J^s(\mathbb{R}^m, \mathbb{R}^n)$ called the ***s-th jet of*** F. ■

In particular, $J^0(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times \mathbb{R}^n$ and a change of coordinates in the first $m + n$-natural coordinates of $J^k(\mathbb{R}^m, \mathbb{R}^n)$ induces a canonical change in $\mathbf{p}$.

Definition 2.4.2. The ***contact system*** on $J^k(\mathbb{R}^m, \mathbb{R}^n)$ is the Pfaffian system $\mathrm{C}^k_{m,n}$ generated by $\{\alpha \in \Omega^1(J^k(\mathbb{R}^m, \mathbb{R}^n)) \mid \mathrm{j}^k(F)^*\alpha = 0,\ \forall F \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)\}$.

By definition all k-jets are integral manifolds of the contact system $\mathrm{C}^k_{m,n}$. A basis for $\mathrm{C}^k_{m,n}$ is

$$
\begin{aligned}
&\theta^j = d\,z^j - \sum_{i=1}^m p^j_i\,d\,x^i, \quad j = 1, \ldots, n \\
&\theta^j_{\mathrm{J}} = d p^j_{\mathrm{J}} - \sum_{i=1}^m p^j_{i\sharp\mathrm{J}}\,d\,x^i, \quad
\begin{cases}
j = 1, \ldots, n, \\
1 \leq s < k, \\
\mathrm{J} = (j_1, \ldots, j_s) \in \mathcal{A}^m_k, \\
i\sharp\mathrm{J} = (j_1, \ldots, i, \ldots, j_s) \in \mathcal{A}^m_k.
\end{cases}
\end{aligned}
\tag{2.16}
$$

Therefore $\dim \mathrm{C}^k_{m,n} = n\left(1 + \sum_{s=1}^{k-1}\binom{m}{s}\right) = n\left(1 + m!\sum_{s=1}^{k-1}\frac{1}{s!\,(m-s)!}\right)$. Several examples of contact systems will be provided in the next sections.

We can now rephrase Corollary 2.3.2 in the context of contact systems.

Theorem 2.4.3 (Pfaff normal form, Jet)**.** *Let I be a rank-1 Pfaffian system on an m-manifold M. Assume $\mathbf{DT}(I) = [1, 0]$ and $\mathbf{Car}(I) = \mathbf{Eng}(I) = [r, 0]$. Then $\mathbf{Cau}(I) = [m - (2r + 1), m]$ and about every point of M there is a coordinate neighborhood U and a submersion $\phi : U \to J^1(\mathbb{R}^r, \mathbb{R})$ such that $I = \phi^*\mathrm{C}^1_{r,1}$.*

Following [1], let $\Delta = 0$ be a system of k-th order partial differential equations (PDE) in m-variables and n-unknown functions. We assume that this system is of maximal rank p, that is, $\Delta = 0$ consists of p functionally independent equations. Therefore we think of such a system as a submersion $\overline{\Delta} : J^k(\mathbb{R}^m, \mathbb{R}^n) \to \mathbb{R}^p$. The ***equation manifold*** of the PDE system is $M_\Delta = \overline{\Delta}^{-1}(0)$, which is an embedded submanifold $\iota : M_\Delta \to J^k(\mathbb{R}^m, \mathbb{R}^n)$ (by inclusion). The ***PDE system*** $\Delta = 0$ ***is canonically associated to the Pfaffian system*** $I = \iota^*\mathrm{C}^k_{m,n}$ defined on M_Δ. Usually one refers to I as to the ***contact system restricted to*** M_Δ. A ***solution*** of $\Delta = 0$ is a smooth map $F : N \to M_\Delta$ such that $F^*I = 0$ and $F^*(dx^1 \wedge \ldots \wedge dx^m) \neq 0$. It is therefore crucial to find a way of characterizing the contact systems so to be able to identify which jet space (if any) describes the solutions of a given system.

2.5 The jet space $J^1(\mathbb{R}^m, \mathbb{R}^n)$

With respect to natural coordinates $(x^1, \ldots, x^m, z, p_1, \ldots, p_m)$ on $J^1(\mathbb{R}^m, \mathbb{R})$, the expressions (2.16) give the contact system

$$\mathrm{C}^1_{m,1} = \{dz - p_1\,dx^1 - \ldots - p_m\,dx^m\}. \tag{2.17}$$

Therefore $\mathrm{C}^1_{m,1}$ is the Pfaff normal form of rank-1 Pfaffian system I with $\mathrm{Eng}(I) = m$. This was treated in Corollary 2.3.2.

Example 2.5.1. We are going to show the importance of obtaining a normal form for a Pfaffian system associated to a given PDE. Consider the single first order PDE in the plane

$$u_x + u_y = 0. \tag{2.18}$$

With natural coordinates (x, y, z, p, q) on $J^1(\mathbb{R}^2, \mathbb{R})$, we have $\mathrm{C}^1_{2,1} = \{dz - p\,dx - q\,dy\}$. On the equation manifold M we take coordinates (x, y, u, u_x) and $\mathrm{C}^1_{2,1}$ restricts to $I = \{\theta\}$, where $\theta = du - u_x\,dx + u_x\,dy = du - u_x\,(dx - dy)$. Since $d\theta = (dx - dy) \wedge du_x$, we have $d\theta \wedge \theta \neq 0$ (thus $I' = 0$) and for dimensional reasons it is $(d\theta)^{[1+1]} \wedge \theta = 0$. Applying Proposition 2.2.8 we conclude that $\mathbf{DT}(I) = [1, 0]$ and $\mathbf{Eng}(I) = [1, 0]$. Notice that $X = \partial_x + \partial_y$ is the only Cauchy characteristic of I on M, thus $\dim \mathrm{Cau}(I) = 1$ and $\mathbf{Cau}(I) = [1, 4]$. Consequently we can apply Theorem 2.4.3. Take natural coordinates (t, s, v) on $J^1(\mathbb{R}, \mathbb{R})$ and define $\phi : M \to J^1(\mathbb{R}, \mathbb{R})$ by

$$t = x - y, \quad s = u, \quad v = u_x.$$

Then $\mathrm{C}^1_{1,1} = \{ds - v\,dt\}$ and $I = \phi^* \mathrm{C}^1_{1,1}$. Therefore the solutions of (2.18) are parameterized by a smooth map $g : t \in \mathbb{R} \to s \in \mathbb{R}$, that is, $u = F(x - y)$. ■

On $J^1(\mathbb{R}^m, \mathbb{R}^n)$ use the natural coordinates $(x^1, \ldots, x^m, z^1, \ldots, z^n, p^1_1, \ldots, p^n_m)$, then (2.16) gives the contact system

$$\mathrm{C}^1_{m,n} = \{dz^j - \sum_{i=1}^{n} p^j_i\,dx^i, \quad j = 1, \ldots, m\}. \tag{2.19}$$

For instance, on $J^1(\mathbb{R}, \mathbb{R}^2)$ with natural coordinates (t, u, v, u', v') the contact system (2.19) is

$$\mathrm{C}^1_{1,2} = \{du - u'\,dt, dv - v'\,dv\}.$$

The following result is due to Robert Bryant [7, Theorem 4.4].

Theorem 2.5.2 (Bryant normal form)**.** *Let I be a rank-s Pfaffian system on an m-manifold M. If*

$$\mathbf{DT}(I) = [s, 0], \quad \mathbf{Car}(I) = \mathbf{Eng}(I) = [r, 0], \quad \mathbf{Cau}(I) = [m - (s + rs + r), m], \quad s \geq 3, \tag{2.20}$$

then about every point of M there is a coordinate neighborhood U and a submersion $\phi : U \to J^1(\mathbb{R}^r, \mathbb{R}^s)$ such that $I = \phi^* \mathrm{C}^1_{r,s}$.

When $s = 1$, actually Theorem 2.5.2 is equivalent to the Pfaff Theorem (2.17). In the case $s = 2$ and $m = 5$ we will show a counter example for which the numerical invariants of $\mathrm{C}^1_{1,2}$ are not enough to characterize $\mathrm{C}^1_{1,2}$, see Section 3.3.

2.6 The jet space $J^k(\mathbb{R}, \mathbb{R})$

This case has been classically treated for $k = 1$ (Pfaff), $k = 2$ (Engel), and $k \geq 3$ (Goursat). We already considered the Pfaff case, in the previous section. Let's take natural coordinates $(x, y_0, y_1, \dots, y_k)$ on $J^k(\mathbb{R}, \mathbb{R})$. From (2.16) the contact system is

$$\mathrm{C}^k_{1,1} = \{\theta^i = dy_{i-1} - y_i\, dx, \quad i = 1, \dots, k\}. \tag{2.21}$$

For instance when $k = 2$ we have

$$\mathrm{C}^2_{1,1} = \{\theta^1 = dy_0 - y_1\, dx, \quad \theta^2 = dy_1 - y_2\, dx\}.$$

In general $(\mathrm{C}^k_{1,1})^{(k)} = 0$ and $(\mathrm{C}^k_{1,1})^{(k-i)} = \{\theta^1, \dots, \theta^i\}$ for $i = 1, \dots, k$. Thus $\mathbf{DT}\big(\mathrm{C}^k_{1,1}\big) = [k, k-1, \dots, 2, 1, 0]$.

A famous example due to Giaro, Kumpera, and Ruiz [22], shows that the derived type alone does not characterize $\mathrm{C}^k_{1,1}$. We work it out here.

Example 2.6.1. The jet space $J^3(\mathbb{R}, \mathbb{R})$ has dimension $1 + 1\binom{1+3}{3} = 1 + 4 = 5$. The natural coordinates (x, y, y_1, y_2, y_3) in (2.21) give

$$\mathrm{C}^3_{1,1} = \{\theta^1 = dy - y_1\, dx, \theta^2 = dy_1 - y_2\, dx, \theta^3 = dy_2 - y_3\, dx\}.$$

Now, on $J^3(\mathbb{R}, \mathbb{R})$ consider the Pfaffian system

$$I = \{\eta^1 = \theta^1, \eta^2 = \theta^2, \eta^3 = y_3\, dy_2 - dx\}.$$

There is not any neighborhood of $y_3 = 0$ on which $\mathrm{C}^3_{1,1}$ pullbacks to I. The structure equations of $\mathrm{C}^3_{1,1}$ with respect to the local coframe θ^1, θ^2, θ^3, $\theta^4 = dy_3$, $\pi^1 = -dx$ are

$$\begin{aligned} d\theta^1 &= \theta^2 \wedge \pi^1, \\ d\theta^2 &= \theta^3 \wedge \pi^1, \\ d\theta^3 &= \theta^4 \wedge \pi^1. \end{aligned}$$

The structure equations of I with respect to the local coframe η^1, η^2, η^3, $\eta^4 = -dy_3$, $\pi^2 = -dy_2$ are

$$\begin{aligned} d\eta^1 &= \eta^2 \wedge (y_3\, \pi^2 + \eta^3), \\ d\eta^2 &= \eta^3 \wedge \pi^2, \\ d\eta^3 &= \eta^4 \wedge \pi^2. \end{aligned}$$

Then $I^{(1)} = \{\eta^1, \eta^2\}$, $I^{(2)} = \{\eta^1\}$, and $I^{(3)} = 0$, and thus $\mathbf{DT}(I) = [3, 2, 1, 0] = \mathbf{DT}\big(\mathrm{C}^3_{1,1}\big)$.

Let's compute the other numerical invariants, using Proposition 2.2.8. It is not too hard to compute $\mathbf{Cau}(I) = [0, 1, 2, 5] = \mathbf{Cau}\big(\mathrm{C}^3_{1,1}\big)$. From the structure equations of $\mathrm{C}^3_{1,1}$ we notice that for $i = 0, 1, 2$ one can write

$$\begin{aligned} d\theta^{3-i} &\not\equiv 0 \mod (\mathrm{C}^3_{1,1})^{(i)}, \\ d\theta^h \wedge \pi^1 &\equiv 0 \mod (\mathrm{C}^3_{1,1})^{(i)}, \quad h = 1, \ldots, 3-i. \end{aligned} \tag{2.22}$$

Therefore $\mathbf{Car}\big(\mathrm{C}^3_{1,1}\big) = [1, 1, 1, 0]$. For I we have a similar situation, that is, for $i = 0, 1, 2$

$$\begin{aligned} d\theta^{3-i} &\not\equiv 0 \mod I^{(i)}, \\ d\theta^h \wedge \tau^i &\equiv 0 \mod I^{(i)}, \quad h = 1, \ldots, 3-i, \quad \tau^0 = \tau^1 = \pi^2, \quad \tau^2 = y_3\, \pi^2 + \eta^3, \end{aligned} \tag{2.23}$$

and thus we have $\mathbf{Car}(I) = [1, 1, 1, 0] = \mathbf{Car}\big(\mathrm{C}^3_{1,1}\big)$.

Finally, for $i = 1, 2, 3$ we have $d\theta^i \wedge d\theta^i = 0$ and $d\eta^3 \wedge d\eta^3 = 0$, thus $\mathbf{Eng}(I) = \mathbf{Eng}\big(\mathrm{C}^3_{1,1}\big) = [1, 1, 1, 0]$.

Therefore we can not distinguish I and $\mathrm{C}^3_{1,1}$ by means only of their numerical invariants. To solve this problem Giaro et al. introduced the notion of *weak derived systems*. Alternatively, following Tilbury and Sastry [39], we point out a visible difference between (2.22) and (2.23). Equations (2.22) show that at every step of the derived flag of $\mathrm{C}^3_{1,1}$, the same 1-form π^1 can be used to compute the Cartan-rank. In (2.23), I does not admit such a unique 1-form. ■

Definition 2.6.2. Let I be a rank-s Pfaffian system on M_m. Assume that $\mathbf{DT}(I) = [s = s_0, s_1, \ldots, s_N]$. An ***adapted basis*** of I is a basis $\{\theta^1, \ldots, \theta^s\}$ of I such that $I^{(i)} = \{\theta^1, \ldots, \theta^{s_i}\}$ for $0 \leq i \leq N$. Obviously one has $I^{(N)} = \{0\}$ when $s_N = 0$. A local coframe $\theta^1, \ldots, \theta^s, \ldots, \theta^m$ is said an ***adapted coframe*** of I if $\{\theta^1, \ldots, \theta^s\}$ is an adapted basis of I. ■

The characterization of the contact system $\mathrm{C}^k_{1,1}$ is given in the following two theorems (see [7, Theorems 5.1 and 5.3]).

Theorem 2.6.3 (Engel normal form)**.** *Let I be a rank-2 Pfaffian system on a 4-manifold M. Suppose $\mathbf{DT}(I) = [2, 1, 0]$. Then about every point of M there is a coordinate neighborhood U and a submersion $\phi : U \to J^2(\mathbb{R}, \mathbb{R})$ such that $I = \phi^* \mathrm{C}^2_{1,1}$.*

It is instructive to remark that the Engel normal form is obtained by first writing the derived system $I' = \{\theta^1\}$ in Pfaff normal form.

Theorem 2.6.4 (Special Goursat normal form)**.** *Let $k \geq 2$ and I be a rank-k Pfaffian system on a $(k+2)$-manifold M. Assume that*

$$\mathbf{DT}(I) = [k, k-1, k-2, \ldots, 2, 1, 0], \tag{2.24}$$

and that $\{\theta^1, \ldots, \theta^k\}$ is an adapted basis of I. Furthermore, assume that there exist two linearly independent 1-forms $\theta^{k+1}, \pi \in \Omega^1(M)/I$ such that

$$d\theta^i \equiv \theta^{i+1} \wedge \pi \mod I^{(k-i)}, \quad i = 1, \ldots, k. \tag{2.25}$$

Then we have

$$\mathbf{Car}(I) = \mathbf{Eng}(I) = [1, 1, \ldots, 1, 0], \quad \mathbf{Cau}(I) = [0, 1, \ldots, k-3, k]. \tag{2.26}$$

Moreover, about every point of M there is a coordinate neighborhood U and a submersion $\phi : U \to J^k(\mathbb{R}, \mathbb{R})$ such that $I = \phi^* \mathrm{C}^k_{1,1}$.

The denomination of "special" is in Goursat [26]. For completeness, we just notice that for the basis of induction of Theorem 2.6.4, which is the Engel normal form, condition (2.24) implies the so called ***Goursat congruences*** (2.25).

We end this section by simply recalling a generalization of Theorem 2.6.4, which does not need the codimension-2 requirement. Gardner and Shadwick in [21] studied Pfaffian systems which admit a *Brunovsky normal form*, that is, which are locally a partial prolongation of $\mathrm{C}^1_{1,n}$. In Murray [33] these systems were called Pfaffian system with *special extended Goursat normal form.* The Goursat congruences (2.25) in this "extended" case consist of more than one congruence at each step, since the derived type is not assumed to be a step-1 sequence.

The special extended Goursat normal form and the characterization of the contact system on the general jet space $J^k(\mathbb{R}^m, \mathbb{R}^n)$ are beyond our subject of studies, 5-dimensional manifolds. We refer to [41] for further studies on the general jet space.

2.7 Frobenius, Bryant, and special Goursat normal forms on a 5-manifold

Let's now summarize the results so far presented for Pfaffian systems on a 5-manifold. We already noticed that rank-4 and -5 Pfaffian systems on a 5-manifold are Frobenius. If I is a rank-s Frobenius Pfaffian system on a 5-manifold, then I admits the normal form $I = \{dx^1, \dots, dx^s\}$, with $\mathbf{DT}(I) = [s]$ and $\mathbf{Cau}(I) = [5 - s]$.

Now, we apply the results exposed in this chapter to the remaining derived types listed in Example 2.2.13, and thus we obtain Table 2.1 (page 27), where we provide an adapted basis for the Pfaffian systems I.

For the derived type $[3, 2, 1, 0]$ we assume the special Goursat normal form can be obtained, according to Theorem 2.6.4.

We can not consider yet the derived types $[2, 0]$ and $[3, 2, 0]$, to which none of the previous theorems apply. In particular, *the tools developed in this chapter do not provide a characterization of* $\mathrm{C}^1_{1,2} = \{dx^2 - x^4\, dx^1, dx^3 - x^5\, dx^1\} = \{du - u'\, dt, dv - v'\, dt\}$. We will treat both derived types in the next chapter.

Table 2.1: Jet-related normal forms on a 5-manifold. F_s is a rank-s Frobenius Pfaffian system.

$\mathbf{DT}(I)$	$\mathbf{Cau}(I)$	$\mathrm{C}^k_{m,n}$	I	Reference
$[1,0]$	$[2,5]$	$C^1(\mathbb{R}^2,\mathbb{R})+F_1$	$dx^2 - x^3\,dx^1$	2.2.18 and 2.4.3
$[1,0]$	$[0,5]$	$C^1(\mathbb{R}^2,\mathbb{R})$	$dx^5 - x^3\,dx^1 - x^4\,dx^2$	2.4.3 (Pfaff)
$[2,1]$	$[1,4]$	$C^1(\mathbb{R},\mathbb{R})+F_1$	$dx^4, dx^2 - x^3\,dx^1$	2.2.18 and 2.4.3
$[2,1,0]$	$[1,2,5]$	$C^2(\mathbb{R},\mathbb{R})$	$dx^2 - x^3\,dx^1, dx^3 - x^4\,dx^1$	2.6.3 (Engel)
$[3,2]$	$[0,3]$	$C^1(\mathbb{R},\mathbb{R})+F_2$	$dx^5, dx^4, dx^3 - x^2\,dx^1$	2.2.18 and 2.4.3
$[3,2,1]$	$[0,1,4]$	$C^2(\mathbb{R},\mathbb{R})+F_1$	$dx^5, dx^2 - x^3\,dx^1, dx^3 - x^4\,dx^1$	2.2.18 and 2.6.3
$[3,2,1,0]$	$[0,1,4,5]$	$C^3(\mathbb{R},\mathbb{R})$	$dx^2 - x^3\,dx^1,\ dx^3 - x^4\,dx^1,$ $dx^4 - x^5\,dx^1$	2.6.4 (Goursat)

CHAPTER 3

RANK-2 AND -3 PFAFFIAN SYSTEMS ON A 5-MANIFOLD

In this chapter we define the main object of our study, general rank-3 (or GR_3D_5) Pfaffian systems in five variables. We will provide new proofs of some results obtained by Cartan [10] and Goursat [26]. An algorithm will be described to write any GR_3D_5 Pfaffian system in general Goursat normal form and in general Monge normal form. This algorithm will be widely used in Chapter 5.

Sections 3.2 and 3.3 are devoted to the characterization of those rank-2 Pfaffian systems on a 5-manifold to which one can not apply the Bryant normal form Theorem 2.5.2.

In Sections 3.4 through 3.6 we define and characterize GR_3D_5 Pfaffian systems.

Section 3.7 is a summary of all the normal forms of rank-2 and -3 Pfaffian systems in five variables obtained in Chapters 2 and 3.

3.1 Antiderived systems

We begin by providing an example of two inequivalent rank-2 Pfaffian systems on a 5-manifold with identical numerical invariants (see also Example 2.6.1). This will motivate the introduction of a new notion, that of an antiderived system of a Pfaffian system.

Remark 3.1.1. To construct the desired example, we first consider the *Hilbert-Cartan equation*, which is the underdetermined ODE

$$z' = y''^2. \tag{3.1}$$

With respect to the coordinates (x, y, z, y', y'') on the equation manifold M, (3.1) generates the rank-3 Pfaffian system

$$I = \begin{cases} \theta^1 = dy - y'\,dx, \\ \theta^2 = dy' - y''\,dx, \\ \theta^3 = dz - y''^2\,dx. \end{cases} \tag{3.2}$$

We shall compute the numerical invariants of I. Another basis for I, more suitable for our calculation, is given by

$$\begin{aligned}
\omega^1 &= -\theta^1 = y'\,dx - dy,\\
\omega^2 &= \frac{1}{2}\,\theta^3 - y''\,\theta^2 = \frac{1}{2}\,dz - y''\,dy' + \frac{1}{2}y''^2\,dx,\\
\omega^3 &= \theta^2 = dy' - y''\,dx.
\end{aligned} \tag{3.3}$$

We complete (3.3) to a coframe $\omega^1, \omega^2, \omega^3, \pi^1, \pi^2$ on M by setting $\pi^1 = dx$ and $\pi^2 = dy''$. The structure equations of I are then

$$\begin{aligned}
d\omega^1 &= dy' \wedge dx = (dy' - y''\,dx) \wedge dx = \omega^3 \wedge \pi^1,\\
d\omega^2 &= -dy'' \wedge dy' + y''\,dy'' \wedge dx = (dy' - y''\,dx) \wedge dy'' = \omega^3 \wedge \pi^2,\\
d\omega^3 &= -dy'' \wedge dx = \pi^1 \wedge \pi^2.
\end{aligned} \tag{3.4}$$

Therefore the derived systems are $I^{(1)} = \{\omega^1, \omega^2\}$ and $I^{(2)} = 0$, so that $\{\omega^1, \omega^2, \omega^3\}$ is an adapted basis for I (see Definition 2.6.2) and $\mathbf{DT}(I) = [3, 2, 0]$.

Now we prove that $\mathbf{Cau}(I) = [0, 0, 5]$, by using Proposition 2.2.8. Let the local basis of vector fields dual to our coframe be $\partial_{\omega^1}, \partial_{\omega^2}, \partial_{\omega^3}, \partial_{\pi^1}, \partial_{\pi^2}$. From the first equation in (3.4), we see that $\mathrm{Cau}\big(I^{(1)}\big) \subseteq \langle \partial_{\omega^2}, \partial_{\pi^2} \rangle$ but, by definition, $\partial_{\omega^2} \lrcorner\, \omega^2 = 1 \neq 0$, and thus we have $\mathrm{Cau}\big(I^{(1)}\big) \subseteq \langle \partial_{\pi^2} \rangle$. From the second equation in (3.4), we see that $\partial_{\pi^2} \lrcorner\, d\omega^2 = -\omega^3 \notin I^{(1)}$. Therefore we have $\mathrm{Cau}\big(I^{(1)}\big) = 0$. Similarly, from the first and second equations in (3.4), we see that $\mathrm{Cau}(I) \subseteq \langle \partial_{\pi^1}, \partial_{\pi^2} \rangle$. But we have $(\partial_{\pi^1} + k\,\partial_{\pi^2}) \lrcorner\, d\omega^3 = \pi^2 - k\,\pi^1 \notin I$ and therefore $\mathrm{Cau}(I) = 0$. We have thus proved that

$$\mathbf{DT}(I) = [3, 2, 0], \quad \mathbf{Cau}(I) = [0, 0, 5]. \tag{3.5}$$

Next, we calculate the Cartan and Engel ranks of I and $I^{(1)}$. Because $I \neq I^{(1)}$, the Pfaffian system I is not Frobenius and, by (2.4), we have both $\mathrm{Car}(I) > 0$ and $\mathrm{Eng}(I) > 0$. On the other hand, applying Proposition 2.2.8, the conditions $dI \wedge \pi^1 \equiv 0 \mod I$ imply $\mathrm{Car}(I) \leq 1$ and thus $\mathrm{Car}(I) = 1$. Consequently $\mathrm{Eng}(I) = 1$, since, by (2.6), we have $\mathrm{Eng}(I) \leq \mathrm{Car}(I)$.

In the same way, because $I^{(1)} \neq I^{(2)}$ and $dI^{(1)} \wedge \omega^3 = 0$, one has $\mathrm{Car}\big(I^{(1)}\big) = 1$ and thus $\mathrm{Eng}\big(I^{(1)}\big) = 1$. Then we can conclude

$$\mathbf{Car}(I) = \mathbf{Eng}(I) = [1, 1, 0]. \tag{3.6}$$

■

At this point we can turn to our example.

Example 3.1.2. From Remark 3.1.1 let's consider the first derived system of I. Set $K = I' = \{\omega^1, \omega^2\}$. From equations (3.5) and (3.6) we get

$$\mathbf{DT}(K) = [2, 0], \quad \mathbf{Cau}(K) = [0, 5], \quad \mathbf{Car}(K) = \mathbf{Eng}(K) = [1, 0]. \tag{3.7}$$

We show that equations (3.7) do not characterize the rank-2 Pfaffian system K.

On $J^1(\mathbb{R}, \mathbb{R}^2)$ with natural coordinates (t, u, v, u', v'), the (rank-2) contact system is

$$H = \mathrm{C}^1_{1,2} = \begin{cases} \eta^1 = du - u'\,dt, \\ \eta^2 = dv - v'\,dt. \end{cases} \tag{3.8}$$

Setting $\eta^3 = dt$, $\tau^1 = du'$ and $\tau^2 = dv'$, we have the coframe $\eta^1, \eta^2, \eta^3, \tau^1, \tau^2$ on $J^1(\mathbb{R}, \mathbb{R}^2)$. The structure equations of H are

$$\begin{aligned} d\eta^1 &= \eta^3 \wedge \tau^1, \\ d\eta^2 &= \eta^3 \wedge \tau^2. \end{aligned} \tag{3.9}$$

The arguments leading to (3.7) from (3.4) apply to (3.9) as well. Therefore the numerical invariants of H are the same as those of K, that is

$$\mathbf{DT}(H) = [2, 0], \quad \mathbf{Cau}(H) = [0, 5], \quad \mathbf{Car}(H) = \mathbf{Eng}(H) = [1, 0]. \tag{3.10}$$

Nevertheless we can distinguish K and H in the following way. In accordance with $\mathbf{Car}(K) = \mathbf{Car}(H) = [1, 0]$, there exist 1-forms $\omega^3 \in \Omega(M)/K$ and $\eta^3 \in \Omega(J^1(\mathbb{R}, \mathbb{R}^2))/H$ such that $dK \wedge \omega^3 \equiv 0 \mod K$ and $dH \wedge \eta^3 \equiv 0 \mod H$. Consider the Pfaffian systems obtained by adjoining each of these 1-forms to their respective rank-2 Pfaffian systems. In the case of K we obtain the rank-3

Pfaffian system I, which is not Frobenius. In the case of H we obtain the rank-3 Pfaffian system $L = \{\eta^1, \eta^2, \eta^3\}$, which is completely integrable since $d\eta^3 = 0$. These constructions are invariant under diffeomorphism and thus H and K can not be equivalent. ■

The example we just worked out motivates the construction of another invariant structure for Pfaffian systems. The following definition formalizes that given by Stormark [38, pages 35, 454].

Definition 3.1.3. ***An antiderived system*** of a Pfaffian system K is a Pfaffian system J such that

[i] $K \subseteq J$;

[ii] $dK \equiv 0 \mod J$;

[iii] if I is a Pfaffian system such that $K \subseteq I \subseteq J$ and $dK \equiv 0 \mod I$, then $I = J$;

[iv] if $\tilde{J}$ is a Pfaffian system satisfying conditions **[i]** to **[iii]** then $\dim \tilde{J} = \dim J$.

When K has a unique antiderived system, we denote it by $K^{(-1)}$. ■

Example 3.1.4. On the 5-dimensional jet space $J^1(\mathbb{R}^2, \mathbb{R})$ with natural coordinates (x, y, z, p, q) consider the Pfaffian systems

$$K = \{dz - p\,dx - q\,dy\},$$

$$J_1 = \{dz - p\,dx - q\,dy, dx, dy\} = \{dz, dy, dx\},$$

$$J_2 = \{dz - p\,dx - q\,dy, dq, dp\},$$

$$J_3 = \{dz - p\,dx - q\,dy, dx, dq\},$$

$$J_4 = \{dz - p\,dx - q\,dy, dy, dp\}.$$

It is easy to check that $\mathrm{Car}(K) = 2$ and that every J_i is an antiderived systems of K. For every i, J_i is Frobenius and $K \neq J_i{}'$. ■

Returning to Example 3.1.2, we see that K and H have respectively antiderived systems I and L. It is the goal of this section to prove that these are indeed the uniquely defined antiderived systems $K^{(-1)} = I$ and $H^{(-1)} = L$.

Proposition 3.1.5. Let K be a rank-s Pfaffian system on M and $\mathrm{Car}(K) = r$. Assume $K = \{\theta^1, \ldots, \theta^s\}$ and let $\pi^1, \ldots, \pi^r \in \Omega^1(M)/K$ be such that $d\theta^i \wedge \pi^1 \wedge \ldots \wedge \pi^r \equiv 0 \bmod K$, for $i = 1, \ldots, s$. Then the following hold.

[i] $\tilde{J} = \{\theta^1, \ldots, \theta^s, \pi^1, \ldots, \pi^r\}$ is an antiderived system of K.

[ii] If J is an antiderived system of K, then $\dim J = \dim K + \mathrm{Car}(K)$ and $K \subseteq J'$.

Proof. **[i]** By hypothesis and the Proposition 2.2.8, $\{\pi^1, \ldots, \pi^r\}$ is a minimal set of linearly independent 1-forms such that $\tilde{J} = \{\theta^1, \ldots, \theta^s, \pi^1, \ldots, \pi^r\}$ is a rank-$(s + r)$ Pfaffian system and $d\theta^i \wedge \pi^1 \wedge \ldots \wedge \pi^r \equiv 0 \bmod K$, for $i = 1, \ldots, s$.

Clearly $K \subseteq \tilde{J}$ and $dK \equiv 0 \mod \tilde{J}$. Moreover, if $K \subseteq I \subseteq \tilde{J}$ and $dK \equiv 0 \mod I$, then I has to be generated by a basis of K together with some of the π^i. By the minimality of $r = \mathrm{Car}(K)$ it follows $I = \tilde{J}$. Therefore $\tilde{J}$ is an antiderived system of K.

[ii] By the last defining condition of antiderived systems we have $\dim J = \dim \tilde{J} = s + r = \dim K + \mathrm{Car}(K)$. By definition of antiderived system we have $K \subseteq J$ and $dK \equiv 0 \bmod J$ and thus $K \subseteq J'$. $\square$

Remark 3.1.6. Example 3.1.2 shows that an antiderived system of $H = \{du - u'\,dt,\, dv - v'\,dt\}$ is the Frobenius system $L = \{du,\, dv,\, dt\}$, while $H \neq L' = L$, that is, the derived system of an antiderived system is not necessarily the original Pfaffian system. We considered H, L, and $\mathcal{CS}(H)$ in Example 2.2.11 as well. ∎

Theorem 3.1.7. *Let K be a Pfaffian system with numerical invariants*

$$\mathbf{DT}(K) = [2, 0], \quad \mathbf{Cau}(K) = [0, 5]. \tag{3.11}$$

Then K has the following properties.

[i] *K is a rank-2 Pfaffian system defined on a 5-manifold.*

[ii] $\mathbf{Car}(K) = \mathbf{Eng}(K) = [1, 0]$.

[iii] *There exist a local coframe* $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ *on* M *and a function* $a \in C^\infty(M)$ *such that* $K = \{\omega^1, \omega^2\}$, $I = \{\omega^1, \omega^2, \omega^3\}$ *is an antiderived system of* K, *and*

$$\begin{aligned} d\omega^1 &\equiv \omega^3 \wedge \omega^4 && \mod K, \\ d\omega^2 &\equiv \omega^3 \wedge \omega^5 && \mod K, \\ d\omega^3 &\equiv a\,\omega^4 \wedge \omega^5 && \mod I. \end{aligned} \tag{3.12}$$

[iv] *There is a unique rank-3 antiderived system* $K^{(-1)}$.

[v] $K^{(-1)}$ *is Frobenius if and only if about every point of* M *there is a coframe as in* **[iii]** *for which* $a = 0$. *This is equivalent to the condition* $K \neq (K^{(-1)})'$.

[vi] $K^{(-1)}$ *is not Frobenius if and only if about every point of* M *there is a coframe as in* **[iii]** *for which* $a = 1$, *that is, if and only if there exists a local coframe* $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ *on* M *such that* $K = \{\omega^1, \omega^2\}$, $K^{(-1)} = \{\omega^1, \omega^2, \omega^3\}$ *and*

$$\begin{aligned} d\omega^1 &\equiv \omega^3 \wedge \omega^4 && \mod (\omega^1, \omega^2), \\ d\omega^2 &\equiv \omega^3 \wedge \omega^5 && \mod (\omega^1, \omega^2), \\ d\omega^3 &\equiv \omega^4 \wedge \omega^5 && \mod (\omega^1, \omega^2, \omega^3). \end{aligned} \tag{3.13}$$

This is equivalent to the condition $K = (K^{(-1)})'$.

Proof. **[i]** Trivially, the Cauchy type and the derived type dictate K to be a rank-2 Pfaffian system defined on a 5-manifold M. Let $K = \{\omega^1, \omega^2\}$.

[ii] Let's prove that $\mathbf{Car}(K) = \mathbf{Eng}(K) = [1, 0]$. First, since $K' = 0$, then $\mathrm{Eng}(K') = \mathrm{Car}(K') = 0$. Now, let's use Proposition 2.2.12. We have that $s = 2$ and $\mathrm{Cau}(K) = 0$; moreover, we just proved that $m = 5$. Therefore $c = \operatorname{codim} \mathrm{Cau}(K) = 5$ and $p = c - s = 5 - 2 = 3$. Hence (2.8) becomes

$$2\,\mathrm{Eng}(K) \leq 3 \leq 3\mathrm{Eng}(K), \tag{3.14}$$

thus $\mathrm{Eng}(K) = 1$ and we therefore proved $\mathbf{Eng}(K) = [1, 0]$.

We want to prove that $\mathrm{Car}(K) = 1$. The inequality (2.6) becomes

$$1 \leq \mathrm{Car}(K) \leq 2. \tag{3.15}$$

On account of (3.15) and Proposition 2.2.8, we only need to show that there exists $\omega^3 \in \Omega^1(M)/K$ such that $d\omega^i \wedge \omega^3 \equiv 0 \mod K$ for $i = 1, 2$.

Let's take three 1-forms π^1, π^2, π^3, so that $\omega^1, \omega^2, \pi^1, \pi^2, \pi^3$ is a local coframe on M. Then one has

$$d\omega^1 \equiv \frac{1}{2} \sum_{i,j=1}^{3} a_{ij}\, \pi^i \wedge \pi^j \mod K, \tag{3.16}$$

where A $= [a_{ij}]$ is a skew-symmetric 3×3 matrix. If $A = 0$ then

$$d\omega^1 \equiv 0 \mod K, \tag{3.17}$$

which implies $\omega^1 \in K'$. But $K' = 0$, thus (3.17) is not possible and $A \neq 0$. By our regularity assumption, one must have $r = \operatorname{rank} \mathrm{A} \neq 0$ on M. A well known theorem of linear algebra states that r is even, thus $r = 2$ on M. This means that A is conjugate to the matrix $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, therefore there is a change of local coframe on M which is the identity on K and which turns (3.16) to

$$d\omega^1 \equiv \pi_1^1 \wedge \pi_1^2 \mod K, \tag{3.18}$$

where π_1^1 and π_1^2 are independent 1-forms in $\{\pi^1, \pi^2, \pi^3\}$. By similar arguments, there are independent 1-forms $\pi_2^1, \pi_2^2 \in \{\pi^1, \pi^2, \pi^3\}$ such that

$$d\omega^2 \equiv \pi_2^1 \wedge \pi_2^2 \mod K. \tag{3.19}$$

Clearly, the set of 1-forms $\{\pi_1^1, \pi_1^2, \pi_2^1, \pi_2^2\}$ is dependent. Without loss of generality we can assume $\pi_1^1 = \pi^1$, $\pi_1^2 = \pi^2$, $\pi_2^2 = \pi^3$ to be independent, and set $\pi_2^1 = b_1\, \pi^1 + b_2\, \pi^2 + b_3\, \pi^3$, so that we can rewrite (3.18) and (3.19) as

$$\begin{aligned} d\omega^1 &\equiv \pi^1 \wedge \pi^2 \mod K, \\ d\omega^2 &\equiv (b_1\, \pi^1 + b_2\, \pi^2 + b_3\, \pi^3) \wedge \pi^3 = (b_1\, \pi^1 + b_2\, \pi^2) \wedge \pi^3 \mod K. \end{aligned} \tag{3.20}$$

Again, since $K' = 0$, it follows that ${b_1}^2 + {b_2}^2 \neq 0$. By means of scaling π^3 and/or interchanging π^1

and π^2, we can assume $b_1 = 1$ and thus $d\omega^2 \equiv (\pi^1 + b_2\,\pi^2) \wedge \pi^3 \mod K$. Let's set $\omega^3 = \pi^1 + b_2\,\pi^2$, so that ω^1, ω^2, ω^3, π^2, π^3 is a local coframe on M. Equations (3.20) become

$$\begin{aligned} d\omega^1 &\equiv \omega^3 \wedge \pi^2 \mod K, \\ d\omega^2 &\equiv \omega^3 \wedge \pi^3 \mod K. \end{aligned} \tag{3.21}$$

As we claimed, there is $\omega^3 \in \Omega^1(M)/K$ such that $d\omega^i \wedge \omega^3 \equiv 0 \mod K$ for $i = 1, 2$ and therefore $\mathrm{Car}(K) \leq 1$. By (3.15) we can conclude that $\mathrm{Car}(K) = 1$ and the proof of **[ii]** is completed.

[iii] We have established in **[ii]** that there exists $\omega^3 \in \Omega^1(M)/K$ such that

$$d\omega \wedge \omega^3 \equiv 0 \mod K, \quad \text{for all } \omega \in K.$$

By Proposition 3.1.5, the rank-3 Pfaffian system $I = \{\omega^1, \omega^2, \omega^3\}$ is an antiderived system of K.

From (3.21) it follows that there is a coframe ω^1, ω^2, ω^3, ω^4, ω^5 on M such that

$$\begin{aligned} d\omega^1 &\equiv \omega^3 \wedge \omega^4 \mod K, \\ d\omega^2 &\equiv \omega^3 \wedge \omega^5 \mod K. \end{aligned} \tag{3.22}$$

For dimensional reasons we have

$$d\omega^3 \equiv a\,\omega^4 \wedge \omega^5 \mod I \tag{3.23}$$

for some $a \in C^\infty(M)$. Thus **[iii]** is proved.

[iv] We must prove that $I = \{\omega^1, \omega^2, \omega^3\}$ is the unique antiderived system of K, which we denote by $K^{(-1)}$. Using Proposition 3.1.5, let's suppose that $J = \{\omega^1, \omega^2, \pi\}$ is another antiderived system of K, so that

$$\begin{aligned} d\omega^1 \wedge \pi &\equiv 0 \mod K, \\ d\omega^2 \wedge \pi &\equiv 0 \mod K. \end{aligned} \tag{3.24}$$

We want to show that $\pi \equiv b\,\omega^3 \mod K$.

With respect to the local coframe $\omega^1, \ldots, \omega^5$ we must have $\pi \equiv b_3\,\omega^3 + b_4\,\omega^4 + b_5\,\omega^5 \mod K$, so that (3.22) and (3.24) give

$$\begin{aligned} 0 &\equiv \omega^3 \wedge \omega^4 \wedge (b_3\,\omega^3 + b_4\,\omega^4 + b_5\,\omega^5) = b_5\,\omega^3 \wedge \omega^4 \wedge \omega^5 \quad \mod K, \\ 0 &\equiv \omega^3 \wedge \omega^5 \wedge (b_3\,\omega^3 + b_4\,\omega^4 + b_5\,\omega^5) = b_4\,\omega^3 \wedge \omega^5 \wedge \omega^4 \quad \mod K. \end{aligned} \tag{3.25}$$

Since $\omega^1, \omega^2, \omega^3, \omega^4$ and ω^5 are independent, (3.25) implies $b_4 = b_5 = 0$. Therefore we have $\pi \equiv b_3\,\omega^3 \mod K$, as claimed. Thus $K^{(-1)} = I = \{\omega^1, \omega^2, \omega^3\}$ is uniquely defined.

[v] $K^{(-1)}$ is Frobenius if and only if $K^{(-1)} = (K^{(-1)})'$. From (3.12) we see that $K^{(-1)} = (K^{(-1)})'$ if and only if $d\omega^3 \equiv 0 \mod K^{(-1)}$, that is, if and only if $a = 0$ on M.

[vi] Let's assume that $K^{(-1)}$ is not Frobenius, which implies $a \neq 0$. Starting from the local coframe of **[iii]**, we can therefore define a new local coframe by replacing ω^2 with $\tilde{\omega}^2 = a\,\omega^2$ and ω^5 with $\tilde{\omega}^5 = a\,\omega^5$. Consequently $K = \{\omega^1, \tilde{\omega}^2\}$, $K^{(-1)} = \{\omega^1, \tilde{\omega}^2, \omega^3\}$ and the congruences (3.12) become

$$\begin{aligned} d\omega^1 &\equiv \omega^3 \wedge \omega^4 && \mod K, \\ d\tilde{\omega}^2 &= d(a\,\omega^2) = da \wedge \omega^2 + a\,d\omega^2 && \\ &\equiv a\,d\omega^2 \equiv a\,\omega^3 \wedge \omega^5 && \\ &\equiv \omega^3 \wedge \tilde{\omega}^5 && \mod K, \\ d\omega^3 &\equiv \omega^4 \wedge \tilde{\omega}^5 && \mod K^{(-1)}. \end{aligned} \tag{3.26}$$

We obtain the desired result dropping the tildes. □

3.2 Characterization of the contact system $\mathrm{C}^1_{1,2}$

As noticed in Section 2.7, the Bryant normal form Theorem 2.5.2 does not characterize the contact system $\mathrm{C}^1_{1,2} = \{du - u'\,dt,\ dv - v'\,dt\}$ on $J^1(\mathbb{R}, \mathbb{R}^2)$. The results proved in Section 3.1 will enable us to resolve this case.

Theorem 3.2.1 (Characterization of $\mathrm{C}^1_{1,2}$). *Let K be a Pfaffian system on M such that* $\mathbf{DT}(K) = [2, 0]$ *and* $\mathbf{Cau}(K) = [0, 5]$. *Then the antiderived system $K^{(-1)}$ is Frobenius if and only if there are* *local coordinates (T, U, V, U', V') on M such that*

$$K = \{dU - U'\,dT,\, dV - V'\,dT\}\,. \tag{3.27}$$

Equivalently, $K^{(-1)}$ is Frobenius if and only if about every point of M there exist a coordinate neighborhood W and a local diffeomorphism $\Phi : W \to J^1(\mathbb{R}, \mathbb{R}^2)$ such that $\Phi^*\mathrm{C}^1_{1,2} = K$.

Proof. By Proposition 3.1.5 the antiderived system $K^{(-1)}$ is uniquely defined.

[$\Leftarrow$] If $K = \{dU - U'\,dT,\, dV - V'\,dT\} = \Phi^*\mathrm{C}^1_{1,2}$, then by Example 3.1.2 the Pfaffian system $K^{(-1)} = \Phi^*({\mathrm{C}^1_{1,2}}^{(-1)})$ is Frobenius.

[$\Rightarrow$] By the Frobenius Theorem 2.2.9, $K^{(-1)}$ is Frobenius if and only if there are local coordinates $(X^1, \ldots, X^5)$ on M such that $K^{(-1)} = \{dX^1, dX^2, dX^3\}$.

First, we prove that $K = \{dX^2 + H^2\,dX^1,\, dX^3 + H^3\,dX^1\}$ for some $H^2, H^3 \in C^\infty(M)$. Since $K \subset \{dX^1, dX^2, dX^3\}$, then $K = \{\eta^2 = F^2_1\,dX^1 + F^2_2\,dX^2 + F^2_3\,dX^3,\, \eta^3 = F^3_1\,dX^1 + F^3_2\,dX^2 + F^3_3\,dX^3\}$ for $F^i_j \in C^\infty(M)$. Since η^2 and η^3 are independent, then we must have

$$\begin{aligned}\eta^2 \wedge \eta^3 = (F^2_1F^3_2 - F^2_2F^3_1)dX^1 \wedge dX^2 + (F^2_1F^3_3 - F^2_3F^3_1)dX^1 \wedge dX^3 \\ + (F^2_2F^3_3 - F^2_3F^3_2)dX^2 \wedge dX^3 \neq 0,\end{aligned} \tag{3.28}$$

so that the 2×3 matrix $[F^i_j]$ has maximal rank 2. By means of a change of coordinates which fixes X^4 and X^5, we may assume $F^2_2 = 1$ and $F^3_3 = 1$. Thus $\eta^2 = dX^2 + F^2_1\,dX^1 + F^2_3\,dX^3$ and $\eta^3 = dX^3 + F^3_1\,dX^1 + F^3_2\,dX^2$.

Define

$$\begin{aligned}\bar\theta^2 &= \eta^2 - F^2_3\eta^3 = (1 - F^2_3)\,dX^2 + (F^2_1 - F^2_3F^3_1)\,dX^1 = G^2_2\,dX^2 + G^2_1\,dX^1,\\ \bar\theta^3 &= \eta^3 - F^3_2\eta^2 = (1 - F^3_2)\,dX^3 + (F^3_1 - F^3_2F^2_1)\,dX^1 = G^3_3\,dX^3 + G^3_1\,dX^1,\end{aligned}$$

so that $K = \{\bar\theta^2, \bar\theta^3\}$. With the same computations as in (3.28), we have

$$\bar\theta^2 \wedge \bar\theta^3 = -G^2_2G^3_1\,dX^1 \wedge dX^2 + G^2_1G^3_3\,dX^1 \wedge dX^3 + G^2_2G^3_3\,dX^2 \wedge dX^3. \tag{3.29}$$

By (3.29), the hypothesis $K' = 0$ implies that

$$\begin{aligned} 0 \neq d\bar{\theta}^2 \wedge \bar{\theta}^2 \wedge \bar{\theta}^3 &= G^3_3 (G^2_2 \, dG^2_1 - G^2_1 \, dG^2_2) \wedge dX^1 \wedge dX^2 \wedge dX^3, \\ 0 \neq d\bar{\theta}^3 \wedge \bar{\theta}^2 \wedge \bar{\theta}^3 &= G^2_2 (G^3_3 \, dG^3_1 - G^3_1 \, dG^3_3) \wedge dX^1 \wedge dX^2 \wedge dX^3, \end{aligned} \tag{3.30}$$

therefore $G^2_2 G^3_3 \neq 0$ and, as claimed, there are $H^2, H^3 \in C^\infty(M)$ such that

$$K = \{\theta^2 = dX^2 + H^2 \, dX^1, \, \theta^3 = dX^3 + H^3 \, dX^1\}. \tag{3.31}$$

We now prove that H^2 and H^3 complete X^1, X^2, X^3 to a coordinate system. Let's rewrite (3.30) as

$$\begin{aligned} 0 \neq d\theta^2 \wedge \theta^2 \wedge \theta^3 &= dH^2 \wedge dX^1 \wedge dX^2 \wedge dX^3, \\ 0 \neq d\theta^3 \wedge \theta^2 \wedge \theta^3 &= dH^3 \wedge dX^1 \wedge dX^2 \wedge dX^3, \end{aligned} \tag{3.32}$$

from which we infer that H^2 and H^3 are not constants and each of them is functionally independent from X^1, X^2 and X^3. Suppose H^2 and H^3 are functionally dependent. Without loss in generality, we can assume that there exists $h \in C^\infty(M)$ such that $dH^2 = h \, dH^3$. Because H^2 and H^3 are not constants, $h \neq 0$ and $\theta = \theta^2 - h\,\theta^3$ is a non-zero 1-form in K. Using (3.31) we compute

$$d\theta = dH^2 \wedge dX^1 - dh \wedge \theta^3 - h \, dH^3 \wedge dX^1 \equiv (dH^2 - h \, dH^3) \wedge dX^1 \equiv 0 \mod K,$$

which implies that $\theta \in K'$, contrary to our hypothesis $K' = 0$.

This proves that H^2 and H^3 complete X^1, X^2, X^3 to a coordinate system. Define the coordinate change $\phi : M \to M$ by

$$T = X^1, \quad U = X^2, \quad V = X^3, \quad U' = -H^2, \quad V' = -H^3,$$

according to which

$$\phi^* K = \{dU - U' \, dT, \, dV - V' \, dT\}. \tag{3.33}$$

This proves (3.27). $\square$

3.3 $\mathbf{GR_2D_5}$ Pfaffian systems

In this section we obtain a local normal form for rank-2 Pfaffian systems K in five variables for which $K' = 0$. From this normal form we calculate and characterize the antiderived system $K^{(-1)}$. Then we define the notion of general rank-2 Pfaffian systems in five variables.

We start from the following lemma, see [26, §76].

Lemma 3.3.1. *Let K be a Pfaffian system on M such that $\mathbf{DT}(K) = [2, 0]$ and $\mathbf{Cau}(K) = [0, 5]$. Then there are local coordinates $\left(y^1, \ldots, y^5\right)$ on M such that*

$$dy^2 - y^3\, dy^1 \in K. \tag{3.34}$$

Proof. From Theorem 3.1.7 we know that $\mathrm{Eng}(K) = 1$ and therefore (Definition 2.2.6) there exists a 1-form $\theta \in K$ such that $\mathrm{Eng}(\theta) = 1$, that is, $\theta \wedge d\theta \wedge d\theta = 0$ and $\theta \wedge d\theta \neq 0$. By the Pfaff Theorem 2.3.1, there exist coordinates $\left(y^1, \ldots, y^5\right)$ on M such that the Pfaffian system $\{\theta\}$ has basis $\{dy^2 - y^3\, dy^1\}$, which implies $dy^2 - y^3\, dy^1 \in K$. $\square$

Remark 3.3.2. Let's recall how to obtain $\{\theta\} = \{dy^2 - y^3\, dy^1\}$ in a coordinate system $\left(y^1, \ldots, y^5\right)$, as described in [11, page 261] or [7, page 38]. This will be the first step of the Goursat Algorithm 3.4.5.

[1.1] Let $K = \{\omega^1, \omega^2\}$ as in Theorem 3.1.7 and suppose $\theta = f_1\, \omega^1 + f_2\, \omega^2 \in K$ to be such that $\mathrm{Eng}(\theta) = 1$, then

$$\begin{cases} \theta \wedge d\theta \wedge d\theta = 0, \\ \theta \wedge d\theta \neq 0, \end{cases} \tag{3.35}$$

and the functions $f_1, f_2 \in C^\infty(M)$ can not both be zero.

By interchanging ω^1 and ω^2 (and thus ω^4 and ω^5) in (3.12) if needed, we can assume that $f_1 \neq 0$. Normalizing, we may as well assume that $\theta = \omega^1 + k\, \omega^2 \in K$ for some function $k \in C^\infty(M)$.

Write $\theta = \omega^1 + k\, \omega^2$ in a coordinate system $\left(z^1, \ldots, z^5\right)$ and let $g = g\left(z^1, \ldots, z^5\right)$. Equation (3.35) assures that the system

$$\theta \wedge d\theta \wedge dg = 0 \tag{3.36}$$

admits three functionally independent solutions (see [7, page 38] for details). Call one of these solutions y^1.

[1.2] Consider the PDE system in g given by

$$\theta \wedge d y^1 \wedge d g = 0. \tag{3.37}$$

The system (3.37) admits two functionally independent solutions, one of which is the function y^1; call the other solution y^2, so that the functions y^1 and y^2 are independent solutions of (3.36).

[1.3] The algebraic equation for g given by

$$\theta \wedge \left(d y^2 - g\, d y^1\right) = 0$$

admits a unique solution y^3. Now we have three independent solutions of (3.36), namely the functions y^1, y^2, and y^3.

[1.4] Complete y^1, y^2, and y^3 to a coordinate system $\left(y^1, \ldots, y^5\right)$ on M. Writing K in these coordinates we have $K = \left\{d y^2 - y^3\, d y^1, \omega^2\right\}$. ■

Proposition 3.3.3. Let K be a Pfaffian system on M such that $\mathbf{DT}(K) = [2, 0]$ and $\mathbf{Cau}(K) = [0, 5]$. Then there are local coordinates $\left(x^1, x^2, x^3, x^4, x^5\right)$ on M and $f \in C^{\infty}(M)$ such that

$$K = \left\{d x^2 - x^3\, d x^1, \quad d x^4 - x^5\, d x^3 + f\, d x^1\right\}. \tag{3.38}$$

Proof. By Lemma 3.3.1 there are local coordinates $\left(y^1, \ldots, y^5\right)$ on M such that

$$K = \left\{\theta^1 = d y^2 - y^3\, d y^1, \omega^2\right\}.$$

Let's proceed as follows.

[2.1] In the $\left(y^1, \ldots, y^5\right)$ coordinate system we can write

$$\omega^2 = Y_0^1\, d y^1 + Y_0^2\, d y^2 + Y^3\, d y^3 + Y^4\, d y^4 + Y^5\, d y^5. \tag{3.39}$$

Define $\theta_0^2 = \omega^2 - Y_0^2\,\theta^1$, so that $K = \left\{\theta^1, \theta_0^2\right\}$ with

$$\theta_0^2 = Y^1\,dy^1 + Y^3\,dy^3 + Y^4\,dy^4 + Y^5\,dy^5. \tag{3.40}$$

[2.2] Let's prove that ${Y^4}^2 + {Y^5}^2 \neq 0$.

Assume $Y^4 = Y^5 = 0$. Then $\theta_0^2 = Y^1\,dy^1 + Y^3\,dy^3$ and $K = \left\{dy^2 - y^3\,dy^1,\, Y^1\,dy^1 + Y^3\,dy^3\right\}$. If $Y^3 = 0$, then $dy^1 \in K'$. But $K' = 0$, therefore $Y^3 \neq 0$ and $K = \left\{dy^2 - y^3\,dy^1,\, \theta_0^2 = dy^3 - Y\,dy^1\right\}$. Then

$$d\theta^1 = dy^1 \wedge dy^3 = dy^1 \wedge \theta_0^2 \equiv 0 \mod K,$$

which is again contrary to our hypothesis that $K' = 0$. Hence ${Y^4}^2 + {Y^5}^2 \neq 0$.

We can thus define $\alpha = Y^4\,dy^4 + Y^5\,dy^5 \neq 0$ so that

$$\theta_0^2 = Y^1\,dy^1 + Y^3\,dy^3 + \alpha.$$

[2.3] Let's prove that there exists $dU - V^1\,dy^1 - V^3\,dy^3 \in K$. Consider a non trivial integrating factor A of α, that is, a function $A = A\left(y^1, \dots, y^5\right) \neq 0$ such that

$$\begin{cases} \dfrac{\partial U}{\partial y^4} = AY^4, \\ \dfrac{\partial U}{\partial y^5} = AY^5, \end{cases} \tag{3.41}$$

for some function $U = U\left(y^1, \dots, y^5\right)$. Such an integrating factor A always exists. Using the notation $U_i = \dfrac{\partial U}{\partial y^i}$, we have

$$A\,\alpha = U_4\,dy^4 + U_5\,dy^5 = dU - U_1\,dy^1 - U_2\,dy^2 - U_3\,dy^3. \tag{3.42}$$

From (3.40) and (3.42) one obtains

$$\begin{aligned} A\,\theta_0^2 &= dU - \left(U_1 - AY_1\right)\,dy^1 - U_2\,dy^2 - \left(U_3 - AY_3\right)\,dy^3 \\ &= dU - \left(U_1 - AY_1 + U_2 y^3\right)\,dy^1 - \left(U_3 - AY_3\right)\,dy^3 - U_2\,\theta^1 \\ &= dU - V^1\,dy^1 - V^3\,dy^3 - U_2\,\theta^1, \end{aligned} \tag{3.43}$$

where we have set $V^1 = U_1 - AY_1 + U_2 y^3$ and $V^3 = U_3 - AY_3$. Then

$$\theta^2 = A\,\theta_0^2 + U_2\,\theta^1 = dU - V^1\,dy^1 - V^3\,dy^3, \tag{3.44}$$

and $K = \{\theta^1, \theta^2\}$.

[2.4] Here we will prove that U together with either one of V^1 or V^3 complete y^1, y^2, y^3 to a coordinate system.

By construction $A \neq 0$, thus, looking back at (3.41), we must have $U_4 = \dfrac{\partial U}{\partial y^4} \neq 0$ or $U_5 = \dfrac{\partial U}{\partial y^5} \neq 0$. In either case we have

$$dy^1 \wedge dy^2 \wedge dy^3 \wedge dU \neq 0.$$

In particular, the vector field $C = U_5\,\partial_{y^4} - U_4\,\partial_{y^5}$ is non singular. We clearly have $C \lrcorner \theta^1 = C \lrcorner \theta^2 = 0$. Moreover, from $d\theta^1 = dy^1 \wedge dy^3$, we can conclude that $C \lrcorner d\theta^1$ is the zero 1-form. Finally, let's compute

$$\begin{aligned} d\theta^2 &= -dV^1 \wedge dy^1 - dV^3 \wedge dy^3 \\ &= -dy^2 \wedge (V_2^1 dy^1 + V_2^3 dy^3) - (V_1^3 - V_3^1)\,dy^1 \wedge dy^3 \\ &\quad - dy^4 \wedge (V_4^1 dy^1 + V_4^3 dy^3) - dy^5 \wedge (V_5^1 dy^1 + V_5^3 dy^3), \end{aligned} \tag{3.45}$$

so that

$$C \lrcorner d\theta^2 = (U_4 V_5^1 - U_5 V_4^1)\,dy^1 + (U_4 V_5^3 - U_5 V_4^3)\,dy^3. \tag{3.46}$$

By hypothesis $\dim \mathrm{Cau}(K) = 0$, consequently (3.46) implies that we have $U_4 V_5^1 - U_5 V_4^1 \neq 0$ or $U_4 V_5^3 - U_5 V_4^3 \neq 0$. By means of interchanging y^1 and y^3, and consequently y^4 and y^5, we may assume that $U_4 V_5^3 - U_5 V_4^3 \neq 0$, which implies

$$dy^1 \wedge dy^2 \wedge dy^3 \wedge dU \wedge dV^3 \neq 0.$$

We can thus define a new coordinate system by

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U, \quad x^5 = V^3.$$

In these new coordinates we have

$$\theta^1 = dx^2 - x^3\,dx^1,$$

$$\theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1,$$

where f is the expression for $-V^1$ in terms of $\left(x^1, \ldots, x^5\right)$. □

Remark 3.3.4. Let's provide explicit equations for the functions U, V^1 and V^3 appearing in (3.44).

[2.3bis] Given the expression (3.39), a solution of (3.41)

$$U_4 = AY^4, \qquad U_5 = AY^5, \tag{3.47}$$

will provide the function U. From the equation

$$W\,\theta^1 + A\,\omega^2 = dU - V^1\,dy^1 - V^3\,dy^3,$$

we obtain the relations

$$W = U_2 - AY_0^2, \qquad V^3 = U_3 - AY^3, \qquad V^1 = U_1 + y^3(U_2 - AY_0^2) - AY_0^1, \tag{3.48}$$

which provide the required V^1 and V^3. ■

We can now obtain a normal form crucial to Section 3.4 and Chapter 5.

Theorem 3.3.5. *Let K be a Pfaffian system on M such that $\mathbf{DT}(K) = [2,0]$ and $\mathbf{Cau}(K) = [0,5]$. Then there are local coordinates $\left(x^1, x^2, x^3, x^4, x^5\right)$ on M and a function $f \in C^\infty(M)$ such that $K = \{\omega^1, \omega^2\}$, we have a uniquely defined antiderived system $K^{(-1)} = \{\omega^1, \omega^2, \omega^3\}$, and*

$$\omega^1 = dx^2 - x^3\,dx^1, \qquad \omega^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \qquad \omega^3 = dx^3 - f_{x^5}\,dx^1. \tag{3.49}$$

Moreover, $K^{(-1)}$ is Frobenius if and only if f is linear in x^5.

Proof. We recall from Theorem 3.1.7 that $\mathbf{DT}(K) = [2,0]$ and $\mathbf{Cau}(K) = [0,5]$ implies that $K^{(-1)}$ is unique.

By Proposition 3.3.3 there are local coordinates $\left(x^1, x^2, x^3, x^4, x^5\right)$ on M and $f \in C^\infty(M)$ such that

$$K = \begin{cases} \omega^1 = dx^2 - x^3\, dx^1, \\ \omega^2 = dx^4 - x^5\, dx^3 + f\, dx^1. \end{cases}$$

The 1-form $\omega^3 = dx^3 - f_{x^5}\, dx^1$ is not in the Pfaffian system K, since

$$\omega^1 \wedge \omega^2 \wedge \omega^3 = dx^2 \wedge dx^4 \wedge dx^3 + * \neq 0.$$

We can define a local coframe $\omega^1, \ldots, \omega^5$ on M by setting

$$\begin{aligned} \omega^4 &= -dx^1, \\ \omega^5 &= dx^5 + f_{x^3}\, dx^1. \end{aligned}$$

The structure equations of K with respect to this coframe are

$$\begin{aligned} d\omega^1 &= dx^1 \wedge dx^3 = dx^1 \wedge \omega^3 \\ &= \omega^3 \wedge \omega^4, \\ d\omega^2 &= -dx^5 \wedge dx^3 + \left(f_{x^1}\, dx^1 + f_{x^2}\, dx^2 + f_{x^3}\, dx^3 + f_{x^4}\, dx^4 + f_{x^5}\, dx^5\right) \wedge dx^1 \\ &= -dx^5 \wedge \omega^3 + \left(f_{x^2}\, dx^2 + f_{x^3}\, dx^3 + f_{x^4}\, dx^4\right) \wedge dx^1 \\ &= -\left(dx^5 + f_{x^3}\, dx^1\right) \wedge \omega^3 + f_{x^2}\, \omega^1 \wedge dx^1 + f_{x^4}\, \omega^2 \wedge dx^1 \\ &= \omega^3 \wedge \omega^5 - \left(f_{x^2}\, \omega^1 + f_{x^4}\, \omega^2\right) \wedge \omega^4 \\ &\equiv \omega^3 \wedge \omega^5 \quad \operatorname{mod} K. \end{aligned} \tag{3.50}$$

These prove that $K^{(-1)} = \{\omega^1, \omega^2, \omega^3\}$ is the antiderived system of K.

Let's compute

$$
\begin{aligned}
d\omega^3 &= dx^1 \wedge \left(f_{x^1x^5}\,dx^1 + f_{x^2x^5}\,dx^2 + f_{x^3x^5}\,dx^3 + f_{x^4x^5}\,dx^4 + f_{x^5x^5}\,dx^5\right)\\
&= f_{x^2x^5}\,dx^1 \wedge dx^2 + f_{x^3x^5}\,dx^1 \wedge dx^3 + f_{x^4x^5}\,dx^1 \wedge dx^4 + f_{x^5x^5}\,dx^1 \wedge dx^5\\
&= f_{x^2x^5}\,dx^1 \wedge \omega^2 + f_{x^3x^5}\,dx^1 \wedge \omega^3 + f_{x^4x^5}\,dx^1 \wedge (\omega^2 + x^5\,\omega^3) + f_{x^5x^5}\,dx^1 \wedge dx^5\\
&\equiv -f_{x^5x^5}\,\omega^4 \wedge \omega^5 \mod K^{(-1)}.
\end{aligned}
\tag{3.51}
$$

Equations (3.50) and (3.51) show that $K^{(-1)}$ is Frobenius if and only if $f_{x^5x^5} = 0$, that is, f is linear in x^5. □

This last theorem justifies the following.

Definition 3.3.6. A ***general rank-*2 *Pfaffian system in five variables***, or simply a GR_2D_5 Pfaffian system, is a Pfaffian system K with numerical invariants $\mathbf{DT}(K) = [2, 0]$ and $\mathbf{Cau}(K) = [0, 5]$, and whose antiderived system is not completely integrable. ■

As a consequence of the results proved in this section and Section 3.1 we obtain the following characterization of GR_2D_5 Pfaffian systems.

Remark 3.3.7. Let K be a rank-2 Pfaffian system on a 5-manifold M. The following conditions are equivalent.

[**i**] K is a GR_2D_5 Pfaffian system.

[**ii**] There exists a coframe $\omega^1, \omega^2, \omega^3, \omega^4, \omega^5$ on M such that

[**a**] $K = \{\omega^1, \omega^2\}$,

[**b**] the antiderived system $K^{(-1)} = \{\omega^1, \omega^2, \omega^3\}$ is not Frobenius,

[**c**] $d\omega^1 \equiv \omega^3 \wedge \omega^4$ and $d\omega^2 \equiv \omega^3 \wedge \omega^5 \mod K$.

[**iii**] There are local coordinates $(x^1, x^2, x^3, x^4, x^5)$ on M and $f \in C^\infty(M)$ such that

[**a**] $K = \left\{dx^2 - x^3\,dx^1,\quad dx^4 - x^5\,dx^3 + f\,dx^1\right\}$,

[**b**] $f_{x^5x^5} \neq 0$. ■

3.4 $\mathbf{GR_3D_5}$ Pfaffian systems and their general Goursat normal form

In this section we first consider an application of the results obtained in the Section 3.3. Then we define the main object of our study, that is, the notion of general rank-3 Pfaffian systems in five variables. Finally, we obtain the (local) general Goursat normal form of such systems and, at the same time, we describe an algorithm for obtaining this normal form.

Proposition 3.4.1. Let K be a GR_2D_5 Pfaffian system on M. Then the antiderived system $K^{(-1)}$ has the following properties.
[i] $K^{(-1)}$ is a rank-3 Pfaffian system defined on a 5-manifold.
[ii] $(K^{(-1)})' = K$.
[iii] $\mathbf{DT}\big(K^{(-1)}\big) = [3, 2, 0]$ and $\mathbf{Cau}\big(K^{(-1)}\big) = [0, 0, 5]$.
[iv] $\mathbf{Car}\big(K^{(-1)}\big) = \mathbf{Eng}\big(K^{(-1)}\big) = [1, 1, 0]$.

Proof. **[i]** This is a trivial consequence of Theorem 3.1.7.

[ii] According to Definition 3.3.6 $K^{(-1)}$ is not Frobenius, consequently, as shown in Example 2.2.13, we must have $\dim (K^{(-1)})' = 2$. By Proposition 3.1.5, one has $K \subseteq (K^{(-1)})'$. However $\dim K = 2$ and thus $(K^{(-1)})' = K$.

[iii] From **[ii]** and the hypothesis $K' = 0$ we have $\mathbf{DT}\big(K^{(-1)}\big) = [3, 2, 0]$. Because $K \subseteq K^{(-1)}$ and $\mathrm{Cau}(K) = 0$, we have $\mathrm{Cau}\big(K^{(-1)}\big) = 0$. Thus, again from the definition of K, we have $\mathbf{Cau}\big(K^{(-1)}\big) = [0, 0, 5]$.

[iv] This is an easy consequence of (3.13). □

Definition 3.4.2. A ***general rank-3 Pfaffian system in five variables***, or simply a GR_3D_5 Pfaffian system, is a system I defined on a 5-manifold M with numerical invariants $\mathbf{DT}(I) = [3, 2, 0]$ and $\mathrm{Cau}(I') = 0$. Note that $\mathrm{Cau}(I') = 0$ is equivalent to $\Omega(M) = \mathcal{CS}(I') = \mathcal{CS}(I)$. ■

The following will clarify the relation between GR_2D_5 and GR_3D_5 Pfaffian systems.

Proposition 3.4.3. I is a GR_3D_5 Pfaffian system if and only if I' is a GR_2D_5 Pfaffian system. Equivalently, K is a GR_2D_5 Pfaffian system if and only if $K^{(-1)}$ is a GR_3D_5 Pfaffian system.

Proof. Let's prove the first form of our statement.

[⇒] By Definition 3.4.2, we have $\mathbf{DT}(I) = [3, 2, 0]$ and thus $\mathbf{DT}(I') = [2, 0]$. Moreover, by the same definition, $\mathrm{Cau}(I') = 0$ and because I is defined on a 5-manifold and $I'' = 0$, then $\mathrm{Cau}(I'') = 5$. Thus $\mathbf{Cau}(I') = [0, 5]$. Finally, because $\dim I' < \dim I$, I is not Frobenius and therefore I' is a GR_2D_5 Pfaffian system.

[$\Leftarrow$] This is a consequence of Remark 3.3.7. $\square$

Theorem 3.4.4 (General Goursat normal form of GR_3D_5)**.** *Let I be a GR_3D_5 Pfaffian system on M. Then there are local coordinates $\left(x^1,\dots,x^5\right)$ on M and a function $f \in C^\infty(M)$ with $f_{x^5\,x^5} \neq 0$ such that*

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases} \tag{3.52}$$

An adapted coframe of I (see Definition 2.6.2) is given by

$$\begin{cases} \omega^1 = dx^2 - x^3\,dx^1, \\ \omega^2 = -f_{x^5\,x^5}\,(dx^4 - x^5\,dx^3 + f\,dx^1), \\ \omega^3 = dx^3 - f_{x^5}\,dx^1, \\ \omega^4 = -dx^1, \\ \omega^5 = -f_{x^5\,x^5}\,(dx^5 + f_{x^3}\,dx^1), \end{cases} \tag{3.53}$$

for which the structure equations of I (see (3.13)*) are*

$$\begin{aligned} d\omega^1 &\equiv \omega^3 \wedge \omega^4 \quad \operatorname{mod}(\omega^1, \omega^2), \\ d\omega^2 &\equiv \omega^3 \wedge \omega^5 \quad \operatorname{mod}(\omega^1, \omega^2), \\ d\omega^3 &\equiv \omega^4 \wedge \omega^5 \quad \operatorname{mod}(\omega^1, \omega^2, \omega^3). \end{aligned}$$

Proof. This is a consequence of Theorem 3.3.5 and Proposition 3.4.1. $\square$

The adapted coframe (3.53) is said to be a ***normalized adapted coframe*** of the GR_3D_5 Pfaffian system I.

I is said to be in ***general Goursat normal form*** when written as in (3.52). The general Goursat normal form of a GR_3D_5 Pfaffian system is of great relevance to Chapter 5. By collecting the results and proofs of Sections 3.1 and 3.3, we arrive at the following algorithm, which will be used in the next chapters.

Algorithm 3.4.5 (Goursat)**.** To obtain the general Goursat normal form

$$I = \{dx^2 - x^3\,dx^1,\, dx^4 - x^5\,dx^3 + f\,dx^1,\, dx^3 - f_{x^5}\,dx^1\}$$

of a GR_3D_5 Pfaffian system I we can proceed as follows.

[0] Write I in an adapted basis $\{\eta^1, \eta^2, \eta^3\}$, that is, a set of 1-forms such that $I = \{\eta^1, \eta^2, \eta^3\}$ and $I' = \{\eta^1, \eta^2\}$.

[1] Compute $dy^2 - y^3\,dy^1 \in I'$ in a coordinate system $\left(y^1, \ldots, y^5\right)$ (Lemma 3.3.1).

[2] Obtain a coordinate system $\left(x^1, \ldots, x^5\right)$ in which the two generators of I' are $\theta^1 = dx^2 - x^3\,dx^1$ and $\theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1$ for some function $f = f\left(x^1, \ldots, x^5\right)$ such that $f_{x^5x^5} \neq 0$ (Proposition 3.3.3).

[3] Set $\theta^3 = dx^3 - f_{x^5}\,dx^1$ to obtain $I = \left\{\theta^1, \theta^2, \theta^3\right\}$ (Theorem 3.3.5).

Example 3.4.6. As a consequence of Example 3.1.2 and Proposition 3.4.3, the Hilbert-Cartan equation $z' = y''^2$ considered in Remark 3.1.1 gives rise to a GR_3D_5 Pfaffian system, namely

$$I = \begin{cases} \eta^1 = dy - y'\,dx, \\ \eta^2 = dy' - y''\,dx, \\ \eta^3 = dz - y''^2\,dx. \end{cases} \tag{3.54}$$

Now we can start the Goursat algorithm.

[0] It was proved in Remark 3.1.1 that $I' = \{\theta^1, \theta^2\}$, where here we set

$$\begin{aligned} \theta^1 &= \eta^1 = dy - y'\,dx, \\ \theta^2 &= \eta^3 - 2y''\,\eta^2 = dz - 2y''\,dy' + y''^2\,dx. \end{aligned} \tag{3.55}$$

[1] Consider the change of coordinates

$$y^1 = x, \quad y^2 = y, \quad y^3 = y', \quad y^4 = z, \quad y^5 = y'',$$

according to which we have

$$\begin{aligned}\theta^1 &= dy^2 - y^3\,dy^1,\\ \theta^2 &= dy^4 - 2y^5\,dy^3 + {y^5}^2\,dy^1.\end{aligned} \tag{3.56}$$

[2] Here the change of coordinates described in the proof Proposition 3.3.3 is quite simple, since we only have to set

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = y^4, \quad x^5 = 2y^5.$$

In view of these new coordinates, we can define the function $f = \dfrac{{x^5}^2}{4}$, whose expression in the previous coordinates was $f = \dfrac{(2y^5)^2}{4} = {y^5}^2$. Therefore

$$I' = \begin{cases}\theta^1 = dx^2 - x^3\,dx^1,\\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1.\end{cases} \tag{3.57}$$

[3] Because $f_{x^5} = \dfrac{x^5}{2}$, we set

$$\theta^3 = dx^3 - f_{x^5}\,dx^1.$$

Thus we obtain $I = \left\{\theta^1, \theta^2, \theta^3\right\}$ in general Goursat normal form. ■

3.5 Monge equations and $\mathbf{GR_3D_5}$ Pfaffian systems

In this section we will consider second order Monge equations which give rise to GR_3D_5 Pfaffian systems. We shall show, conversely, that any GR_3D_5 Pfaffian system is locally the realization of a general Monge equation. Some terminology introduced here is taken from control theory. Motivated by [18, page 148], we give the following.

Definition 3.5.1. An ***n-th-order Monge equation*** is an underdetermined n-th-order ordinary differential equation in one independent variable x and two unknown functions $y = f(x)$ and $z = g(x)$ of the form

$$z' = H\Big(x, y, z, y', \ldots, y^{(n)}\Big). \tag{3.58}$$

An equation (3.58) is said to be a n-th-order ***general Monge equation*** if $\dfrac{\partial^2 H}{\partial {y^{(n)}}^2} \neq 0$, that is, H is an expression nonlinear in $y^{(n)}$. ■

A well known example of an autonomous second order general Monge equation is the Hilbert-Cartan equation (see for instance [13, p. 13] and [7, p. 57])

$$z' = y''^2. \tag{HC}$$

We considered this equation in Remark 3.1.1 and in Example 3.4.6. The "prime" or "dot" notations in equations like (HC) are cumbersome. Henceforth we shall use jet notation. On the split jet space $J^{n,1}(\mathbb{R}, \mathbb{R}^2)$, with natural coordinates $(X, Y, Z, Y_1, Z_1, Y_2, Y_3, \ldots, Y_n)$, the equation (3.58)

$$Z_1 = H(X, Y, Z, Y_1, \ldots, Y_n),$$

defines an hypersurface N_{n+3}, of dimension $(n+3)$. Restricting the contact ideal of $J^{n,1}(\mathbb{R}, \mathbb{R}^2)$ to N_{n+3}, we obtain the rank-$(n+1)$ Pfaffian system

$$J = \begin{cases} \mu^1 = dY - Y_1\, dX, \\ \quad\vdots \\ \mu^n = dY_{n-1} - Y_n\, dX, \\ \mu^{n+1} = dZ - H\, dX. \end{cases} \tag{3.59}$$

Example 3.5.2. Consider the linear second order Monge equation $z' = y''$. In terms of the above notation, this Monge equation can be written as

$$Z_1 = Y_2. \tag{3.60}$$

This gives rise to the rank-3 Pfaffian system

$$J = \begin{cases} \mu^1 = dY - Y_1\, dX, \\ \mu^2 = dY_1 - Y_2\, dX, \\ \mu^3 = dZ - Y_2\, dX, \end{cases} \tag{3.61}$$

on the 5-manifold $N = N_5$. Another basis for J, more suitable for our calculation, is given by

$$\begin{aligned} \theta^1 &= \mu^3 - \mu^2 = d(Z - Y_1), \\ \theta^2 &= \mu^1 = dY - Y_1\, dX, \\ \theta^3 &= \mu^2 = dY_1 - Y_2\, dX. \end{aligned} \tag{3.62}$$

Because

$$\begin{aligned} d\theta^1 &= 0, \\ d\theta^2 &= dX \wedge \theta^3, \\ d\theta^3 &= dX \wedge dY_2, \end{aligned} \tag{3.63}$$

the numerical invariants of J are $\mathbf{DT}(J) = [3, 2, 1]$ and $\mathbf{Cau}(J) = [0, 1, 4]$. Consider the change of variables

$$x^1 = X, \quad x^2 = Y, \quad x^3 = Y_1, \quad x^4 = Y_2, \quad x^5 = Z - Y_1. \tag{3.64}$$

With respect to these new variables, we have

$$J = \{dx^5, dx^2 - x^3\, dx^1, dx^3 - x^4\, dx^1\}, \tag{3.65}$$

in accordance with Table 2.1. Now, using (3.64) and (3.65), we can express the closed form general solution of (3.60) by

$$Y = f(X), \quad Z = f'(X) + k, \tag{3.66}$$

where f is any differentiable function and k any constant. ■

Example 3.5.3. Consider the second order Monge equation

$$Z_1 = Y_2 + {Y_1}^2. \tag{3.67}$$

The rank-3 Pfaffian system associated to this equation is

$$J = \begin{cases} \mu^1 = dY - Y_1\, dX, \\ \mu^2 = dY_1 - Y_2\, dX, \\ \mu^3 = dZ - (Y_2 + {Y_1}^2)\, dX, \end{cases} \tag{3.68}$$

In this case, the structure equations of J are

$$\begin{aligned} d\mu^1 &= dX \wedge \mu^2, \\ d\mu^2 &= dX \wedge dY_2, \\ d\mu^3 &= dX \wedge dY_2 + 2Y_1\, dX \wedge \mu^2. \end{aligned} \tag{3.69}$$

As usual, we define an adapted basis of J by

$$\begin{aligned} \theta^1 &= \mu^3 - \mu^2 + 2Y_1\, \mu^1, \\ \theta^2 &= -2\, \mu^1, \\ \theta^3 &= -\frac{2}{Y_2}\, \mu^2, \end{aligned} \tag{3.70}$$

with structure equations

$$\begin{aligned} d\theta^1 &\equiv dY_1 \wedge \theta^2 && \mod \theta^1, \\ d\theta^2 &\equiv dY_1 \wedge \theta^3 && \mod (\theta^1, \theta^2), \\ d\theta^3 &\equiv dY_1 \wedge d\left(\frac{2}{Y_2}\right) && \mod J. \end{aligned} \tag{3.71}$$

It is clear from (3.71) that we have $\mathbf{DT}(J) = [3, 2, 1, 0]$. Moreover, on account of (3.71), we can apply Theorem 2.6.4 to conclude that J admits the special Goursat normal form of a rank-3 Pfaffian system, that is, J is locally the contact ideal on $J^3(\mathbb{R}, \mathbb{R})$. The following local expression of (3.70) unveils this geometric structure:

$$\begin{aligned} \theta^1 &= d(Z - Y_1) + 2Y_1\, dY - {Y_1}^2\, dX \\ &= d(Z - Y_1 + 2Y_1 Y - {Y_1}^2 X) - 2(Y - Y_1 X)\, dY_1, \\ \theta^2 &= -2\, dY + 2Y_1\, dX = d(2(Y - Y_1 X)) - 2X\, dY_1, \\ \theta^3 &= d(2X) - \frac{2}{Y_2}\, dY_1. \end{aligned} \tag{3.72}$$

With respect to the new variables

$$x^1 = Y_1, \quad x^2 = Z - Y_1 + 2Y_1Y - {Y_1}^2X, \quad x^3 = 2(Y - Y_1X), \quad x^4 = 2X, \quad x^5 = \frac{2}{Y_2}, \qquad (3.73)$$

we have

$$J = \{dx^2 - x^3\,dx^1, dx^3 - x^4\,dx^1, dx^4 - x^5\,dx^1\}, \qquad (3.74)$$

as expected. As we described in Section 2.4, we can write down the solutions of (3.67) using the normal form (3.74) and the change of variables (3.73). Indeed, using the inverse of (3.73), namely

$$X = -\frac{1}{2}x_4, \quad Y = \frac{1}{2}(x_3 - x_1x_4), \quad Z = -x_2 + x_1(-\frac{1}{2}x_1x_4 + x_3), \quad Y_1 = x_1, \quad Y_2 = -\frac{2}{x_5},$$

we can write the solutions of (3.67) in the parametric form

$$X = f'', \quad Y = \bar{x}f'' - f', \quad Z = \bar{x}^2f'' - 2\bar{x}f' + 2f,$$

where $\bar{x}$ is a parameter and $f = f(\bar{x})$ is a generic function. ■

We consider the second order Monge equation

$$Z_1 = H(X, Y, Z, Y_1, Y_2). \qquad (3.75)$$

On the 5-manifold $N = N_5$, with natural coordinates (X, Y, Z, Y_1, Y_2), (3.75) determines the rank-3 Pfaffian system

$$J = \begin{cases} \mu^1 = dY - Y_1\,dX, \\ \mu^2 = dY_1 - Y_2\,dX, \\ \mu^3 = dZ - H\,dX. \end{cases} \qquad (3.76)$$

We are particularly concerned with Monge equations for which (3.76) is a GR_3D_5 Pfaffian system. To this end, our next goal is the computation of the structure equations of J.

Since we want (3.75) to be of the second order, we assume $H_{Y_2} \neq 0$, where we start using the subscript notation for the partial derivation. Then we can define a new basis for J and write

$$J = \begin{cases} \theta^1 = -\mu^1 = Y_1\, dX - dY, \\ \theta^2 = \mu^3 - H_{Y_2}\, \mu^2 = dZ - (H - Y_2 H_{Y_2})\, dX - H_{Y_2}\, dY_1, \\ \theta^3 = \mu^2 = dY_1 - Y_2\, dX. \end{cases} \tag{3.77}$$

It is easy to see that θ^1, θ^2, θ^3, dX, and dY_2 constitute a local coframe on N. Let's compute

$$\begin{aligned} d\theta^2 = &- \left[(H_Y - Y_2 H_{Y_2 Y})\, dY + (H_Z - Y_2 H_{Y_2 Z})\, dZ + (H_{Y_1} - Y_2 H_{Y_2 Y_1})\, dY_1 \right. \\ &\quad \left. - Y_2 H_{Y_2 Y_2}\, dY_2\right] \wedge dX \\ &- (H_{Y_2 X}\, dX + H_{Y_2 Y}\, dY + H_{Y_2 Z}\, dZ + H_{Y_2 Y_1}\, dY_1 + H_{Y_2 Y_2}\, dY_2) \wedge (\theta^3 + Y_2\, dX) \\ = &-(H_Y\, dY + H_Z\, dZ + H_{Y_1}\, dY_1) \wedge dX \\ &- (H_{Y_2 X}\, dX + H_{Y_2 Y}\, dY + H_{Y_2 Z}\, dZ + H_{Y_2 Y_1}\, dY_1 + H_{Y_2 Y_2}\, dY_2) \wedge \theta^3 \\ = &- \left[-H_Y\, \theta^1 + H_Z\, (\theta^2 + H_{Y_2} dY_1) + H_{Y_1}\, dY_1 - H_{Y_2 X}\, \theta^3\right] \wedge dX \\ &- \left\{H_{Y_2 Y}\, (Y_1\, dX - \theta^1) + H_{Y_2 Z}\, \left[(H - Y_2 H_{Y_2})\, dX + H_{Y_2}\, dY_1\right] \right. \\ &\quad \left. + H_{Y_2 Y_1}\, dY_1 + H_{Y_2 Y_2}\, dY_2\right\} \wedge \theta^3 \\ = &\, (H_Y\, \theta^1 - H_Z\, \theta^2) \wedge dX \\ &- \left[H_Z\, H_{Y_2} + H_{Y_1} - H_{Y_2 X} - Y_1 H_{Y_2 Y} - H_{Y_2 Z}\, (H - Y_2 H_{Y_2})\right] \theta^3 \wedge dX \\ &- \left[-H_{Y_2 Y}\, \theta^1 + (H_{Y_2 Z} H_{Y_2} + H_{Y_2 Y_1}) Y_2\, dX + H_{Y_2 Y_2}\, dY_2\right] \wedge \theta^3 \\ = &\, H_{Y_2 Y}\, \theta^1 \wedge \theta^3 + (H_Y\, \theta^1 - H_Z\, \theta^2) \wedge dX + H_{Y_2 Y_2}\, \theta^3 \wedge dY_2 \\ &- (H_Z H_{Y_2} + H_{Y_1} - H_{Y_2 X} - Y_1 H_{Y_2 Y} - H H_{Y_2 Z} - Y_2 H_{Y_2 Y_1})\, \theta^3 \wedge dX \\ = &\, H_{Y_2 Y}\, \theta^1 \wedge \theta^3 + (H_Y\, \theta^1 - H_Z\, \theta^2) \wedge dX + H_{Y_2 Y_2}\, \theta^3 \wedge dY_2 - L\, \theta^3 \wedge dX, \end{aligned}$$

where we set

$$L = H_Z H_{Y_2} + H_{Y_1} - H_{Y_2 X} - Y_1 H_{Y_2 Y} - H H_{Y_2 Z} - Y_2 H_{Y_2 Y_1}.$$

Finally, the structure equations of J are

$$\begin{aligned}
d\theta^1 &= \theta^3 \wedge dX, \\
d\theta^2 &= H_{Y_2Y}\,\theta^1 \wedge \theta^3 + (H_Y\,\theta^1 - H_Z\,\theta^2) \wedge dX + H_{Y_2Y_2}\,\theta^3 \wedge dY_2 - L\,\theta^3 \wedge dX, \\
d\theta^3 &= dX \wedge dY_2.
\end{aligned} \tag{3.78}$$

From these we see that $\theta^3 \notin J^{(1)}$ and that $J^{(1)} = \{\theta^1, \theta^2\}$.

Concerning the second derived system $J^{(2)}$, we prove the following.

Proposition 3.5.4. Let J be the rank-3 Pfaffian system defined at (3.77). Then $J^{(2)} = 0$ if and only if $H_{Y_2Y_2} \neq 0$.

Proof. First we notice that since $J^{(1)} = \left\{\theta^1, \theta^2\right\}$, one has $J^{(2)} \neq 0$ if and only if there is a pair of smooth functions k_1 and k_2 on N such that ${k_1}^2 + {k_2}^2 \neq 0$ and $k_1\,\theta^1 + k_2\,\theta^2 \in J^{(2)}$. Using (3.78) we can compute

$$\begin{aligned}
d(k_1\,\theta^1 + k_2\,\theta^2) &\equiv k_1\,d\theta^1 + k_2\,d\theta^2 \\
&\equiv (k_1 - k_2 L)\,\theta^3 \wedge dX + k_2 H_{Y_2Y_2}\,\theta^3 \wedge dY_2 \mod J^{(1)}.
\end{aligned}$$

By definition $k_1\,\theta^1 + k_2\,\theta^2 \in J^{(2)}$ if and only if $d(k_1\,\theta^1 + k_2\,\theta^2) \equiv 0 \mod J^{(1)}$ and hence

$$k_1\,\theta^1 + k_2\,\theta^2 \in J^{(2)} \Leftrightarrow k_1 = k_2 L \text{ and } k_2 H_{Y_2Y_2} = 0. \tag{3.79}$$

Let's consider each case.

[$\Leftarrow$] If $H_{Y_2Y_2} \neq 0$ and $k_1\,\theta^1 + k_2\,\theta^2 \in J^{(2)}$, then from (3.79) it follows that $k_2 = 0$ and thus $k_1 = 0$. Consequently we have $J^{(2)} = 0$.

[$\Rightarrow$] If $H_{Y_2Y_2} = 0$, plugging $k_2 = 1$ in (3.79) we would have a non zero 1-form in $J^{(2)}$, that is to say $J^{(2)} \neq 0$. □

Remark 3.5.5. Definition 3.5.1 and Proposition 3.5.4 combine to show that a second order Monge equation $Z_1 = H(X, Y, Z, Y_1, Y_2)$ gives rise to a GR_3D_5 Pfaffian system if and only if it is general, that is, $H_{Y_2Y_2} \neq 0$. ■

Remark 3.5.6. According to equations (3.78) and as shown in the proof of Theorem 3.1.7.[**vi**], the GR_3D_5 Pfaffian system J associated to a general second order Monge equation admits the normalized adapted coframe

$$\begin{aligned}
\omega^1 &= Y_1\,dX - dY,\\
\omega^2 &= \frac{1}{H_{Y_2Y_2}}\left(dZ - (H - Y_2H_{Y_2})\,dX - H_{Y_2}\,dY_1\right),\\
\omega^3 &= dY_1 - Y_2\,dX,\\
\omega^4 &= dX,\\
\omega^5 &= dY_2 + \frac{1}{H_{Y_2Y_2}}\left(H_{Y_2X} + Y_1H_{Y_2Y} + Y_2H_{Y_2Y_1} + HH_{Y_2Z} - H_ZH_{Y_2} - H_{Y_1}\right)\,dX,
\end{aligned} \tag{3.80}$$

which satisfies the structure equations (3.13). ■

The relation between Monge equations and general rank-3 Pfaffian systems in 5-dimensions is described by the following consequence of Theorem 3.4.4.

Theorem 3.5.7. *Every GR_3D_5 Pfaffian system I on a 5-manifold M is locally the realization of a general second order Monge equation $Z_1 = H(X,Y,Z,Y_1,Y_2)$. In other words, there exist local coordinates (X,Y,Z,Y_1,Y_2) on M and a function $H(X,Y,Z,Y_1,Y_2) \in C^\infty(M)$ such that*

$$I = \begin{cases} \mu^1 = dY - Y_1\,dX,\\ \mu^2 = dY_1 - Y_2\,dX,\\ \mu^3 = dZ - H\,dX. \end{cases} \tag{3.81}$$

and $H_{Y_2Y_2} \neq 0$.

We will call (3.81) the ***Monge normal form of the*** GR_3D_5 ***Pfaffian system*** I.

Proof. According to Theorem 3.4.4, we can write I in its general Goursat normal form, that is, there are local coordinates $\left(x^1,\ldots,x^5\right)$ on M such that

$$I = \begin{cases} \omega^1 = dx^2 - x^3\,dx^1,\\ \omega^2 = dx^4 - x^5\,dx^3 + f\,dx^1,\\ \omega^3 = dx^3 - f_{x^5}\,dx^1, \end{cases} \tag{3.82}$$

for some $f \in C^\infty(M)$ with $f_{x^5\,x^5} \neq 0$. Another basis of I is

$$\begin{aligned} \mu^1 &= \omega^1 = dx^2 - x^3\,dx^1, \\ \mu^2 &= \omega^3 = dx^3 - f_{x^5}\,dx^1, \\ \mu^3 &= \omega^2 + x^5\,\omega^3 = dx^4 - (x^5 f_{x^5} - f)\,dx^1. \end{aligned} \tag{3.83}$$

Define the mapping $\phi : M \to M$ by

$$X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = f_{x^5}.$$

Since the Jacobian determinant of ϕ is $f_{x^5 x^5} \neq 0$, ϕ is a local diffeomorphism. Set

$$\tilde{H} = x^5 f_{x^5} - f, \quad H = \tilde{H} \circ \phi^{-1}, \tag{3.84}$$

where the function H is just the expression of $\tilde{H}$ in the (X, Y, Z, Y_1, Y_2) coordinates. At this point, from (3.83) we see that the local expression of I is (3.81), that is, I is the realization of a second order Monge equation $Z_1 = H(X, Y, Z, Y_1, Y_2)$. Consequently, by Remark 3.5.5 one must have $H_{Y_2 Y_2} \neq 0$. This last property can also be checked by direct computation. $\square$

Motivated by Theorem 3.5.7, we can add another step to the Goursat Algorithm 3.4.5, so as to arrive at an algorithm which can be used to obtain the Monge normal form of I.

Algorithm 3.5.8 (Monge)**.** To obtain the Monge normal form

$$I = \{dY - Y_1\,dX,\, dY_1 - Y_2\,dX,\, dZ - H\,dX\},$$

of a GR_3D_5 Pfaffian system I we can proceed as follows.

[0] Write I in an adapted basis $\{\eta^1, \eta^2, \eta^3\}$, that is, a set of 1-forms such that $I = \{\eta^1, \eta^2, \eta^3\}$ and $I' = \{\eta^1, \eta^2\}$.

[1] Compute $dy^2 - y^3\,dy^1 \in I'$ in a coordinate system $\left(y^1, \ldots, y^5\right)$ (Lemma 3.3.1).

[2] Obtain a coordinate system $\left(x^1, \ldots, x^5\right)$ in which two generators of I' are $\theta^1 = dx^2 - x^3\,dx^1$ and $\theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1$ for some function $f = f\left(x^1, \ldots, x^5\right)$ such that $f_{x^5 x^5} \neq 0$ (Proposition

3.3.3).

[3] Set $\theta^3 = dx^3 - f_{x^5}\,dx^1$ to obtain $I = \left\{\theta^1, \theta^2, \theta^3\right\}$ (Theorem 3.3.5).

[4] Write I in the new basis

$$\mu^1 = \omega^1 = dx^2 - x^3\,dx^1,$$

$$\mu^2 = \omega^3 = dx^3 - f_{x^5}\,dx^1,$$

$$\mu^3 = \omega^2 + x^5\,\omega^3 = dx^4 - (x^5 f_{x^5} - f)\,dx^1.$$

Then define the local coordinates

$$X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = f_{x^5}.$$

Finally, the desired Monge equation is $Z_1 = H(X, Y, Z, Y_1, Y_2)$, where H is the expression of the function

$$\tilde{H}\left(x^1, x^2, x^3, x^4, x^5\right) = x^5 f_{x^5} - f$$

in the (X, Y, Z, Y_1, Y_2) coordinates (Theorem 3.5.7).

Example 3.5.9. On a 5-manifold M with local coordinates (a, b, c, u, v), consider the Pfaffian system locally defined by

$$I = \begin{cases} \eta^1 = db - u\,da, \\ \eta^2 = dc - v\,da - 2\sqrt{v}\,du, \\ \eta^3 = da + \dfrac{1}{\sqrt{v}}\,du, \end{cases} \tag{3.85}$$

on a neighborhood where $v > 0$. Now we start the Monge algorithm.

[0] Let's complete the basis of I to a coframe by adjoining du and dv. With respect to this coframe, the structure equations of I are

$$I = \begin{cases} d\eta^1 = da \wedge du = \eta^3 \wedge du, \\ d\eta^2 = da \wedge dv + \dfrac{1}{\sqrt{v}}\,du \wedge dv \\ \qquad = \eta^3 \wedge dv, \\ d\eta^3 = \dfrac{v}{2\sqrt{v}}\,du \wedge dv. \end{cases} \tag{3.86}$$

From (3.85) we see immediately that $I' = \{\eta^1, \eta^2\}$, hence I is already expressed in an adapted basis.

[1] Consider the change of coordinates

$$y^1 = a, \quad y^2 = b, \quad y^3 = u, \quad y^4 = c, \quad y^5 = v,$$

according to which we have

$$I = \begin{cases} \eta^1 = dy^2 - y^3\,dy^1, \\ \eta^2 = dy^4 + 2\sqrt{y^5}\,dy^3 - y^5\,dy^1, \\ \eta^3 = dy^1 + \dfrac{1}{\sqrt{y^5}}\,dy^3. \end{cases} \tag{3.87}$$

[2] Here we only have to perform the change of coordinates

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = y^4, \quad x^5 = 2\sqrt{y^5}.$$

In these new coordinates, we can define the function $f = -\dfrac{{x^5}^2}{4}$, whose expression in the previous coordinates is $f = -\dfrac{\left(2\sqrt{y^5}\right)^2}{4} = -y^5$. Therefore

$$I = \begin{cases} \theta^1 = \eta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = \eta^2 = dx^4 - x^5\,dx^3 - \dfrac{{x^5}^2}{4}\,dx^1, \\ \eta^3 = dx^1 + \dfrac{2}{x^5}\,dx^3. \end{cases} \tag{3.88}$$

[3] Because $f_{x^5} = -\dfrac{x^5}{2}$, we set $\theta^3 = dx^3 - f_{x^5}\,dx^1 = dx^3 + \dfrac{x^5}{2}\,dx^1$. Then we have $I = \left\{\theta^1, \theta^2, \theta^3\right\}$ in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 - \dfrac{{x^5}^2}{4}\,dx^1, \\ \theta^3 = dx^3 + \dfrac{x^5}{2}\,dx^1, \end{cases} \tag{3.89}$$

where $f = -\dfrac{{x^5}^2}{4}$ and $f_{x^5x^5} = -\dfrac{1}{2} \neq 0$.

[4] Consider the function

$$\tilde{H}(x^1, x^2, x^3, x^4, x^5) = x^5 f_{x^5} - f = x^5(-\frac{x^5}{2}) + \frac{{x^5}^2}{4} = -\frac{{x^5}^2}{4},$$

Write I in the new basis

$$\begin{aligned}
\mu^1 &= \omega^1 = dx^2 - x^3\,dx^1,\\
\mu^2 &= \omega^3 = dx^3 + \frac{x^5}{2}\,dx^1,\\
\mu^3 &= \omega^2 + x^5\,\omega^3 = dx^4 - \tilde{H}\,dx^1 = dx^4 + \frac{{x^5}^2}{4}\,dx^1.
\end{aligned}$$

Define the local coordinates

$$X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = f_{x^5} = -\frac{x^5}{2}.$$

The expression of $\tilde{H} = -\frac{{x^5}^2}{4}$ in the (X, Y, Z, Y_1, Y_2) coordinates is $\tilde{H} = -{Y_2}^2$. Since

$$I = \begin{cases} \mu^1 = dY - Y_1\,dX,\\ \mu^2 = dY_1 - Y_2\,dX,\\ \mu^3 = dZ - {Y_2}^2\,dX, \end{cases} \tag{3.90}$$

we conclude that I is locally the realization of the Monge equation $Z_1 = {Y_2}^2$, which is the Hilbert-Cartan equation. We can express the initial local form (3.85) of I in the form (3.90) by the change of coordinates

$$X = a, \quad Y = b, \quad Z = c, \quad Y_1 = u, \quad Y_2 = -\sqrt{v}. \tag{3.91}$$

■

3.6 Cartan tensors and particular normal forms obtained by Cartan

In this section we briefly review an invariant of GR_3D_5 Pfaffian systems which is used to partially distinguish them. Then we provide some of the few general Monge normal forms obtained by Cartan in his 1910 paper.

The equivalence method applied in [10] was first introduced by Cartan in [12]. Many works have been dedicated to its foundations and applications, among which we mention Gardner [19], Hsiao [29], and Stormark [38]. We shall provide a few details about the equivalence method in Chapter 8. Using this method, Cartan [10] proved that any GR_3D_5 Pfaffian system I has two fundamental invariants. The first one is a fourth degree homogeneous polynomial in two variables $\mathcal{F}_I(x_1, x_2)$, expressed by

$$\mathcal{F}_I = A_1{x_1}^4 + 4A_2{x_1}^3x_2 + 6A_3{x_1}^2{x_2}^2 + 4A_4x_1{x_2}^3 + A_5{x_2}^4.$$

With an abuse of language, we call $\mathcal{F}_I$ the ***Cartan 2-tensor*** of I. The other invariant is a homogeneous polynomial $\mathcal{G}_I(x_1, x_2, x_3)$ of the fourth degree in three variables such that $\mathcal{G}_I(x_1, x_2, 0) = \mathcal{F}_I$ and whose expression is

$$\begin{aligned}\mathcal{G}_I = \mathcal{F}_I &+ 4\left(B_1{x_1}^3 + 3B_2{x_1}^2x_2 + 3B_3x_1{x_2}^2 + B_4{x_2}^3\right)x_3 \\ &+ 6\left(C_1{x_1}^2 + 2C_2x_1x_2 + C_3{x_2}^2\right){x_3}^2 + 4\left(D_1x_1 + D_2x_2\right){x_3}^3 + E{x_3}^4.\end{aligned}$$

We call $\mathcal{G}_I$ the ***Cartan 3-tensor*** of I.

Explicit formulas for $\mathcal{F}_I$ and $\mathcal{G}_I$ are apparently impossible to provide, except for very special cases. Starting from Hsiao's paper, we were able to implement a Maple package, called *FiveVariables*, that computes both Cartan tensors (see Chapter 8).

If two general rank-3 Pfaffian systems I and J are equivalent, then their Cartan 2-tensors $\mathcal{F}_I$ and $\mathcal{F}_J$ have the same types of roots, that is, the ***root type*** of $\mathcal{F}_I$ is an invariant of I. However, we must remark that if $\mathcal{F}_I$ and $\mathcal{F}_J$ have the same root type, then I and J need not be equivalent (unless $\mathcal{F}_I \equiv 0$).

A GR_3D_5 Pfaffian system I must have one of the following root types:

[∞] $\mathcal{F}_I$ has infinitely many roots, that is, $\mathcal{F}_I \equiv 0$;

[4] $\mathcal{F}_I$ has a root of multiplicity four;

[3,1] $\mathcal{F}_I$ has one triple root and one simple root;

[2,2] $\mathcal{F}_I$ has two double roots;

[2,1,1] $\mathcal{F}_I$ has one double root and two simple roots;

[1,1,1,1] $\mathcal{F}_I$ has four simple roots.

We note that the root types [3,1] and [2,1,1] were treated simultaneously by Cartan [10, §VII] (Stormark did not mention the type [2,1,1]). In Chapter 6 we will provide many examples of Pfaffian systems for all the root types.

In [10] upper bounds for the dimensions of the symmetry algebra $\mathcal{S}ym$ of a general rank-3 Pfaffian system I are given. In particular, *every* GR_3D_5 *Pfaffian system has a finite dimensional symmetry algebra* and the dimension is either 14, or less than or equal to 7. In Table 3.1 (page 63) we report the general Monge normal forms and the data about $\mathcal{S}ym$ that Cartan obtained using the root type of I.

Cartan treated the root type $[2,2]$ in [10, §XI, page 172]. In this case I is determined, up to equivalences, by a constant β and $\mathcal{S}ym$ is either of dimension 5 or 6. If $\dim \mathcal{S}ym = 6$, then, for special values of β, Cartan proves that $\mathcal{S}ym$ is the direct sum of two 3-dimensional algebras, one of which is semisimple, or $\mathcal{S}ym = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$.

3.7 Summary of non-Frobenius rank-2 and -3 Pfaffian systems in five variables

Here, Table 3.2 (page 63) summarizes the normal forms for non-Frobenius rank-2 and -3 Pfaffian systems I on a 5-manifold, for which $\mathrm{Cau}(I) = 0$. An adapted basis of I is provided.

Table 3.1: General Monge normal forms in Cartan 1910 [10].

Type	$\mathcal{S}ym$	Monge normal form $Z_1 = H$
$[\infty]$	$\dim \mathcal{S}ym = 14$ and $\mathcal{S}ym \simeq \mathfrak{g}_2$	${Y_2}^2$.
$[4]$	$\mathcal{S}ym$ is solvable and either $\dim \mathcal{S}ym = 7$ or $\dim \mathcal{S}ym = 6$	$-\frac{1}{2}\left({Y_2}^2 + \frac{10}{3} k {Y_1}^2 + \left(1 + k^2 - k''\right) Y^2\right)$, where $k = k(X)$.
$[2, 2]$	$\mathcal{S}ym$ is not solvable. $\dim \mathcal{S}ym = 6$ or $\dim \mathcal{S}ym = 5$	only two for special cases when $\dim \mathcal{S}ym = 6$.
$[3, 1]$ $[2, 1, 1]$ $[1, 1, 1, 1]$	$\dim \mathcal{S}ym \leq 5$	none provided.

Table 3.2: Normal forms of non-Frobenius rank-2 and -3 Pfaffian systems on a 5-manifold. $f = f(x^1, \ldots, x^5)$ and $f_{x^5 x^5} \neq 0$.

$\mathbf{DT}(I)$	I
$[2, 0]$, $\mathrm{C}^1_{1,2}$	$\{dx^2 - x^3\, dx^1, dx^4 - x^5\, dx^1\}$
$[2, 0]$, GR_2D_5	$\{dx^2 - x^3\, dx^1, dx^4 - x^5\, dx^3 + f\, dx^1\}$
$[3, 2]$	$\{dx^5, dx^4, dx^3 - x^2\, dx^1\}$
$[3, 2, 1]$	$\{dx^5, dx^2 - x^3\, dx^1, dx^3 - x^4\, dx^1\}$
$[3, 2, 1, 0]$	$\{dx^2 - x^3\, dx^1, dx^3 - x^4\, dx^1, dx^4 - x^5\, dx^1\}$
$[3, 2, 0]$, GR_3D_5	$\{dx^2 - x^3\, dx^1, dx^4 - x^5\, dx^3 + f\, dx^1, dx^3 - f_{x^5}\, dx^1\}$

CHAPTER 4

CARTAN INTEGRATION METHOD FOR NONLINEAR INVOLUTIVE SYSTEMS OF PDE AND GENERAL GOURSAT EQUATIONS

In this chapter we shall show how the geometric study of general Monge equations is closely related to that of two different kinds of partial differential equations.

The first kind we deal with is given by the nonlinear involutive systems of two PDE in two variables and one unknown, considered in Section 4.1. An integration method of these nonlinear involutive systems will be discussed. The core of this method is the proof that the reduction by the Cauchy characteristic of the Pfaffian system I_2, generated by a nonlinear involutive system, is indeed a GR_3D_5 Pfaffian system I. Therefore the general Monge normal form of I will relate an underdetermined ordinary differential equation to the initial pair of PDE. This integration method, implicitly described by Cartan in [10], is quite effective when the PDE at hand has a 3-dimensional symmetry algebra, not containing the Cauchy characteristic, in which case we can use the general Monge normal forms that we will compute in Chapter 5. The general solution of a particular family of nonlinear involutive systems of PDE in the plane will be computed using this integration method, and some examples will be provided.

The second kind of PDE considered consists of a class of non-Monge-Ampère parabolic PDE in the plane, referred to by Cartan as Goursat equations. This kind is studied in Section 4.2.

In Section 4.3 we will analyze a bijective mapping between these two types of PDE, which is defined by means of a particular parametric representation of them.

Section 4.4 is devoted to the elaboration of a canonical procedure to associate a nonlinear involutive system to a given GR_3D_5 Pfaffian system.

4.1 Involutive systems and $\mathbf{GR_3D_5}$ Pfaffian systems

In this section we describe how, in his five-variables paper [10], Cartan resolves a problem considered by Goursat in [25, Chapter VI] and [23], namely, the integration of an involutive system

$\mathcal{S}_2$ of two scalar partial differential equations of the second order in the plane.

We will provide a convenient parametric realization of nonlinear involutive systems $\mathcal{S}_2$. The general solution of nonlinear involutive systems for which this parametrization is especially simple will be given, together with some examples.

A system of two scalar second order PDE in the plane has two independent variables x and y, and one dependent variable z. Using the standard notation $(z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = (p, q, r, s, t)$, such a system can always be written as

$$\mathcal{S}_2 \equiv \begin{cases} r = R(x, y, z, p, q, t), \\ s = S(x, y, z, p, q, t). \end{cases} \tag{4.1}$$

This system defines the rank-3 Pfaffian system

$$I_2 = \begin{cases} dz - p\,dx - q\,dy, \\ dp - R\,dx - S\,dy, \\ dq - S\,dx - t\,dy, \end{cases} \tag{4.2}$$

on the 6-manifold M_6 with local coordinates (x, y, z, p, q, t).

The general definition of involutive Pfaffian systems (with independence condition) is beyond the purpose of this section. We refer, for instance, to Griffiths and Jensen [27, page 43] or to Ivey [30, page 176]. Given for granted this general definition, we can give the following.

Definition 4.1.1. An ***involutive system of two PDE in the plane*** $\mathcal{S}_2$ is a system (4.1) which gives rise to an involutive Pfaffian system I_2 (4.2). Such a system of PDE is said to be ***linear*** if it is linear in t, that is, $R_{tt} = 0$ and $S_{tt} = 0$. ■

We content ourselves with the following characterization of involutive systems of two PDE in the plane, which is proved by Cartan [10, §III]. Here and in the following pages we will use the differential operators

$$\bar{D}_x = \partial_x + p\,\partial_z + R\,\partial_p + S\,\partial_q, \quad \bar{D}_y = \partial_y + q\,\partial_z + S\,\partial_p + t\,\partial_q. \tag{4.3}$$

defined on M_6.

Theorem 4.1.2. *Let $\mathcal{S}_2$ be a system of two PDE in the plane* (4.1)*. Let I_2 be the rank-*3 *Pfaffian system* (4.2) *defined by $\mathcal{S}_2$ on M_6. Then the following properties are equivalent.*

[i] *S_2 is involutive.*

[ii] *I_2 admits a (Frobenius) rank-*5 *antiderived system, that is, the Cartan-rank of I is* $\mathrm{Car}(I_2) = 2$ *(see [10, equation (4) at page 123]).*

[iii] $\dim \mathrm{Cau}(I_2) = 1$.

[iv] *The functions R and S satisfy the conditions*

$$R_t = S_t^2, \quad \textit{and} \quad \bar{D}_x(S) = \bar{D}_y(R) - S_t\,\bar{D}_y(S). \tag{4.4}$$

[v] *I_2 is reduced by its Cauchy characteristic*

$$C_2 = \partial_x - S_t\,\partial_y + (p - qS_t)\;\partial_z + (R - SS_t)\;\partial_p + (S - tS_t)\;\partial_q + \bar{D}_y(S)\,\partial_t \tag{4.5}$$

*to a rank-*3 *Pfaffian system on a* 5*-manifold.*

Proposition 4.1.3. Let $\mathcal{S}_2$ be an involutive system of two PDE in the plane given by (4.1). Then $\mathcal{S}_2$ is linear if and only if $S_{tt} = 0$.

Proof. The first of conditions (4.4) implies $R_{tt} = 2S_tS_{tt}$. Consequently, if $S_{tt} = 0$, then $R_{tt} = 0$ and $\mathcal{S}_2$ is linear. □

Proposition 4.1.4. Let $\mathcal{S}_2$ be a system of two PDE in the plane

$$r = R(t), \quad s = S(t). \tag{4.6}$$

Then $\mathcal{S}_2$ is involutive if and only if $R_t = {S_t}^2$.

Proof. This is just a trivial application of Theorem 4.1.2, because (4.6) reduces (4.4) to $R_t = {S_t}^2$. □

Henceforth, we will consider an involutive system of PDE in the plane (4.1). Let's examine closely the structure equations of I_2 and the consequences of Theorem 4.1.2. First, let's consider another basis for I_2, writing

$$I_2 = \begin{cases} \alpha = dz - p\,dx - q\,dy, \\ \alpha^1 = dp - R\,dx - S\,dy - S_t\,(dq - S\,dx - t\,dy) \\ \quad = dp - S_t\,dq - (R - SS_t)\,dx - (S - tS_t)\,dy, \\ \alpha^2 = dq - S\,dx - t\,dy. \end{cases} \tag{4.7}$$

Let's complete this basis of I_2 to a local coframe $\alpha, \alpha^1, \alpha^2, \pi, \sigma, \eta$ on M_6 by setting

$$\pi = dy + S_t\,dx, \quad \sigma = dt - \bar{D}_y(S)\,dx, \quad \eta = dx.$$

We readily see that the Cauchy characteristic considered in (4.5) is actually $C_2 = \partial_\eta$. Next, define the operator

$$V_y = \left[\partial_t, \bar{D}_y\right] - \bar{D}_y \circ \partial_t = \partial_t \bar{D}_y - 2\,\bar{D}_y \partial_t.$$

As shown in [10, pages 122-124], condition (4.4) implies that the structure equations of I_2 are

$$\begin{aligned} d\alpha &\equiv \alpha^2 \wedge \pi && \operatorname{mod}(\alpha, \alpha^1), \\ d\alpha^1 &\equiv \alpha^2 \wedge (V_y(S)\,\pi - S_{tt}\,\sigma) && \operatorname{mod}(\alpha, \alpha^1), \\ d\alpha^2 &\equiv \pi \wedge \sigma && \operatorname{mod}(\alpha, \alpha^1, \alpha^2). \end{aligned} \tag{4.8}$$

From (4.8) we see that ${I_2}' = \{\alpha, \alpha^1\}$ and that $\Xi = \{\alpha, \alpha^1, \alpha^2, \pi, \sigma\} = {\partial_\eta}^\perp = \mathcal{CS}(I_2)$. In particular, ∂_η is also a Cauchy characteristic of Ξ. Consequently, as we discussed in Corollary 2.2.17, I_2 is reduced by ∂_η to a rank-3 Pfaffian system I, on the quotient manifold M_5, such that $\mathbf{DT}(I_2) = \mathbf{DT}(I)$, and thus $\mathbf{DT}\big({I_2}'\big) = \mathbf{DT}(I')$.

Moreover, because of (4.8), we know that both I' and I are non-Frobenius Pfaffian systems on M_5. Applying the results of Example 2.2.13 concerning rank-3 Pfaffian systems on a 5-manifold, we conclude that

$$\mathbf{DT}(I_2) \in \{[3,2,1],\ [3,2,1,0],\ [3,2,0]\}. \tag{4.9}$$

In order to further characterize involutive systems, we now consider ${I_2}''$.

Lemma 4.1.5. *Let* I_2 *be the rank-*3 *Pfaffian system* (4.7) *associated to an involutive system* (4.1)*. Assume* $k\,\alpha + k_1\,\alpha^1 \in {I_2}'$ *is a* 1*-form. Then we have*

$$k\,\alpha + k_1\,\alpha^1 \in {I_2}'' - \{0\} \Leftrightarrow \begin{cases} k + k_1 V_y(S) = 0, \\ k_1 S_{tt} = 0, \\ k^2 + {k_1}^2 \neq 0. \end{cases} \tag{4.10}$$

Proof. Because α and α^1 are independent, $\theta = k\,\alpha + k_1\,\alpha^1$ is non-zero if and only if $k^2 + {k_1}^2 \neq 0$.

According to (4.8), we have

$$\begin{aligned} d\theta &\equiv k\,d\alpha + k_1\,d\alpha^1 \\ &\equiv \alpha^2 \wedge [(k + k_1 V_y(S))\,\pi - k_1 S_{tt}\,\sigma] \quad \mod(\alpha, \alpha^1). \end{aligned} \tag{4.11}$$

Because α^2, π, and σ are independent, we conclude, from (4.11), that (4.10) holds. □

Using Lemma 4.1.5 and the results of Chapter 3 we obtain the following two propositions.

Proposition 4.1.6 (Linear Involutive Systems)**.** Let $\mathcal{S}_2$ be an involutive system of two PDE in the plane (4.1). Let I_2 be the rank-3 Pfaffian system (4.7) defined by $\mathcal{S}_2$ on M_6. Then the following properties are equivalent.

[i] $\mathcal{S}_2$ is linear in t (see Proposition 4.1.3).

[ii] $\mathbf{DT}(I_2) \in \{[3,2,1],\ [3,2,1,0]\}$.

[iii] About every point of M_6 there are local coordinates $(X^1, \ldots, X^5, X^6)$ such that I_2 can be written in one of the following normal forms

$$I_2 = \{d X^5,\ d X^2 - X^3\, d X^1,\ d X^3 - X^4\, d X^1\}, \tag{4.12}$$

or

$$I_2 = \{d X^2 - X^3\, d X^1,\ d X^3 - X^4\, d X^1,\ d X^4 - X^5\, d X^1\}. \tag{4.13}$$

Proof. Before proving the series of implications, let's first recall, from Proposition 4.1.3, that $\mathcal{S}_2$ is linear in t if and only if $S_{tt} = 0$. Moreover, by (4.9), we know that $\mathbf{DT}(I_2) \in \{[3,2,1],\ [3,2,1,0]\}$ if and only if ${I_2}'' \neq 0$.

[i ⇒ ii] Assume $S_{tt} = 0$. Then in (4.10) we may choose $k_1 = -1$, and $k = V_y(S)$, so that $\theta \in {I_2}'' \neq 0$.

[i ⇐ ii] Assume ${I_2}'' \neq 0$. Then (4.10) holds for some $k, k_1 \in C^\infty(M_6)$ such that $k^2 + {k_1}^2 \neq 0$. If $k_1 = 0$ then, from the first equation on the right-hand-side of (4.10), one has $k = 0$, thus we must have $k_1 \neq 0$. Consequently, by the second equation on the right-hand-side of (4.10), we conclude that $S_{tt} = 0$.

[ii ⇔ iii] By Corollary 2.2.17, there are local coordinates $(X^1, \ldots, X^5, X^6)$ on M_6 such that a Cauchy characteristic of I_2 is ∂_{X^6}. Moreover, take local coordinates $(x^1, \ldots, x^5)$ on the quotient manifold M_5, then the standard projection $\mathbf{q} : M_6 \to M_5$ is locally defined by

$$x^i = X^i, \quad i = 1, \ldots, 5.$$

Now, it is enough to pullback the normal forms in Table 2.1 by $\mathbf{q}$ in order to obtain (4.12) and (4.12). The converse is trivial. □

Proposition 4.1.7 (nonlinear Involutive Systems)**.** With the same hypothesis and notation of Proposition 4.1.6, the following properties are equivalent.

[i] $\mathcal{S}_2$ is nonlinear in t (see Proposition 4.1.3).

[ii] $\mathbf{DT}(I_2) = [3,2,0]$, that is, the reduction I of I_2 is a GR_3D_5 Pfaffian system.

[iii] About every point of M_6 there are local coordinates $(X^1, \ldots, X^5, X^6)$ and a function $F = F(X^1, \ldots, X^5) \in C^\infty(M_6)$ such that I_2 can be written in the general Goursat normal form

$$I_2 = \{dX^2 - X^3\,dX^1,\ dX^4 - X^5\,dX^3 + F\,dX^1,\ dX^3 - F_{X^5}\,dX^1\}, \quad F_{X^5X^5} \neq 0. \tag{4.14}$$

[iv] About every point of M_6 there are local coordinates (X, Y, Z, Y_1, Y_2, X^6) and a function $H = H(X, Y, Z, Y_1, Y_2) \in C^\infty(M_6)$ such that I_2 can be written in the Monge normal form

$$I_2 = \{dY - Y_1\,dX,\ dY_1 - Y_2\,dX,\ dZ - H\,dX\}, \quad H_{Y_2Y_2} \neq 0. \tag{4.15}$$

[**v**] The general solution of $\mathcal{S}_2$ can be expressed in terms of an arbitrary function $F(X)$, its derivatives $F'(X)$ and $F''(X)$, and the function $G(X)$ such that

$$G'(X) = H(X, F(X), F'(X), F''(X), G(X)).$$

Proof. [**i** $\Leftrightarrow$ **ii**] This is just a consequence of Propositions 4.1.3 and 4.1.6 together with (4.9).

[**ii** $\Leftrightarrow$ **iii** $\Leftrightarrow$ **iii**] These equivalences are proved by considering the projection $\mathbf{q} : M_6 \to M_5$ defined by ∂_η and any of its sections $\mathbf{s} : M_5 \to M_6$. When I is written in one of the normal forms proved in Section 3.4, respectively the general Goursat or the general Monge, then $I_2 = \mathbf{q}^* I$ has the desired local expressions. Conversely, if I_2 admits one of the local expressions in (4.14) or (4.15), then $I = \mathbf{s}^* I_2$ is a GR_3D_5 Pfaffian system.

[**v** $\Leftrightarrow$ **iv**] Evidently one is a restatement of the other. $\square$

Remark 4.1.8. The proof of Theorem 3.1.7 showed us how to obtain a normalized adapted coframe when the structure equations are as in (4.8). Because we are assuming $S_{tt} \neq 0$, we can define the change of local coframe

$$\begin{aligned}
\sigma &\to \frac{-1}{S_{tt}}(V_y(S)\,\pi - S_{tt}\,\sigma) = dt - \frac{V_y(S)}{S_{tt}}\,dy - \left(\frac{S_t V_y(S)}{S_{tt}} + \bar{D}_y(S)\right) dx,\\
\alpha^1 &\to -\frac{1}{S_{tt}}\,\alpha^1.
\end{aligned}$$

We thus consider the local coframe

$$\begin{aligned}
\alpha &= dz - p\,dx - q\,dy,\\
\alpha^1 &= -\frac{1}{S_{tt}}\,[dp - S_t\,dq - (R - SS_t)\,dx - (S - tS_t)\,dy]\,,\\
\alpha^2 &= dq - S\,dx - t\,dy,\\
\pi &= dy + S_t\,dx,\\
\sigma &= dt - \frac{V_y(S)}{S_{tt}}\,dy - \left(\frac{S_t V_y(S)}{S_{tt}} + \bar{D}_y(S)\right) dx,\\
\eta &= dx,
\end{aligned} \tag{4.16}$$

with respect to which we have the structure equations

$$
\begin{aligned}
d\alpha &\equiv \alpha^2 \wedge \pi && \operatorname{mod}(\alpha, \alpha^1),\\
d\alpha^1 &\equiv \alpha^2 \wedge \sigma && \operatorname{mod}(\alpha, \alpha^1),\\
d\alpha^2 &\equiv \pi \wedge \sigma && \operatorname{mod} I_2,\\
d\pi &\equiv 0 && \operatorname{mod}\{\alpha, \alpha^1, \alpha^2, \pi, \sigma\},\\
d\sigma &\equiv 0 && \operatorname{mod}\{\alpha, \alpha^1, \alpha^2, \pi, \sigma\}.
\end{aligned}
\tag{4.17}
$$

These prove that (4.16) is a normalized adapted coframe of I_2. ■

The final goal of this section is to obtain an alternative parametric realization of nonlinear involutive systems $\mathcal{S}_2$. This parametrization is not only useful for examples and calculations, but it will relate nonlinear involutive systems to a special class of parabolic PDE in the plane, as we will see in Sections 4.2 and 4.3.

Theorem 4.1.9. *A system of two PDE in the plane*

$$
r = R(x, y, z, p, q, t) \quad \text{and} \quad s = S(x, y, z, p, q, t), \tag{4.18}
$$

is nonlinear and involutive if and only if it can be written in the parametric form

$$
\begin{cases}
r = -2\psi + 2\lambda\dot{\psi} - \lambda^2\ddot{\psi},\\
s = -\dot{\psi} + \lambda\ddot{\psi},\\
t = -\ddot{\psi}.
\end{cases}
\tag{4.19}
$$

Here $\psi = \psi(x, y, z, p, q, \lambda)$, $\dot{\psi} = \dfrac{\partial\psi}{\partial\lambda}$, *and* ψ *satisfies the conditions*

$$
2\left(\psi_y + q\psi_z - \left(\dot{\psi} - \lambda\ddot{\psi}\right)\psi_p - \ddot{\psi}\psi_q\right) = \dot{\psi}_x + \lambda\dot{\psi}_y + (p + \lambda q)\,\dot{\psi}_z - \left(2\psi - \lambda\dot{\psi}\right)\dot{\psi}_p - \dot{\psi}\dot{\psi}_q \tag{4.20}
$$

and

$$
\dddot{\psi} \neq 0. \tag{4.21}
$$

Before proving this theorem, we first consider some useful remarks.

Remark 4.1.10. **[i]** Eliminating λ from (4.19) one obtains (4.18).

[ii] The system (4.19) is equivalent to the equation

$$r + 2\lambda s + \lambda^2 t + 2\psi = 0, \tag{4.22}$$

together with its first two derivatives with respect to λ.

[iii] Suppose a parametric realization (4.19) is obtained in terms of x, y, z, p, q, and λ, say

$$\begin{cases} r = \rho(x, y, z, p, q, \lambda), \\ s = \sigma(x, y, z, p, q, \lambda), \\ t = \tau(x, y, z, p, q, \lambda). \end{cases}$$

Then we do not need to solve differential equations to compute ψ, since by (4.22) we have

$$\psi = -\frac{1}{2}(\rho + 2\lambda\sigma + \lambda^2\tau). \tag{4.23}$$

Proof of Theorem 4.1.9. [$\Rightarrow$] We start from (4.18) and derive (4.19). Let's recall that, by Proposition 4.1.7, the system (4.18) is assumed to satisfy the conditions

$$R_t = S_t^2, \quad S_{tt} \neq 0. \tag{4.24}$$

On account of the second condition at (4.24) ($S_{tt} \neq 0$), we can define a new coordinate system (X, Y, Z, P, Q, λ) on M_6 by

$$X = x, \quad Y = y, \quad Z = z, \quad P = p, \quad Q = q, \quad \lambda = -S_t. \tag{4.25}$$

Using the inverse of (4.25), we can express t in terms of (X, Y, Z, P, Q, λ), say $t = \tau(X, Y, Z, P, Q, \lambda)$. Then, express the functions R and S of the system (4.18) in these new variables by

$$R = \Upsilon(X, Y, Z, P, Q, \lambda), \quad S = \Sigma(X, Y, Z, P, Q, \lambda). \tag{4.26}$$

Next, let $\psi = \psi(X, Y, Z, P, Q, \lambda)$ be the function on M_6 such that

$$\tau = -\ddot{\psi} = -\frac{\partial^2 \psi}{\partial \lambda^2}\,, \quad \psi|_{\lambda=0} = -\frac{1}{2}\Upsilon|_{\lambda=0}, \quad \dot{\psi}|_{\lambda=0} = -\Sigma|_{\lambda=0}, \tag{4.27}$$

where $\psi|_{\lambda=0} = \psi(X, Y, Z, P, Q, 0)$. In particular we have

$$\frac{\partial \tau}{\partial \lambda} = -\dddot{\psi} \neq 0. \tag{4.28}$$

Applying the chain rule to the second expression in (4.26) we get

$$\lambda = -\frac{\partial S}{\partial t} = -\frac{\partial \Sigma}{\partial \lambda}\frac{\partial \lambda}{\partial t},$$

hence, from (4.28), we have $\Sigma_\lambda = \lambda\dddot{\psi}$. Integrating by parts once we end up with

$$\Sigma - \Sigma|_{\lambda=0} = \int_0^\lambda \lambda\dddot{\psi}\, d\lambda = \lambda\ddot{\psi} - \int_0^\lambda \ddot{\psi}\, d\lambda = \lambda\ddot{\psi} - \dot{\psi} + \dot{\psi}|_{\lambda=0}. \tag{4.29}$$

By the definition (4.27) of ψ, we have

$$S = \Sigma = \lambda\ddot{\psi} - \dot{\psi}. \tag{4.30}$$

Analogously, from the first of (4.24), we obtain

$$\lambda^2 = \frac{\partial R}{\partial t} = \frac{\partial \Upsilon}{\partial \lambda}\frac{\partial \lambda}{\partial t},$$

which gives $\Upsilon_\lambda = -\lambda^2\dddot{\psi}$. Integrating by parts twice we have

$$\begin{aligned}\Upsilon - \Upsilon|_{\lambda=0} &= -\int_0^\lambda \lambda^2\dddot{\psi}\, d\lambda = -\lambda^2\ddot{\psi} + 2\int_0^\lambda \lambda\ddot{\psi}\, d\lambda = -\lambda^2\ddot{\psi} + 2\lambda\dot{\psi} - 2\int_0^\lambda \dot{\psi}\, d\lambda \\ &= -\lambda^2\ddot{\psi} + 2\lambda\dot{\psi} - 2\psi + 2\psi|_{\lambda=0}.\end{aligned} \tag{4.31}$$

Using the definition of ψ once more, we have

$$R = \Upsilon = -\lambda^2\ddot{\psi} + 2\lambda\dot{\psi} - 2\psi. \tag{4.32}$$

At this point it has been shown that there exists a function $\psi = \psi(X, Y, Z, P, Q, \lambda)$ such that

$$\dddot{\psi} \neq 0,$$

while, gathering equations (4.27), (4.30), and (4.32), we can rewrite the nonlinear involutive system (4.18) as

$$\begin{cases} r = -2\psi + 2\lambda\dot{\psi} - \lambda^2\ddot{\psi}, \\ s = -\dot{\psi} + \lambda\ddot{\psi}, \\ t = -\ddot{\psi}, \end{cases}$$

which is (4.19).

It remains to derive (4.20), which we shall prove by using the Cauchy characteristic of the given involutive system. In the local coordinates (X, Y, Z, P, Q, λ) the involutive system (4.19) gives rise to the rank-3 Pfaffian system

$$I_2 = \begin{cases} \alpha = dZ - P\,dX - Q\,dY, \\ \alpha^1 = dP - (-\lambda^2\ddot{\psi} + 2\lambda\dot{\psi} - 2\psi)\,dX - (\lambda\ddot{\psi} - \dot{\psi})\,dY, \\ \alpha^2 = dQ - (\lambda\ddot{\psi} - \dot{\psi})\,dX + \ddot{\psi}\,dy. \end{cases} \tag{4.33}$$

Let's define the operators

$$\begin{aligned} \bar{D}_X &= \partial_X + P\,\partial_Z + (-2\psi + 2\lambda\dot{\psi} - \lambda^2\ddot{\psi})\,\partial_P + (\lambda\ddot{\psi} - \dot{\psi})\,\partial_Q, \\ \bar{D}_Y &= \partial_Y + Q\,\partial_Z + (\lambda\ddot{\psi} - \dot{\psi})\,\partial_P - \ddot{\psi}\,\partial_Q. \end{aligned} \tag{4.34}$$

By Theorem 4.1.2, the involutive system (4.33) has a unique Cauchy characteristic directional field, namely

$$C_2 = \partial_x - S_t\,\partial_y + (p - qS_t)\ \partial_z + (R - SS_t)\ \partial_p + (S - tS_t)\ \partial_q + \bar{D}_y(S)\,\partial_t. \tag{4.35}$$

This vector field in the local coordinates (4.25) is written as

$$C_2 = \partial_X + \lambda\,\partial_Y + (P + \lambda Q)\,\partial_Z + (\lambda\dot{\psi} - 2\psi)\,\partial_P - \dot{\psi}\,\partial_Q + \dddot{\psi}\bar{D}_Y(\lambda\ddot{\psi} - \dot{\psi})\,\partial_\lambda. \tag{4.36}$$

Imposing the condition $C_2 \in \mathrm{Cau}(I_2)$ and knowing that $\dddot{\psi} \neq 0$, we obtain the following

$$\bar{D}_X(\dot{\psi}) = \bar{D}_Y(\lambda\dot{\psi} - 2\psi), \tag{4.37}$$

or equivalently

$$2\bar{D}_Y(\psi) = \left[\bar{D}_X + \lambda\,\bar{D}_Y\right](\dot{\psi}). \tag{4.38}$$

In view of (4.25), we can use to the small case variables and denote $\psi = \psi(x, y, z, p, q, \lambda)$. With this abuse of notation we write (4.38) explicitly

$$2\left(\psi_y + q\psi_z - \left(\dot{\psi} - \lambda\ddot{\psi}\right)\psi_p - \ddot{\psi}\psi_q\right) = \dot{\psi}_x + \lambda\dot{\psi}_y + (p + \lambda q)\,\dot{\psi}_z - \left(2\psi - \lambda\dot{\psi}\right)\dot{\psi}_p - \dot{\psi}\dot{\psi}_q,$$

which is (4.20).

As a final consideration, we notice that (4.38) (or (4.20)) is equivalent to the commutativity of the vectors (4.34), that is, $[\bar{D}_X, \bar{D}_Y] = 0$.

[$\Leftarrow$] As noticed in Remark 4.1.10, (4.19) is a parametrization of a system of two scalar second-order PDE in the plane. This system is involutive according to Theorem 4.1.2 because, by means of (4.20), it admits the Cauchy characteristic (4.36). $\square$

As noticed earlier, the parametrization in Theorem 4.1.9 in many instances simplifies the integration of a nonlinear involutive system, as we shall show in a particular case at the end of this section. First, let's consider the following.

Remark 4.1.11. Let $\mathcal{S}_2$ be a nonlinear involutive system parameterized by (4.19). Then we can say that on the 6-manifold M_6 with local coordinates (x, y, z, p, q, λ), $\mathcal{S}_2$ gives rise to the rank-3 Pfaffian system

$$I_{2,\psi} = I_2 = \begin{cases} \alpha = dz - p\,dx - q\,dy, \\ \alpha^1 = dp - (-\lambda^2\ddot{\psi} + 2\lambda\dot{\psi} - 2\psi)\,dx - (\lambda\ddot{\psi} - \dot{\psi})\,dy, \\ \alpha^2 = dq - (\lambda\ddot{\psi} - \dot{\psi})\,dx + \ddot{\psi}\,dy. \end{cases} \tag{4.39}$$

By Theorem 4.1.9, $\dddot{\psi} \neq 0$ and we can define the local coframe on M_6

$$
\begin{aligned}
\alpha &= dz - p\,dx - q\,dy,\\
\alpha^1 &= \dddot{\psi}\left(dp + \lambda\,dq + \left(2\psi - \lambda\dot{\psi}\right)\,dx + \dot{\psi}\,dy\right),\\
\alpha^2 &= dq - \left(\lambda\ddot{\psi} - \dot{\psi}\right)\,dx + \ddot{\psi}\,dy,\\
\pi &= dy - \lambda\,dx\\
\sigma &= -\dddot{\psi}\left(d\lambda - \left(\psi_q - \lambda\dot{\psi}_q - 2\lambda\psi_p + \lambda^2\dot{\psi}_p\right)\,dx - \left(\dot{\psi}_q - \lambda\dot{\psi}_p\right)\,dy\right)\\
\eta &= dx.
\end{aligned}
\tag{4.40}
$$

With respect to this local coframe, $I_{2,\psi}$ satisfies the structure equations

$$
\begin{aligned}
d\alpha &\equiv \alpha^2 \wedge \pi \quad \mathrm{mod}\,(\alpha, \alpha^1),\\
d\alpha^1 &\equiv \alpha^2 \wedge \sigma \quad \mathrm{mod}\,(\alpha, \alpha^1),\\
d\alpha^2 &\equiv \pi \wedge \sigma \quad \mathrm{mod}\, I_{2,\psi}.
\end{aligned}
\tag{4.41}
$$

From these we see that $\Xi_{2,\psi} = \left\{\alpha, \alpha^1, \alpha^2, \pi, \sigma\right\}$ is a rank-5 antiderived system of $I_{2,\psi}$. Moreover, we see that

$$
\partial_\eta = \partial_x + \lambda\,\partial_y + (p + q\lambda)\,\partial_z + \left(\lambda\dot{\psi} - 2\psi\right)\,\partial_p - \dot{\psi}\,\partial_q + 2\left(\psi_q - \lambda\psi_p\right)\,\partial_\lambda,
\tag{4.42}
$$

is a Cauchy characteristic of both $I_{2,\psi}$ and $\Xi_{2,\psi}$. ∎

The following Theorem is the result of our own analysis of Cartan's [10, page 137 and §XII]. First a small remark.

Remark 4.1.12. Any function $\psi = \psi(\lambda)$ is a solution of (4.20). Indeed, when we plug such an expression, equation (4.20) becomes the tautology $0 = 0$.

Theorem 4.1.13. *Let $\mathcal{S}_2$ be a nonlinear involutive system parameterized by*

$$
\begin{cases}
r = -2\psi + 2\lambda\dot{\psi} - \lambda^2\ddot{\psi},\\
s = -\dot{\psi} + \lambda\ddot{\psi},\\
t = -\ddot{\psi},
\end{cases}
\tag{4.43}
$$

where $\psi = \psi(\lambda)$ and $\dddot{\psi} \neq 0$ (see Theorem 4.1.9). The following properties hold.

[i] *The Cauchy characteristic ∂_η (see (4.42)) defines the projection $\mathbf{q} : M_6 \to M_5$*

$$\begin{aligned}
&X = \lambda, \quad Y = (y - \lambda x)\dot{\psi} + q\lambda + p + 2x\psi, \\
&Y_1 = (y - \lambda x)\ddot{\psi} + q + x\dot{\psi}, \quad Y_2 = (y - \lambda x)\dddot{\psi}, \\
&Z = -\frac{1}{2}\left(z - x^2\psi - xp + x^2\lambda\dot{\psi} - \frac{1}{2}y^2\ddot{\psi} + xy\lambda\ddot{\psi} - \frac{1}{2}x^2\lambda^2\ddot{\psi} - qy - xy\dot{\psi}\right),
\end{aligned} \tag{4.44}$$

which reduces $I_{2,\psi}$ to the GR_3D_5 Pfaffian system generated by the general Monge equation

$$Z_1 = \frac{{Y_2}^2}{\dddot{\psi}}. \tag{4.45}$$

[ii] *The 2-dimensional integral manifold $\mathbf{s} : (\lambda, \mu) \in \mathbb{R}^2 \to M_6$ of $\mathcal{S}_2$ is given by*

$$\begin{aligned}
&\lambda = \lambda, \quad x = \mu, \quad y = \lambda\mu + \frac{\ddot{F}}{\dddot{\psi}} \\
&z = -\mu^2\psi + \left(F - \frac{\dot{\psi}\ddot{F}}{\dddot{\psi}}\right)\mu - \frac{1}{2}\left(G + \frac{\ddot{\psi}\ddot{F}^2}{\dddot{\psi}^2} - \frac{2\dot{F}\ddot{F}}{\dddot{\psi}}\right), \\
&p = (\lambda\dot{\psi} - 2\psi)\mu - \frac{1}{\dddot{\psi}}\left[\left(\dddot{\psi}\dot{F} - \ddot{\psi}\ddot{F}\right)\lambda - \dot{\psi}\ddot{F}\right] + F, \quad q = -\dot{\psi}\mu + \dot{F} - \frac{\ddot{\psi}\ddot{F}}{\dddot{\psi}},
\end{aligned} \tag{4.46}$$

where $F = F(\lambda)$ and $G = G(\lambda)$ are related by the equation $\dddot{\psi}\,\dot{G} = \ddot{F}^2$.

Before proceeding to the proof, let's first present some applications of this theorem.

Example 4.1.14 (Hilbert-Cartan)**.** By Proposition 4.1.4, the nonlinear system of PDE in the plane

$$\mathcal{S}_2 \equiv \begin{cases} r = R(t) = \dfrac{1}{3}t^3, \\ s = S(t) = \dfrac{1}{2}t^2. \end{cases} \tag{4.47}$$

is involutive, because $R_t = t^2 = {S_t}^2$. From the proof of Theorem 4.1.9, let's set $\lambda = -S_t = -t$.

To calculate ψ we substitute $t = -\lambda$ in (4.47) to obtain

$$\begin{cases} r = -\dfrac{1}{3}\lambda^3, \\ s = \dfrac{1}{2}\lambda^2, \\ t = -\lambda. \end{cases} \tag{4.48}$$

Plugging (4.48) in (4.23) we arrive at

$$\psi = -\frac{1}{2}(-\frac{1}{3}\lambda^3 + \lambda^3 - \lambda^3) = \frac{1}{6}\lambda^3, \tag{4.49}$$

so that we have $\dddot{\psi} = 1$. Applying Theorem 4.1.13, the solution of (4.47) is expressed in terms of solutions of the Monge equation

$$\dot{G} = \ddot{F}^2, \tag{4.50}$$

which is the Hilbert-Cartan equation. Specifically, from (4.46),

$$x = \mu, \quad y = \lambda\mu + \ddot{F}, \quad z = -\frac{1}{6}\mu^2\lambda^3 + \left(F - \frac{1}{2}\lambda^2\ddot{F}\right)\mu - \frac{1}{2}\left(G + \lambda\ddot{F}^2 - 2\dot{F}\ddot{F}\right), \tag{4.51}$$

is the solution of the involutive system (4.47). For instance, consider the solution of (4.50) given by $F(\lambda) = \frac{1}{12}\lambda^3$, and $G(\lambda) = \frac{1}{12}\lambda^3 + k$, where k is a constant. We have $\dot{F} = \frac{1}{4}\lambda^2$ and $\ddot{F} = \frac{1}{2}\lambda$, and thus (4.51) gives

$$y = \lambda x + \frac{1}{2}\lambda, \quad z = -\frac{1}{6}x^2\lambda^3 + \lambda^3(\frac{1}{12} - \frac{1}{4})x - \frac{1}{2}\left(\frac{1}{12}\lambda^3 + k + \frac{1}{4}\lambda^3 - \frac{1}{4}\lambda^3\right) = -\frac{\lambda^3(2x+1)^2}{24} - \frac{k}{2}.$$

On a neighborhood of $x \neq -\frac{1}{2}$, we find $\lambda = \frac{2y}{2x+1}$ and thus we obtain

$$z = -\frac{y^3}{3(2x+1)} - \frac{k}{2}, \tag{4.52}$$

which is a closed form solution of the involutive system (4.47). ■

Example 4.1.15 (Goursat-Cartan)**.** Assume $m \notin \left\{0, \frac{1}{2}, 1\right\}$ and consider the system

$$\mathcal{S}_2 \equiv \begin{cases} r = R(t) = \dfrac{t^{2m-1}}{2m-1}, \\ s = S(t) = \dfrac{(-1)^m t^m}{m}. \end{cases} \tag{4.53}$$

Because $R_t = t^{2(m-1)} = {S_t}^2$, we can apply Proposition 4.1.4, thus the system (4.53) is involutive. In this case we can set

$$\lambda = -S_t = -(-1)^m t^{m-1} = (-1)^{m-1}t^{m-1}, \tag{4.54}$$

thus we can express $t = -\lambda^{1/(m-1)}$ and we obtain the following parametric expression of (4.53)

$$\begin{cases} r = -\dfrac{\lambda^{(2m-1)/(m-1)}}{2m-1}, \\ s = \dfrac{\lambda^{m/(m-1)}}{m}, \\ t = -\lambda^{1/(m-1)}. \end{cases} \tag{4.55}$$

Using (4.23) we find

$$\psi = \frac{(m-1)^2}{m(2m-1)} \lambda^{(2m-1)/(m-1)}, \tag{4.56}$$

according to which we have $\dddot{\psi} = \frac{1}{m-1}\lambda^{(2-m)/(m-1)}$. Now, applying Theorem 4.1.13, the solution of (4.53) is expressed in terms of solutions of the Monge equation

$$\dot{G} = (m-1)\lambda^{(m-2)/(m-1)} \ddot{F}^2. \tag{4.57}$$

Note that for $m = 2$ this conclusion agrees with that of Exercise 4.1.14. Using (4.46) as showed in Exercise 4.1.14, given a value of m (that is a choice of ψ), for every solution of the Monge equation (4.57) we can compute the corresponding closed form solution of the involutive system (4.53).

A system similar to (4.53) is considered by Cartan in [10, page 113] (and Stormark in [38, page 473]), but the expression of ψ is not provided there. We chose a different normalization to allow continuity with Example 4.1.14. ■

With this little motivation, we can proceed to the proof of Theorem 4.1.13.

Proof of Theorem 4.1.13. To prove both parts **[i]** and **[ii]**, we first built a quotient by the Cauchy characteristic, in order to reduce $I_{2,\psi}$ to a GR_3D_5 Pfaffian system I. Then we use the Monge Algorithm 3.5.8 to obtain the general Monge normal form of I and thus the equation $\dddot{\psi}\, Z_1 = {Y_2}^2$.

To begin, when $\psi = \psi(\lambda)$, the Cauchy characteristic (4.42) of the Pfaffian system $I_{2,\psi}$ (see (4.40)) is

$$\partial_\eta = \partial_x + \lambda\, \partial_y + (p + q\lambda)\; \partial_z + \left(\lambda\dot{\psi} - 2\psi\right)\, \partial_p - \dot{\psi}\, \partial_q.$$

A complete set of invariants for ∂_η is easily checked to be

$$
\begin{aligned}
&z^1 = \lambda, \quad z^2 = y - \lambda x, \quad z^3 = p + 2x\psi - x\lambda\dot{\psi}, \quad z^4 = q + x\dot{\psi}, \\
&z^5 = z + x^2\psi - x(p + 2x\psi - x\lambda\dot{\psi}) - x\lambda(q + x\dot{\psi}).
\end{aligned}
\tag{4.58}
$$

Let's set $z^6 = x$, which together with (4.58) defines a change of local coordinates $\tau_{6,6} : M_6 \to M_6$. Because $z^1 = \lambda$, we can write $\psi(z^1) = \psi(\lambda) = \psi$ and still set $\dot{\psi} = \frac{\partial\psi}{\partial z^1}$. In particular, the inverse change of coordinates is

$$
\begin{aligned}
\tau_{6,6}^{-1} : \quad & x = z^6, \quad y = z^2 + z^1 z^6, \quad z = z^5 - z^6(z^6\psi - z^3 - z^1 z^4), \\
& p = z^3 - z^6(2\psi - z^1\dot{\psi}), \quad q = z^4 - z^6\dot{\psi}, \quad \lambda = z^1.
\end{aligned}
\tag{4.59}
$$

Let's recall the adapted basis of $I_{2,\psi}$ at (4.40)

$$
\begin{aligned}
\alpha &= dz - p\,dx - q\,dy, \\
\alpha^1 &= dp + \lambda\,dq + \left(2\psi - \lambda\dot{\psi}\right)\,dx + \dot{\psi}\,dy, \\
\alpha^2 &= dq - \left(\lambda\ddot{\psi} - \dot{\psi}\right)\,dx + \ddot{\psi}\,dy,
\end{aligned}
\tag{4.60}
$$

where α^1 is considered without the normalizing factor $\dddot{\psi}$. In these new coordinates we can express

$$\begin{aligned}
\alpha &= d\,[z^5 - z^6(z^6\psi - z^3 - z^1 z^4)] - [z^3 - z^6(2\psi - z^1\dot{\psi})]\,dz^6 - (z^4 - z^6\dot{\psi})\,d\,(z^2 + z^1 z^6) \\
&= dz^5 - (z^6\psi - z^3 - z^1 z^4)\,dz^6 - z^6\,d\,(z^6\psi - z^3 - z^1 z^4) - [z^3 - z^6(2\psi - z^1\dot{\psi})]\,dz^6 \\
&\quad - (z^4 - z^6\dot{\psi})\,dz^2 - z^1(z^4 - z^6\dot{\psi})\,dz^6 - z^6(z^4 - z^6\dot{\psi})\,dz^1 \\
&= dz^5 - (z^4 - z^6\dot{\psi})\,dz^2 - z^6(z^4 - z^6\dot{\psi})\,dz^1 \\
&\quad - [z^6\psi - z^3 - z^1 z^4 + z^3 - z^6(2\psi - z^1\dot{\psi}) + z^1(z^4 - z^6\dot{\psi}) + z^6\psi]\,dz^6 \\
&\quad - {z^6}^2\dot{\psi}\,dz^1 + z^6\,dz^3 + z^4 z^6\,dz^1 + z^1 z^6\,dz^4 \\
&= (z^6\dot{\psi} - z^4)\,dz^2 + z^6\,dz^3 + z^1 z^6\,dz^4 + dz^5, \\
\alpha^1 &= d\,[z^3 - z^6(2\psi - z^1\dot{\psi})] + z^1\,d\,(z^4 - z^6\dot{\psi}) + (2\psi - z^1\dot{\psi})\,dz^6 + \dot{\psi}\,d\,(z^2 + z^1 z^6) \\
&= dz^3 - z^6\,(2\dot{\psi} - \dot{\psi} - z^1\ddot{\psi})\,dz^1 - (2\psi - z^1\dot{\psi})\,dz^6 + z^1\,(dz^4 - z^6\ddot{\psi}\,dz^1 - \dot{\psi}\,dz^6) \\
&\quad + (2\psi - z^1\dot{\psi})\,dz^6 + \dot{\psi}\,(dz^2 + z^6\,dz^1 + z^1\,dz^6) \\
&= dz^3 + \dot{\psi}\,dz^2 + z^1\,dz^4, \\
\alpha^2 &= d\,(z^4 - z^6\dot{\psi}) - (z^1\ddot{\psi} - \dot{\psi})\,dz^6 + \ddot{\psi}\,d\,(z^2 + z^1 z^6) \\
&= dz^4 - z^6\ddot{\psi}\,dz^1 - \dot{\psi}\,dz^6 - (z^1\ddot{\psi} - \dot{\psi})\,dz^6 + \ddot{\psi}\,(dz^2 + z^6\,dz^1 + z^1\,dz^6) \\
&= dz^4 + \ddot{\psi}\,dz^2.
\end{aligned}$$

Therefore we have

$$I_{2,\psi} = \begin{cases} \alpha = (z^6\dot{\psi} - z^4)\,dz^2 + z^6\,dz^3 + z^1 z^6\,dz^4 + dz^5, \\ \alpha^1 = dz^3 + \dot{\psi}\;dz^2 + z^1\,dz^4, \\ \alpha^2 = dz^4 + \ddot{\psi}\,dz^2. \end{cases}$$

Now, we just tweak a little bit the basis of $I_{2,\psi}$, defining $\alpha^0 = \alpha - z^6\,\alpha^1$ so that

$$I_{2,\psi} = \begin{cases} \alpha^0 = dz^5 - z^4\,dz^2, \\ \alpha^1 = dz^3 + \dot{\psi}\;dz^2 + z^1\,dz^4, \\ \alpha^2 = dz^4 + \ddot{\psi}\,dz^2. \end{cases}$$

As we expected, $I_{2,\psi}$ *does not depend on* z^6.

Let M_5 be the quotient manifold with local coordinates $(\bar{y}^1, \ldots, \bar{y}^5)$. Using (4.58), the standard projection $\mathbf{q}_{6,5} : M_6 \to M_5$ by ∂_η is locally define by

$$\mathbf{q}_{6,5} : \quad \bar{y}^i = z^i, \quad i = 1, \ldots, 5. \tag{4.61}$$

For any constant μ, a local section $\mathbf{s}_{5,6} : M_5 \to M_6$ of the projection (4.61) is given by

$$\mathbf{s}_{5,6} : \quad z^6 = \mu, \quad z^i = \bar{y}^i, \quad i = 1, \ldots, 5. \tag{4.62}$$

Consequently, if we let $\psi(\bar{y}^1) = \psi(\lambda) = \psi$ denote the function $\mathbf{q}_{6,5} \circ \psi$, the reduction of $I_{2,\psi}$ is the rank-3 Pfaffian system $I = \mathbf{s}_{5,6}{}^*(I_{2,\psi})$ given by

$$I = \begin{cases} \eta^1 = \mathbf{s}_{5,6}{}^*\alpha^1 = d\,\bar{y}^3 + \dot{\psi}\,d\,\bar{y}^2 + \bar{y}^1\,d\,\bar{y}^4, \\ \eta^2 = \mathbf{s}_{5,6}{}^*\alpha^0 = d\,\bar{y}^5 - \bar{y}^4\,d\,\bar{y}^2, \\ \eta^3 = \mathbf{s}_{5,6}{}^*\alpha^2 = d\,\bar{y}^4 + \ddot{\psi}\,d\,\bar{y}^2. \end{cases} \tag{4.63}$$

To arrive at transformations (4.44) and (4.46), we now follow the steps **[0]** to **[4]** of the Monge Algorithm 3.5.8.

[0] By Proposition 4.1.7, we know that I is a GR_3D_5 Pfaffian system and by the above manipulations we know that (4.63) is an adapted basis for I.

[1] We want to built a 1-form θ^1 in $I' = \{\eta^1, \eta^2\}$ whose Engel-rank is 1, that is such that $\theta^1 = dy^2 - y^3\,dy^1$ for independent functions y^1, y^2, and y^3. The form η^2 would work, but we want λ to be the independent variable of the Monge equation (4.45), therefore we wish to set $y^1 = \bar{y}^1 = \mathbf{q}_{6,5}(\lambda)$. Here it is what we can do. Let's rewrite

$$\eta^1 = d\,\bar{y}^3 + d\,(\dot{\psi}\bar{y}^2) - \bar{y}^2\ddot{\psi}\,d\,\bar{y}^1 + d\,\bar{y}^1\bar{y}^4 - \bar{y}^4\,d\,\bar{y}^1 = d\,(\bar{y}^3 + \dot{\psi}\bar{y}^2 + \bar{y}^1\bar{y}^4) - (\bar{y}^2\ddot{\psi} - \bar{y}^4)\,d\,\bar{y}^1.$$

Now define the change of coordinates $\tau_{5,1} : M_5 \to M_5$

$$\tau_{5,1} : \quad y^1 = \bar{y}^1, \quad y^2 = \bar{y}^3 + \dot{\psi}\bar{y}^2 + \bar{y}^1\bar{y}^4, \quad y^3 = \bar{y}^2\ddot{\psi} - \bar{y}^4, \quad y^4 = \bar{y}^5, \quad y^5 = \bar{y}^2, \tag{4.64}$$

with inverse $\tau_{5,1}^{-1} : M_5 \to M_5$

$$\tau_{5,1}^{-1} : \quad \bar{y}^1 = y^1, \quad \bar{y}^2 = y^5, \quad \bar{y}^3 = y^2 - \dot{\psi} y^5 + y^1(y^5\ddot{\psi} - y^3), \quad \bar{y}^4 = y^5\ddot{\psi} - y^3, \quad \bar{y}^5 = y^4. \tag{4.65}$$

With respect to the local coordinates (4.64) (considering $\psi(y^1) = \psi(\lambda) = \psi$) we have

$$I' = \begin{cases} \eta^1 = dy^2 - y^3\, dy^1, \\ \eta^2 = dy^4 - (y^5\ddot{\psi} - y^3)\, dy^5. \end{cases} \tag{4.66}$$

[2] In view of Proposition 3.3.3 and Remark 3.3.4, in this step we want to construct a 1-form $dU - V^3\, dy^3 - V^1\, dy^1 = W\, \eta^1 + A\, \eta^2 \in I'$. From (3.47), we need an integrating factor A for which the PDE system

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= A \cdot 1, \\ \frac{\partial U}{\partial y^5} &= A(y^5\ddot{\psi} - y^3), \end{aligned} \tag{4.67}$$

can be easily solved. We see that for $A = 2$ we have the solution $U = 2y^4 + 2y^3y^5 - {y^5}^2\ddot{\psi}$, note that $\psi = \psi(y^1)$. Remembering the notation $\eta^2 = Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5$ used in Proposition 3.3.3, the expressions (3.48) become

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - 2 \cdot 0 = 0, \\ V^3 &= \frac{\partial U}{\partial y^3} - 2 \cdot 0 = 2y^5, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 \left(\frac{\partial U}{\partial y^2} - 2 \cdot 0 \right) - 2 \cdot 0 = -{y^5}^2 \dddot{\psi}. \end{aligned} \tag{4.68}$$

Indeed, we can directly check that

$$dU - V^3\, dy^3 - V^1\, dy^1 = d(2y^4 + 2y^3y^5 - {y^5}^2\ddot{\psi}) - 2y^5\, dy^3 + {y^5}^2\dddot{\psi}\, dy^1 = 2\eta^2 \in I'. \tag{4.69}$$

Using U and V^3 we define a new change of coordinates $\tau_{5,2} : M_5 \to M_5$

$$\tau_{5,2} : \quad x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = 2y^4 + 2y^3y^5 - {y^5}^2\ddot{\psi}, \quad x^5 = 2y^5. \tag{4.70}$$

The inverse change $\tau_{5,2}^{-1} : M_5 \to M_5$ is given by

$$\tau_{5,2}^{-1}: \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = \frac{1}{2}\left(x^4 - x^3x^5 + \frac{{x^5}^2}{4}\ddot{\psi}\right), \quad y^5 = \frac{x^5}{2}. \tag{4.71}$$

Again, we can consider $\psi = \psi(x^1) = \psi(\lambda)$, thus in the coordinates (4.70) we have

$$I' = \begin{cases} \omega^1 = dx^2 - x^3\,dx^1, \\ \omega^2 = dx^4 - x^5\,dx^3 + \dfrac{{x^5}^2}{4}\dddot{\psi}\,dx^1. \end{cases}$$

In particular the dx^1-component of ω^2 is the function $f = \frac{{x^5}^2}{4}\dddot{\psi}$.

[3] We can compute $f_{x^5} = \frac{x^5}{2}\dddot{\psi}$ and write the 1-form in I

$$\omega^3 = dx^3 - f_{x^5}\,dx^1 = dx^3 - \frac{x^5}{2}\dddot{\psi}\,dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \omega^1 = dx^2 - x^3\,dx^1, \\ \omega^2 = dx^4 - x^5\,dx^3 + \dfrac{{x^5}^2}{4}\dddot{\psi}\,dx^1, \\ \omega^3 = dx^3 - \dfrac{x^5}{2}\dddot{\psi}\,dx^1. \end{cases}$$

[4] In view of (3.84), we set

$$\tilde{H} = x^5 f_{x^5} - f = x^5\frac{x_5}{2}\dddot{\psi} - \frac{{x^5}^2}{4}\dddot{\psi} = \frac{{x^5}^2}{4}\dddot{\psi},$$

so that I can be written in the new basis

$$I = \begin{cases} \mu^1 = \omega^1 = dx^2 - x^3\,dx^1, \\ \mu^2 = \omega^3 = dx^3 - \dfrac{x^5}{2}\dddot{\psi}\,dx^1, \\ \mu^3 = \omega^2 + x^5\,\omega^3 = dx^4 - \tilde{H}\,dx^1. \end{cases} \tag{4.72}$$

Finally, remembering that we have $\dddot{\psi} \neq 0$, we define the change of local coordinates $\tau_{5,4} : M_5 \to M_5$

$$\tau_{5,4} : \quad X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = \frac{x^5}{2}\dddot{\psi}, \tag{4.73}$$

whose inverse $\tau_{5,4}^{-1} : M_5 \to M_5$ is

$$\tau_{5,4}^{-1} : \quad x^1 = X, \quad x^2 = Y, \quad x^3 = Y_1, \quad x^4 = Z, \quad x^5 = \frac{2Y_2}{\dddot{\psi}}. \tag{4.74}$$

Completing the application of (3.84), we set

$$H = \tau_{5,4}^{-1} \circ \tilde{H} = \frac{\left(\frac{2Y_2}{\dddot{\psi}}\right)^2}{4}\dddot{\psi} = \frac{{Y_2}^2}{\dddot{\psi}}.$$

Therefore, according to (4.72) and (4.74), we obtain the Monge normal form of I

$$I = \begin{cases} \mu^1 = dY - Y_1\, dX, \\ \mu^2 = dY_1 - Y_2\, dX, \\ \mu^3 = dZ - H\, dX = dZ - \dfrac{{Y_2}^2}{\dddot{\psi}}\, dX. \end{cases} \tag{4.75}$$

In this way we see that I is locally realized by the Monge equation

$$Z_1 = \frac{{Y_2}^2}{\dddot{\psi}}. \tag{4.76}$$

This concludes the Monge Algorithm.

Now let's go back to the initial involutive system. I is the reduction of $I_{2,\psi}$ by means of the projection $\mathbf{q} = \tau_{5,4} \circ \tau_{5,2} \circ \tau_{5,1} \circ \mathbf{q}_{6,5} \circ \tau_{6,6} : M_6 \to M_5$

$$\begin{aligned} \mathbf{q} : \quad & X = \lambda, \quad Y = (y - \lambda x)\dot{\psi} + q\lambda + p + 2x\psi, \\ & Y_1 = (y - \lambda x)\ddot{\psi} + q + x\dot{\psi}, \quad Y_2 = (y - \lambda x)\dddot{\psi}, \\ & Z = -\frac{1}{2}\left(z - x^2\psi - xp + x^2\lambda\dot{\psi} - \frac{1}{2}y^2\ddot{\psi} + xy\lambda\ddot{\psi} - \frac{1}{2}x^2\lambda^2\ddot{\psi} - qy - xy\dot{\psi}\right). \end{aligned} \tag{4.77}$$

This proves **[i]**.

A section of $\mathbf{q}$ is given by $\mathbf{s} = \tau_{6,6}^{-1} \circ \mathbf{s}_{5,6} \circ \tau_{5,1}^{-1} \circ \tau_{5,2}^{-1} \circ \tau_{5,4}^{-1} : M_5 \to M_6$

$$\begin{aligned} \mathbf{s}: \quad & \lambda = X, \quad x = \mu, \quad y = \lambda\mu + \frac{\ddot{F}}{\dddot{\psi}}, \\ & z = -\mu^2\psi + \left(F - \frac{\dot{\psi}\ddot{F}}{\dddot{\psi}}\right)\mu - \frac{1}{2}\left(G + \frac{\ddot{\psi}\ddot{F}^2}{\dddot{\psi}^2} - \frac{2\dot{F}\ddot{F}}{\dddot{\psi}}\right), \\ & p = (\lambda\dot{\psi} - 2\psi)\mu - \frac{1}{\dddot{\psi}}\left[\left(\dddot{\psi}\dot{F} - \ddot{\psi}\ddot{F}\right)\lambda - \dot{\psi}\ddot{F}\right] + F, \quad q = -\dot{\psi}\mu + \dot{F} - \frac{\ddot{\psi}\ddot{F}}{\dddot{\psi}}. \end{aligned} \tag{4.78}$$

In summary, for every choice of $\psi = \psi(\lambda)$ (such that $\dddot{\psi} \neq 0$), a given solution $Z = G(X) = G(\lambda)$ and $Y = F(X) = F(\lambda)$ of the Monge equation (4.76) provides the 2-dimensional integral manifold of the nonlinear involutive system (4.43)

$$x = \mu, \quad y = \lambda\mu + \frac{\ddot{F}}{\dddot{\psi}}, \quad z = -\mu^2\psi + \left(F - \frac{\dot{\psi}\ddot{F}}{\dddot{\psi}}\right)\mu - \frac{1}{2}\left(G + \frac{\ddot{\psi}\ddot{F}^2}{\dddot{\psi}^2} - \frac{2\dot{F}\ddot{F}}{\dddot{\psi}}\right),$$

which proves **[ii]**. □

4.2 Goursat equations and $\mathbf{GR_3D_5}$ Pfaffian systems

In this section we study scalar second order PDE in the plane

$$F(x, y, z, p, q, r, s, t) = 0, \tag{4.79}$$

which are parabolic, that is, for which

$$F_s{}^2 - 4F_rF_t = 0. \tag{4.80}$$

A geometric definition of parabolic equations will be given, followed by that of Goursat parabolic equations. Then the main object, Goursat parabolic equations of general type, or simply general Goursat equations (as in [10, page 140]), will be considered. Stormark [38, Chapter 16] notices that this is the class of parabolic PDE in the plane with complete Monge characteristics.

As in [4] and [8], we give the following.

Definition 4.2.1. A ***parabolic Pfaffian system*** is a rank-3 Pfaffian system I_1 defined on a 7-dimensional manifold M_7 for which there exists a local coframe α, α^1, α^2, π^1, σ^1, π^2, σ^2 such that

[i] $I_1 = \{\alpha, \alpha^1, \alpha^2\}$, $\pi^1 \wedge \pi^2 \neq 0$, and

[ii] the following structure equations are satisfied

$$\begin{aligned} d\alpha &\equiv \alpha^1 \wedge \pi^1 + \alpha^2 \wedge \pi^2 && \mathrm{mod}\,(\alpha), \\ d\alpha^1 &\equiv G\,\alpha^2 \wedge \pi^1 + \pi^2 \wedge \sigma^2 && \mathrm{mod}\,(\alpha, \alpha^1), \\ d\alpha^2 &\equiv \pi^1 \wedge \sigma^2 + \pi^2 \wedge \sigma^1 && \mathrm{mod}\, I_1, \end{aligned} \tag{4.81}$$

where $G \in C^\infty(M_7)$. ■

Theorem 4.2.2. *[8, page 0.6] Let I_1 be a parabolic Pfaffian system on M_7 and let $\phi : M_7 \to M_7$ be a diffeomorphism. Assume $\bar{I}_1 = \phi^* I_1$, then the following properties hold.*

[i] *$\bar{I}_1$ is a parabolic Pfaffian system.*

[ii] *Assume $\bar{\alpha}$, $\bar{\alpha}^1$, $\bar{\alpha}^2$, $\bar{\pi}^1$, $\bar{\sigma}^1$, $\bar{\pi}^2$, $\bar{\sigma}^2$ is a local coframe adapted to $\bar{I}_1$, with $d\bar{\alpha}^1 \equiv \bar{G}\,\bar{\alpha}^2 \wedge \bar{\pi}^1 + \bar{\pi}^2 \wedge \bar{\sigma}^2$* $\mathrm{mod}\,\bar{I}_1$, *and define $I_1^1 = \{\alpha, \alpha^1\}$. Then*

$$\phi^* I_1^1 = \{\bar{\alpha}, \bar{\alpha}^1\}, \quad \bar{G} = k(G \circ \phi) \text{ with } k \in \mathbb{R} - \{0\}. \tag{4.82}$$

The function G is called the ***Goursat invariant*** of I_1.

Because of (4.81), a parabolic Pfaffian system I_1 has the following antiderived systems:

$$\begin{aligned} \Xi_0 &= \{\alpha, \alpha^1, \alpha^2, \pi^2, \sigma^2\}, \\ \Xi_1 &= \{\alpha, \alpha^1, \alpha^2, \pi^1, \pi^2\}, \\ \Xi_2 &= \{\alpha, \alpha^1, \alpha^2, \sigma^1, \sigma^2\}. \end{aligned}$$

Theorem 4.2.3. *[10, pages 130-132] Let I_1 be a parabolic Pfaffian system on M_7 and assume that α, α^1, α^2, π^1, σ^1, π^2, σ^2 is a local coframe adapted to I_1. Then we have.*

[i] $\Xi_0 = \mathcal{CS}(I_1^1)$ *is the Cartan system of I_1^1 and, in particular,* $\partial_{\sigma^1} \in \mathrm{Cau}(I_1^1)$.

[ii] *$G = 0$ if and only if* (4.81) *reduces to*

$$\begin{aligned} d\alpha &\equiv \alpha^1 \wedge \pi^1 + \alpha^2 \wedge \pi^2 && \operatorname{mod}(\alpha), \\ d\alpha^1 &\equiv \pi^2 \wedge \sigma^2 && \operatorname{mod}(\alpha, \alpha^1), \\ d\alpha^2 &\equiv \pi^1 \wedge \sigma^2 + \pi^2 \wedge \sigma^1 && \operatorname{mod} I_1, \end{aligned} \tag{4.83}$$

see also [10, page 132].

[iii] *$G = 0$ if and only if* $\dim \mathrm{Cau}\big(I_1^1\big) = 2$*.*

[iv] *Ξ_0 is completely integrable if and only if $G = 0$.*

Definition 4.2.4. Ξ_0 is called the ***Monge system*** associated to the parabolic Pfaffian system I_1. A ***Goursat parabolic Pfaffian system*** I_1 has Frobenius Monge system Ξ_0, or equivalently $G = 0$. We say that a PDE in the plane is a ***Goursat parabolic equation*** if it gives rise to a Goursat parabolic Pfaffian system. ■

Proof of Theorem 4.2.3, sketch. Part **[i]** is an easy consequence of equations (4.81).

Part **[ii]** can be inferred by applying Theorem 4.2.2 to the structure equations (4.81).

Part **[iii]** is a consequence of **[ii]**. Indeed, because of (4.83), one has $G = 0$ if and only if there exists an adapted coframe for which $\mathrm{Cau}\big(I_1^1\big) = \langle \partial_{\pi^1}, \partial_{\sigma^1} \rangle$.

The proof of part **[iv]** is based on the construction of an adapted coframe such that

$$\begin{aligned} d\pi^2 &\equiv 0 && \operatorname{mod} \Xi_0, \\ d\sigma^2 &\equiv -G\,\pi^1 \wedge \sigma^1 && \operatorname{mod} \Xi_0. \end{aligned} \tag{4.84}$$

One can derive these equations by taking the exterior derivative of (4.81), or alternatively by using a local coordinate formulation. We use this last approach. In [10, §IV] and [8, page 0.3], it is shown that every parabolic Pfaffian system is locally realized by a parabolic PDE in the plane. Moreover, every parabolic PDE in the plane can be written in the form

$$r = E(x, y, z, p, q, s, t), \tag{4.85}$$

where one has

$$E_s{}^2 + 4E_t = 0. \tag{4.86}$$

On the 7-manifold M_7 with local coordinates (x, y, z, p, q, s, t), the PDE (4.85) defines the rank-3 Pfaffian system

$$I_1 = \{d\,z - p\,d\,x - q\,d\,y,\ d\,p - E\,d\,x - s\,d\,y,\ d\,q - s\,d\,x - t\,d\,y\}. \tag{4.87}$$

Define on M_7 the differential operators

$$\bar{D}_x = \partial_x + p\,\partial_z + E\,\partial_p + s\,\partial_q, \quad \bar{D}_y = \partial_y + q\,\partial_z + s\,\partial_p + t\,\partial_q. \tag{4.88}$$

Then a local adapted coframe of I_1, for which (4.81) holds, is given by

$$\begin{aligned}
\alpha &= d\,z - p\,d\,x - q\,d\,y,\\
\alpha^1 &= d\,p - E\,d\,x - s\,d\,y - \frac{1}{2}E_s(d\,q - s\,d\,x - t\,d\,y),\\
\alpha^2 &= d\,q - s\,d\,x - t\,d\,y,\\
\pi^1 &= -d\,x,\\
\sigma^1 &= -d\,t,\\
\pi^2 &= -d\,y - E_s\,d\,x - \frac{1}{2}E_{ss}\,\alpha^2,\\
\sigma^2 &= -d\,s + \frac{1}{2}E_s\,d\,t + \bar{D}_y(E)\,d\,x + \frac{1}{2}\bar{D}_y(E_s)\,\alpha^2.
\end{aligned} \tag{4.89}$$

Then (see [4, equation (6)] as well) the Goursat invariant of I_1 is explicitly given by

$$G = E_q + \frac{1}{2}E_sE_p + \frac{1}{4}E_s\bar{D}_y(E_s) - \frac{1}{2}\bar{D}_x(E_s) - \frac{1}{2}E_{ss}\bar{D}_y(E). \tag{4.90}$$

By direct computation we see that (4.84) holds. □

Example 4.2.5. Consider the special class of parabolic PDE in the plane $r = E(x, y, z, p)$. Then the Goursat invariant is $G = 0$.

On the other side, consider the heat equation $z_{xx} - z_y = 0$ (here y is the time-variable). With the notation used previously we would write this parabolic PDE as $r = q$. Then the Goursat invariant for this equation is $G = 1$. ■

This example motivates the standard terminology, which calls ***dispersive*** those parabolic PDE in the plane which are non-Goursat, that is, such that $G \neq 0$.

At this point we just want to remark that if I_1 is a parabolic Pfaffian system, then I_1^1 *is not the derived system of* I_1, since $I_1{}' = \{\alpha\}$. We have $\mathbf{DT}(I_1) = [3,1,0]$, $\mathbf{Cau}(I_1) = [0,2,7]$, $\mathbf{DT}\big(I_1^1\big) = [2,0]$, and $\mathbf{Cau}\big(I_1^1\big) = [2,7]$. Moreover, because $\mathrm{Cau}\big(I_1^1\big) = \langle \partial_{\pi^1}, \partial_{\sigma^1}\rangle$, *we can reduce* I_1^1 *to a rank-*2 *Pfaffian system* $\bar{I}_1^1$ *on a* 5*-manifold.* The results of Chapter 3 can then help us to characterize a Goursat parabolic Pfaffian system I_1 by means of its invariant subsystem I_1^1. This is the goal of the rest of this section.

Authors like Cartan [10, section IV], or Goursat [24, page 93], [25, page 166], and [23, section IV], or Stormark [38, section 16.1]), carry out quite few differential algebra computations to provide another useful coordinate representation of Goursat parabolic equations. We show their results in the following statements, whose proofs we just sketch.

Theorem 4.2.6. $\mathcal{S}_1 \equiv F(x,y,z,p,q,r,s,t) = 0$ *is a parabolic equation if and only if it can be written in the parametric form*

$$\begin{cases} r = \lambda^2 t + 2\lambda\dot{\psi} - 2\psi, \\ s = -\lambda t - \dot{\psi}, \end{cases} \tag{4.91}$$

where $\psi = \psi(x,y,z,p,q,\lambda)$ *and* $\dot{\psi} = \frac{\partial\psi}{\partial\lambda}$.

Proof. Starting from the form $r = E(x,y,z,p,q,s,t)$ of $\mathcal{S}_1$ (see (4.85)), where one assumes $E_sE_t \neq 0$, we obtain the parametrization (4.91) by setting

$$\begin{cases} \lambda = -\dfrac{1}{2}E_s, \\ \psi = -\dfrac{1}{2}(E + 2s\lambda + t\lambda^2). \end{cases} \tag{4.92}$$

The actual change from the coordinates (x,y,z,p,q,s,t) to (x,y,z,p,q,λ,t) can be worked out with the same steps as those in the proof of Theorem 4.1.9.

Conversely, we can obtain the parabolic equation $\mathcal{S}_1$ by eliminating λ from (4.91). □

Note that (4.91) is equivalent to the equation

$$r + 2\lambda s + \lambda^2 t + 2\psi = 0, \tag{4.93}$$

together with its first derivative with respect to λ, that is,

$$\begin{cases} r + 2\lambda s + \lambda^2 t + 2\psi = 0, \\ s + \lambda t + \dot{\psi} = 0. \end{cases} \tag{4.94}$$

Theorem 4.2.7. *I_1 is a parabolic Pfaffian system on M_7 if and only if there exist local coordinates $(x, y, z, p, q, \lambda, t)$ on M_7 such that*

$$I_1 = \begin{cases} \alpha = dz - p\,dx - q\,dy, \\ \alpha^1 = dp + \lambda\,dq - (\lambda\dot{\psi} - 2\psi)\,dx + \dot{\psi}\,dy, \\ \alpha^2 = dq + (\lambda t + \dot{\psi})\,dx - t\,dy. \end{cases} \tag{4.95}$$

Here $\psi = \psi(x, y, z, p, q, \lambda)$ and $\dot{\psi} = \frac{\partial \psi}{\partial \lambda}$.

Proof. We already mentioned [10, §IV] and [8, page 0.3], according to which every parabolic Pfaffian system is locally generated by a parabolic PDE in the plane. Consequently, in view of Theorem 4.2.6 and expression (4.94), I_1 is a parabolic Pfaffian system on M_7 if and only if there exist local coordinates $(x, y, z, p, q, \lambda, t)$ on M_7 such that

$$I_1 = \begin{cases} \alpha = dz - p\,dx - q\,dy, \\ \alpha_0^1 = dp - (\lambda^2 t + 2\lambda\dot{\psi} - 2\psi)\,dx + (\lambda t + \dot{\psi})\,dy, \\ \alpha^2 = dq + (\lambda t + \dot{\psi})\,dx - t\,dy. \end{cases} \tag{4.96}$$

Defining

$$\alpha^1 = \alpha_0^1 + \lambda\,\alpha^2 = dp + \lambda\,dq - (\lambda\dot{\psi} - 2\psi)\,dx + \dot{\psi}\,dy,$$

we obtain (4.95). □

Remark 4.2.8. Let I_1 be a parabolic Pfaffian system with local expression (4.95), then clearly $\partial_t \in \mathrm{Cau}\big(I_1^1\big)$. Using Corollary 2.2.17, let M_6 be the qoutient manifold of M_7 by ∂_t and denote $\bar{I}_1^1$ the reduction of I_1^1. We can choose local coordinates (x, y, z, p, q, λ) on M_6, so that

$$\bar{I}_1^1 = \begin{cases} \bar{\alpha} = dz - p\,dx - q\,dy, \\ \bar{\alpha}^1 = dp + \lambda\,dq - (\lambda\dot{\psi} - 2\psi)\,dx + \dot{\psi}\,dy. \end{cases} \tag{4.97}$$

We can still denote $\psi = \psi(x, y, z, p, q, \lambda)$ and $\dot{\psi} = \frac{\partial\psi}{\partial\lambda}$. From Corollary 2.2.17, we know that $\mathbf{DT}\big(\bar{I}_1^1\big) = \mathbf{DT}\big(I_1^1\big) = [2, 0]$. ■

Now we can characterize our Goursat parabolic scalar partial differential equations in the plane.

Theorem 4.2.9. *Let I_1 be a parabolic Pfaffian system with local expression* (4.95)*. With the notation of Remark 4.2.8, the following conditions are equivalent.*

[i] I_1 *is a Goursat parabolic Pfaffian system.*

[ii] *The function $\psi = \psi(x, y, z, p, q, \lambda)$ in Theorem 4.2.7 satisfies the second-order PDE in* 6*-variables*

$$\dot{\psi}_x + \lambda\dot{\psi}_y + (p + \lambda q)\,\dot{\psi}_z - (2\psi - \lambda\dot{\psi})\dot{\psi}_p - \dot{\psi}\dot{\psi}_q - 2\left(\psi_y + q\psi_z - (\dot{\psi} - \lambda\ddot{\psi})\psi_p - \ddot{\psi}\psi_q\right) = 0. \tag{4.98}$$

[iii] $\dim \mathrm{Cau}\big(\bar{I}_1^1\big) = 1$*. In particular, we have*

$$\partial_x + \lambda\,\partial_y + (p + q\lambda)\,\partial_z + \left(\lambda\dot{\psi} - 2\psi\right)\,\partial_p - \dot{\psi}\,\partial_q + 2\,(\psi_q - \lambda\psi_p)\,\partial_\lambda \in \mathrm{Cau}\big(\bar{I}_1^1\big). \tag{4.99}$$

Proof. **[i ⇔ ii]** In view of Theorems 4.2.3 and 4.2.7, one has to prove that (4.98) is equivalent to the condition $G = 0$. This can be shown, for instance, by using the local expression (4.90) together with the change of coordinates pointed out in the proof of Theorem 4.2.6.

[i ⇔ iii] According to Theorem 4.2.3, I_1 is a Goursat parabolic Pfaffian system if an only if $\dim \mathrm{Cau}\big(I_1^1\big) = 2$. Because of Remark 4.2.8, this last condition is equivalent to $\dim \mathrm{Cau}\big(\bar{I}_1^1\big) = 1$.

We would like to note that, by direct computation, one can show that (4.99) is a Cauchy characteristic of $\bar{I}_1^1$ if and only if ψ satisfies (4.98). □

We can finally obtain our goal.

Theorem 4.2.10. *Let I_1 be a Goursat parabolic Pfaffian system. Then I_1^1 is reduced by its two Cauchy characteristics to a rank-2 Pfaffian system (on a 5-manifold) which is either the contact system $\mathrm{C}^1_{1,2}$ or a GR_2D_5 Pfaffian system.*

In particular, using the notations of Remark 4.2.8, only two cases are possible.

[Monge-Ampère] $\dddot{\psi} = 0$, *I_1^1 is reduced to $\mathrm{C}^1_{1,2}$ and I_1 is generated by a Monge-Ampère equation*

$$\mathcal{S}_1 \equiv (r + A)\,(t + C) - (s + B)^2 = 0, \tag{4.100}$$

where A, B, C are functions of x, y, z, p, q such that $\psi = \frac{1}{2}C\lambda^2 + B\lambda + \frac{1}{2}A$ satisfies (4.98).

[General] $\dddot{\psi} \neq 0$, *I_1^1 reduces to a GR_2D_5 Pfaffian system and I_1 is generated by a non-Monge-Ampère equation.*

Proof. As a consequence of Remark 4.2.8, I_1^1 is reduced by one of its Cauchy characteristics to the rank-2 Pfaffian system

$$\bar{I}_1^1 = \begin{cases} \bar{\alpha} = d\,z - p\,d\,x - q\,d\,y, \\ \bar{\alpha}^1 = d\,p + \lambda\,d\,q - (\lambda\dot{\psi} - 2\psi)\,d\,x + \dot{\psi}\,d\,y, \end{cases}$$

on the 6-manifold M_6 and $\mathbf{DT}(K) = \mathbf{DT}\big(\bar{I}_1^1\big) = [2, 0]$.

By Theorem 4.2.9, we have $\dim \mathrm{Cau}\big(\bar{I}_1^1\big) = 1$, so that $\bar{I}_1^1$ has a 5-dimensional Cartan system $\mathcal{CS}\big(\bar{I}_1^1\big)$, which we shall now determine. Define

$$\begin{aligned} \bar{\alpha}^2 &= d\,q - (\lambda\ddot{\psi} - \dot{\psi})\,d\,x + \ddot{\psi}\,d\,y, \\ \bar{\pi} &= d\,y - \lambda\,d\,x, \\ \bar{\sigma} &= d\lambda - \Big(2\psi_q - \lambda\dot{\psi}_q - 2\lambda\psi_p + \lambda^2\dot{\psi}_p\Big)\,d\,x - \Big(\dot{\psi}_q - \lambda\dot{\psi}_p\Big)\,d\,y, \\ \bar{\eta} &= d\,x, \end{aligned} \tag{4.101}$$

which complete $\bar{I}_1^1$ to a local coframe on M_6. Then we have the following structure equations

$$\begin{aligned} d\bar{\alpha} &\equiv 0 && \mod\{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2\}, \\ d\bar{\alpha}^1 &\equiv 0 && \mod\{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2\}, \\ d\bar{\alpha}^2 &\equiv \dddot{\psi}\,\bar{\pi}\wedge\bar{\sigma} && \mod\{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2\}, \\ d\bar{\pi} &\equiv 0 && \mod\{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2, \bar{\pi}, \bar{\sigma}\}, \\ d\bar{\sigma} &\equiv 0 && \mod\{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2, \bar{\pi}, \bar{\sigma}\}. \end{aligned} \tag{4.102}$$

Therefore we have $\mathcal{CS}(\bar{I}_1^1) = \{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2, \bar{\pi}, \bar{\sigma}\}$ and, by the first two equations of (4.102), $\bar{I} = \{\bar{\alpha}, \bar{\alpha}^1, \bar{\alpha}^2\}$ is an antiderived system of $\bar{I}_1^1$. Moreover, in accordance with (4.99), we see that

$$\partial_{\bar{\eta}} = \partial_x + \lambda\,\partial_y + (p + q\lambda)\;\partial_z + \left(\lambda\dot{\psi} - 2\psi\right)\,\partial_p - \dot{\psi}\,\partial_q + 2\,(\psi_q - \lambda\psi_p)\;\partial_\lambda \in \mathrm{Cau}(\bar{I}_1^1).$$

Another consequence of (4.102) is that $\dddot{\psi}$ is an invariant of $\partial_{\bar{\eta}}$ (this can be checked using (4.98)). We can conclude that $\bar{I}_1^1$ (and thus I_1^1) reduces to a rank-2 Pfaffian system K on a 5-manifold M_5 such that $\mathbf{DT}(K) = \mathbf{DT}(\bar{I}_1^1) = [2,0]$ and $\mathbf{Cau}(K) = [0,5]$.

Consequently, we can apply Proposition 3.1.5 and say that K has a unique antiderived system $K^{(-1)}$, which implies that $K^{(-1)}$ is the reduction of $\bar{I}$. The results of Sections 3.2 and 3.3 apply and only two cases are possible.

[**Monge-Ampère type**] One can have $\dddot{\psi} = 0$ and equivalently the systems $\bar{I}$ and $K^{(-1)}$ are completely integrable. By Theorem 3.2.1 (page 36), we can conclude that $\bar{I}_1^1$ is equivalent to the contact system $\mathrm{C}^1_{1,2}$. Accordingly, $\bar{I}$ can be reduced to the normal form

$$\bar{I} = \{dU - U'\,dT,\; dV - V'\,dT, dT\}. \tag{4.103}$$

Moreover, because $\dddot{\psi} = 0$, then $\psi = \frac{1}{2}C\lambda^2 + B\lambda + \frac{1}{2}A$, where A, B, and C, are some functions depending on x, y,z,p, and q, such that (4.98) holds. Plugging this expression in (4.93) we obtain

$$\mathcal{S}_1 \equiv (r + A)\,(t + C) - (s + B)^2 = 0,$$

that is, a parabolic Monge-Ampère equation, which can be solved by quadratures as shown by (4.103).

[General type] We can have $\dddot{\psi} \neq 0$. Equivalently the systems $\bar{I}$ and $K^{(-1)}$ are not completely integrable, thus by Remark 3.3.7 (page 45) K is a GR_2D_5 Pfaffian system and I_1 is not generated by a Monge-Ampère equation. $\square$

Following Cartan's terminology, [10, page 140], we call ***general Goursat equations*** those Goursat parabolic equations which are not Monge-Ampère equations.

We end this section with a final remark. Looking at Remark 4.1.11 (page 75) and at the Pfaffian system

$$\bar{I} = \begin{cases} \bar{\alpha} = d\,z - p\,d\,x - q\,d\,y, \\ \bar{\alpha}^1 = d\,p - (-\lambda^2\ddot{\psi} + 2\lambda\dot{\psi} - 2\psi)\,d\,x - (\lambda\ddot{\psi} - \dot{\psi})\,d\,y, \\ \bar{\alpha}^2 = d\,q - (\lambda\ddot{\psi} - \dot{\psi})\,d\,x + \ddot{\psi}\,d\,y, \end{cases}$$

constructed in the proof of Theorem 4.2.10, we can see that if $\mathcal{S}_1$ is a general Goursat equation then $\bar{I} = I_{2,\psi}$, that is, every general Goursat equation can be associated to a nonlinear involutive system. This relation is the subject of the next section.

4.3 A special bijection

As anticipated in sections 4.1 and 4.2, here we will look closer at the relation between nonlinear involutive systems and general Goursat equations. A clear statement of this relation is given and some examples are provided. What follows is stated in [10, §IV.24].

Theorem 4.3.1. *There is a* 1*-to-*1 *correspondence between nonlinear involutive systems of two second order PDE in the plane $\mathcal{S}_2$, general Goursat parabolic equations $\mathcal{S}_1$, and the solutions $\psi = \psi(x, y, z, p, q, \lambda)$ of the PDE in* 6*-variables*

$$2\left(\psi_y + q\psi_z - \left(\dot{\psi} - \lambda\ddot{\psi}\right)\psi_p - \ddot{\psi}\psi_q\right) = \dot{\psi}_x + \lambda\dot{\psi}_y + (p + \lambda q)\,\dot{\psi}_z - \left(2\psi - \lambda\dot{\psi}\right)\dot{\psi}_p - \dot{\psi}\dot{\psi}_q \qquad (4.104)$$

for which $\dddot{\psi} = \frac{\partial^3\psi}{\partial\lambda^3} \neq 0$.

Proof. We just have to recall Theorem 4.1.9 for $\mathcal{S}_2$ (see page 71) and Theorems 4.2.6, 4.2.9 and 4.2.10 for $\mathcal{S}_1$ (see pages 90 and 92). In particular, let $\psi = \psi(x, y, z, p, q, \lambda)$ be a solution of (4.104) for which $\dddot{\psi} \neq 0$. Then we can proceed as follows.

[1] Eliminating λ from the system

$$\begin{cases} r = -2\psi + 2\lambda\dot{\psi} - \lambda^2\ddot{\psi}, \\ s = -\dot{\psi} + \lambda\ddot{\psi}, \\ t = -\ddot{\psi}, \end{cases} \tag{4.105}$$

we obtain a nonlinear involutive system $\mathcal{S}_2$.

[2] Eliminating λ from the system

$$\begin{cases} r = \lambda^2 t + 2\lambda\dot{\psi} - 2\psi, \\ s = -\lambda t - \dot{\psi}, \end{cases} \tag{4.106}$$

we obtain a Goursat equation $\mathcal{S}_1$, which is general because $\dddot{\psi} \neq 0$.

The choice of ψ is thus a bijective mapping between nonlinear involutive systems and general Goursat equations. $\square$

We will say that $\mathcal{S}_2$ is the involutive system associated to the general Goursat equation $\mathcal{S}_1$ and conversely.

Let's show some example.

Example 4.3.2 (Hilbert-Cartan)**.** From Example 4.1.14, let $\mathcal{S}_2$ be the nonlinear involutive system

$$\begin{cases} r = \dfrac{1}{3}t^3, \\ s = \dfrac{1}{2}t^2, \end{cases} \tag{4.107}$$

which is parameterized by $\psi = \frac{1}{6}\lambda^3$. In order to find the general Goursat equation associated to (4.107), let's write (4.106) for $\psi = \frac{1}{6}\lambda^3$

$$\begin{cases} r = \lambda^2 t + 2\lambda\dfrac{1}{2}\lambda^2 - \dfrac{1}{3}\lambda^3 = \lambda^2 t + \dfrac{2}{3}\lambda^3, \\ s = -\lambda t - \dfrac{1}{2}\lambda^2. \end{cases} \tag{4.108}$$

Combining (4.108), we have $3r+4s\lambda=-t\lambda^2$, then solving the second equation for λ, we obtain the general Goursat parabolic equation $\mathcal{S}_1$

$$(3r-6ts+2t^3)^2-4(t^2-2s)^3=0,$$

which can be written as

$$32s^3-12t^2s^2+9r^2-36rts+12rt^3=0. \tag{4.109}$$

Conversely, we can, for instance, write $\mathcal{S}_1$ in the form

$$r=E(x,y,z,p,q,s,t)=-\frac{2}{3}(t^3-3ts+(t^2-2s)^{3/2}).$$

Using (4.92), we can set $\lambda=-\frac{1}{2}E_s=-t-\sqrt{t^2-2s}$ and compute $\psi=-\dfrac{1}{2}(E+2s\lambda+t\lambda^2)$. After few steps, one finds $\psi=\frac{1}{6}\lambda^3$, as expected. ■

A generalization of the previous example is the following.

Example 4.3.3 (Goursat-Cartan)**.** Starting from Example 4.1.15, let $m\notin\left\{0,\frac{1}{2},1\right\}$, then

$$\begin{cases} r=\dfrac{t^{2m-1}}{2m-1}, \\ s=\dfrac{(-1)^m t^m}{m}, \end{cases} \tag{4.110}$$

is a nonlinear involutive system parameterized by $\psi=\frac{(m-1)^2}{m(2m-1)}\lambda^{(2m-1)/(m-1)}$. Plugging this function in (4.106) we obtain

$$\begin{cases} r=\lambda^2 t+2\lambda\dfrac{(m-1)\lambda^{(2m-1)/(m-1)}}{m\lambda}-\dfrac{2(m-1)^2}{m(2m-1)}\lambda^{(2m-1)/(m-1)}, \\ s=-\lambda t-\dfrac{(m-1)\lambda^{(2m-1)/(m-1)}}{m\lambda}, \end{cases} \tag{4.111}$$

which is the parametric expression of the general Goursat parabolic equation associated to (4.110). It is not an easy task to eliminate λ from (4.111). A little manipulation shows that λ is a root of

$$2tm\lambda^2+\lambda s+(1-2m)r=0. \tag{4.112}$$

If we take $m = 2$ then we have the previous Example 4.3.2. It can be shown that for $m \in \left\{\pm 2, \pm \frac{3}{2}\right\}$ the Cartan 2-tensor is identically zero and the general Goursat equations obtained are (contact) equivalent to (4.109).

4.4 Lifting $\mathbf{GR_3D_5}$ Pfaffian systems to a 6-manifold

In the previous sections 4.1 and 4.2 we saw how to canonically associate a GR_3D_5 Pfaffian system to both nonlinear involutive systems and general Goursat equations. Moreover, in Section 4.3 we presented a 1-to-1 mapping between these last two. In this section we will close the loop, showing how a nonlinear involutive system can be constructed starting from a GR_3D_5 Pfaffian system.

Roughly speaking, we outline the procedure for canonically lifting a GR_3D_5 Pfaffian System to a nonlinear involutive system of two PDE in the plane. We start from Cartan's considerations at [10, page 142].

Let I be a GR_3D_5 Pfaffian system on M_5, with local coordinates $(x^1, x^2, x^3, x^4, x^5)$. Assume that $\omega^1, \ldots, \omega^5$ is a local coframe on M_5 adapted to I and set

$$\begin{aligned} d\omega^1 &= a^1 \wedge \omega^1 + a^2 \wedge \omega^2 + \omega^3 \wedge \omega^4, \\ d\omega^2 &= b^1 \wedge \omega^1 + b^2 \wedge \omega^2 + \omega^3 \wedge \omega^5, \\ d\omega^3 &= c^1 \wedge \omega^1 + c^2 \wedge \omega^2 + c^3 \wedge \omega^3 + \omega^4 \wedge \omega^5, \end{aligned} \tag{4.113}$$

for some 1-forms a^i, b^i, and c^i, on M_5. If we consider M_5 immersed in $M_6 = M_5 \times \mathbb{R}$, where local coordinates $(x^1, \ldots, x^5, x^6)$ are chosen, then I is "lifted" to the rank-3 Pfaffian system $\tilde{I} = \{\omega^1, \omega^2, \omega^3\}$ on M_6. Our goal is to show that $\tilde{I}$ is generated by an involutive system of two PDE in the plane.

Proposition 4.4.1. The 1-form $\alpha_0 = \omega^1 + x^6\, \omega^2 \in \tilde{I}'$ has Engel-rank $\mathrm{Eng}(\alpha) = 2$.

Proof. By means of Proposition 2.2.8 (see page 14), all we need to prove is that $0 \neq \alpha_0 \wedge (d\alpha_0)^{[2]} = \alpha_0 \wedge d\alpha_0 \wedge d\alpha_0$, since for dimensional reasons we have $\alpha_0 \wedge (d\alpha_0)^{[2+1]} = 0$. Indeed, we have

$$\begin{aligned} d\alpha_0 &= d\omega^1 + x^6\, d\omega^2 + dx^6 \wedge \omega^2 \\ &= \left(a^1 + x^6\, b^1\right) \wedge \omega^1 + \left(a^2 + x^6\, b^2 + dx^6\right) \wedge \omega^2 + \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right) \\ &= \beta^1 \wedge \omega^1 + \beta^2 \wedge \omega^2 + \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right), \end{aligned} \tag{4.114}$$

where we define

$$\beta^1 = a^1 + x^6\, b^1, \quad \beta^2 = a^2 + x^6\, b^2 + dx^6.$$

In particular, (4.114) can be written as

$$d\alpha_0 = \beta^1 \wedge \alpha_0 + (\beta^2 - x^6\, \beta^1) \wedge \omega^2 + \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right), \tag{4.115}$$

which proves already that we have $\alpha_0 \in \tilde{I}'$. Let's continue our computations, starting from (4.114).

$$\begin{aligned} (d\alpha_0)^{[2]} &= \left(\beta^1 \wedge \omega^1 + \beta^2 \wedge \omega^2\right) \wedge \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right) \\ &= \left(\beta^1 \wedge \alpha_0 + (\beta^2 - x^6\, \beta^1) \wedge \omega^2\right) \wedge \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right). \end{aligned}$$

Therefore

$$\begin{aligned} \alpha_0 \wedge (d\alpha_0)^{[2]} &= \left(\beta^2 - x^6\, \beta^1\right) \wedge \omega^2 \wedge \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right) \\ &= \left(dx^6 + a^2 + x^6\, b^2 - x^6\, a^1 - {x^6}^2\, b^1\right) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right) \\ &= dx^6 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \left(\omega^4 + x^6\, \omega^5\right) + (\text{terms without } dx^6). \end{aligned} \tag{4.116}$$

Because by construction $\omega^1, \ldots, \omega^5$, and dx^6, are independent, equation (4.116) proves that $\alpha_0 \wedge (d\alpha_0)^{[2]} \neq 0$. □

Consequently, by the Pfaff normal form Theorem 2.3.1, we have the following.

Remark 4.4.2. There are five functionally independent functions x, y, z, p, q on M_6 such that

$$\alpha = dz - p\, dx - q\, dy \in \tilde{I}', \tag{4.117}$$

and $\alpha = f(x^1, \ldots, x^6)\,\alpha_0$. ■

Moreover, as a consequence of (4.115), we can write

$$d\alpha \equiv \alpha_0^1 \wedge \left(dx^6 + a^2 + x^6\, b^2 - x^6\, a^1 - {x^6}^2\, b^1\right) + \alpha_0^2 \wedge \left(\omega^4 + x^6\, \omega^5\right) \mod \alpha, \tag{4.118}$$

where $\alpha_0^1 = -f\,\omega^2$ and $\alpha_0^2 = f\,\omega^3$, thus $\tilde{I} = \{\alpha, \alpha_0^1, \alpha_0^2\}$.

Let's now define $\pi^1 = dx^6 + a^2 + x^6\, b^2 - x^6\, a^1 - {x^6}^2\, b^1$ and $\pi^2 = \omega^4 + x^6\, \omega^5$. Then we can write

$$d\alpha \equiv \alpha_0^1 \wedge \pi^1 + \alpha_0^2 \wedge \pi^2 \mod \alpha. \tag{4.119}$$

With computations similar to those in the proof of Proposition 4.4.1 and using (4.113), one proves that the rank-5 Pfaffian system $\Xi = \{\alpha, \alpha_0^1, \alpha_0^2, \pi^1, \pi^2\}$ is completely integrable and, because of (4.117), we have $\Xi = \{dx, dy, dz, dp, dq\}$ (see [10, page 142]). Consequently, there are four functions R,S,$\bar{S}$, and T, depending on $x^1, \ldots, x^6$, such that

$$\tilde{I} = \begin{cases} \alpha = dz - p\,dx - q\,dy, \\ \alpha^1 = dp - R\,dx - S\,dy, \\ \alpha^2 = dq - \bar{S}\,dx - T\,dy. \end{cases} \tag{4.120}$$

At this point, because the condition $\alpha \in \tilde{I}'$ is equivalent to the equation $d\alpha \wedge \alpha \wedge \alpha^1 \wedge \alpha^2 = 0$, we can conclude that $\bar{S} = S$.

Finally, we prove the following.

Theorem 4.4.3. *Let I be a GR_3D_5 Pfaffian system. Assume $\tilde{I}$ is the lifted rank-3 Pfaffian system on the 6-manifold M_6 constructed above. Then $\tilde{I}$ is locally generated by a nonlinear involutive system of two PDE in the plane.*

Proof. Continuing the above construction, assume that τ completes x, y, z, p, q, to a local coordinate system on M_6 and assume R, S, and T, are expressed in these new coordinates. Then we already proved that $\tilde{I}$ admits the local expression

$$\tilde{I} = \begin{cases} \alpha = dz - p\,dx - q\,dy, \\ \alpha^1 = dp - R\,dx - S\,dy, \\ \alpha^2 = dq - S\,dx - T\,dy. \end{cases} \tag{4.121}$$

Now, since we are on a 6-manifold and x, y, z, p, and q, are functionally independent, then the functions R, S, and T, are related by two independent equations. Without loss in generality we can assume $T = \tau$. Taking standard coordinates (x, y, z, p, q, r, s, t) on $J^2(\mathbb{R}^2, \mathbb{R})$, we can conclude that (4.121) is generated by a pair $\mathcal{S}_2$ of PDE in the plane

$$r = R, \quad s = S,$$

after the obvious mapping $x = x$, $y = y$, $z = z$, $p = p$, $q = q$, $t = \tau$.

Finally, because $\tilde{I}$ has a unique Cauchy characteristic, namely ∂_{x^6}, by Theorem 4.1.2 the system $\mathcal{S}_2$ is involutive, and because $\tilde{I}$ reduces to the GR_3D_5 Pfaffian system I then by Proposition 4.1.7 $\mathcal{S}_2$ is nonlinear. $\square$

In order for our lifting to be considered "canonical," we rephrase the following (unnumbered) Theorem at [10, page 144].

Remark 4.4.4. Let $\mathcal{S}_2$ and $\tilde{\mathcal{S}}_2$ be two nonlinear involutive systems of PDE in the plane. Assume that I and $\tilde{I}$ are the GR_3D_5 Pfaffian systems respectively associated to $\mathcal{S}_2$ and $\tilde{\mathcal{S}}_2$. Then $\mathcal{S}_2$ and $\tilde{\mathcal{S}}_2$ are contact equivalent if and only if I is transformed to $\tilde{I}$ by a change of coordinates.

In particular, different lifting of a fixed GR_3D_5 Pfaffian system will produce equivalent nonlinear involutive systems of PDE in the plane.

CHAPTER 5

NORMAL FORMS OF GR_3D_5 PFAFFIAN SYSTEMS WITH A 3-DIMENSIONAL TRANSVERSE AND FREE SYMMETRY ALGEBRA

In this chapter we provide normal forms for all GR_3D_5 Pfaffian systems with a transverse free-acting 3-dimensional symmetry algebra, see [15]. We shall use the classification of the 3-dimensional real Lie algebras given by Patera and Winternitz in [36]. We shall focus on two kinds of normal forms, one in coordinates related to a particular representation of the symmetry algebra and one related to general Monge equations. To obtain this latter kind, we shall use the Monge Algorithm 3.5.8, which will provide the Goursat normal form as well.

Section 5.1 is divided in two parts. In the first we introduce some new notations which will go along with those used in Chapters 2 and 3. Using these notations we describe the procedure that will take us first to the symmetry normal form and then to the general Monge normal form, for each algebraic type. The second part is dedicated to a more detailed formalization of our assumptions.

In Section 5.2 we relate the matrix realization of 3-dimensional Lie algebras that we shall use to the classification provided in [36].

From Section 5.3 through 5.10 we implement the forementioned procedure to obtain our normal forms.

5.1 Gauge formalism

The main purpose of this section is the description of the procedure that will lead us to the desired normal forms computed from Section 5.3 on. We formalize our assumptions and introduce some notations about moving frames, mainly following [34], [28], and [14].

Let I a GR_3D_5 Pfaffian system on the 5-dimensional manifold M and let $I = \{\theta^1, \theta^2, \theta^3\}$ be an adapted basis, that is, the derived flag of I is $I' = \{\theta^1, \theta^2\}$, $I'' = 0$. In particular, from Chapter 3 we know that the following ***derived flag conditions*** must be satisfied

$$d\theta^1 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = 0, \tag{5.1a}$$

$$d\theta^2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = 0, \tag{5.1b}$$

$$d\theta^3 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \neq 0, \tag{5.1c}$$

$$\big(k_1\, d\theta^1 + k_2\, d\theta^2\big) \wedge \theta^1 \wedge \theta^2 = 0 \Rightarrow k_1 = k_2 = 0. \tag{5.1d}$$

We then assume that on M is defined a free action of a 3-dimensional symmetry group G of I. This is equivalent to assume that about every point of M the action of G admits three linearly independent infinitesimal generators E_1, E_2, E_3 and that the Lie algebra associated to G, denoted $\mathfrak{g} = \langle E_1, E_2, E_3 \rangle$, is such that

$$\mathcal{L}_{\mathfrak{g}} I = \{\mathcal{L}_X \theta \mid X \in \mathfrak{g}, \quad \theta \in I\} \subseteq I.$$

Finally we assume that this action is transverse to I, that is, about every point of M the matrix $[F_i^j] = \big[E_i \lrcorner\, \theta^j\big]$ has maximum rank three, which in this case means $\det [F_i^j] \neq 0$.

This assumptions and their consequences are resumed in the following.

Proposition 5.1.1. Let I be a GR_3D_5 Pfaffian system on M. Assume the following are true.

[i] $\mathfrak{g}$ is a 3-dimensional symmetry algebra of I.

[ii] $\mathfrak{g}$ generates a free action on M by means of the group of local transformations G.

[iii] The action of G is transverse to I.

Then the following are true.

[Step 1] Locally $M = G \times \mathbb{R}^2$ and a local coordinate system on M is given by (a, b, c, u, v), where a, b and c provide a parametrization of G, and u, v are invariants of the action of G.

[Step 2] Let $\omega^i = A^i\, da + B^i\, db + C^i\, dc$ for $i = 1, 2, 3$ be the left-invariant Maurer-Cartan forms on G. Assume that $O = \{E_1, \ldots, E_n\}$ is an optimal list of 1-dimensional subalgebras of $\mathfrak{g}$. Then an adapted basis of I is

$$\theta^i = T^i_j(u, v)\, \omega^j + \alpha^i, \quad i = 1, 2, 3, \tag{5.2}$$

where $T^i_j(u, v)$ are such that $(I')^{\perp} \in O$ and $\alpha^i = U^i(u, v)\, du + V^i(u, v)\, dv$ are such that (5.1) hold.

[Step 3] Let the action of $\mathrm{U} \in G$ fix the conjugation-class $(I')^{\perp}$. Define $\mathrm{H} = \left(\alpha^1, \alpha^2, \alpha^3\right)^T$ and $\mathrm{K} = \left(\tau^1, \tau^2, \tau^3\right)^T$, where $\tau^i = \frac{\partial T^i_j}{\partial u}\, du + \frac{\partial T^i_j}{\partial v}\, dv$. Then we can use the vectorial mapping

$$g_{\mathrm{U}} : \mathrm{H} \to \bar{\mathrm{H}} = \mathrm{M}\mathrm{N}^{-1}\left(\mathrm{K} + \mathrm{M}^{-1}\mathrm{H}\right), \tag{5.3}$$

to reduce (5.2) to a normal form. Here M and N are defined by the change in the ω^i components exerted by U (see (5.5) and (5.6)). ■

In Section 5.2 we provide the optimal list of 1-dimensional subalgebras according to [36]. In each of the Sections 5.3 to 5.10 we will follow **[Step 1]** to **[Step 3]** in order to arrive at the symmetry normal forms, then we will proceed with the following.

[Step 4] Apply the Monge Algorithm 3.5.8 to change the symmetry normal form obtained at **[Step 3]** in to a general Monge normal form.

For the convenience of the reader, we resumed these normal forms in Theorems 3 and 4.

Proposition 5.1.1 and its prove can be formalized in the context of connections on fiber bundles, which are defined by expression like (5.2), then (5.3) can be interpreted as a gauge transformation. We call (5.3) an ***admissible gauge transformation***. In some case, like for the abelian type $3A_1$, the exploitation of the gauge transformation to reduce the connection to a normal form is not complete, meaning that there still could be some gauge transformation which leave the normal form invariant and which will change only the expression of generic function. We called this the residual freedom in gauging. We did not explore this part.

The next paragraphs are intended to roughly prove Proposition 5.1.1. We refer to [14, Appendix 1] for details.

To start, about every point of M there are defined two functionally independent invariants of the action of G, that is, locally we have $M = G \times \mathbb{R}^2$ (M is a trivial principal fiber bundle). Thus about every point of M there are defined coordinates (a, b, c, u, v), where u and v are invariant under the action of G (see [34, Theorem 2.18]). This will allow us to define a canonical coframe on M which will be used to provide our normal forms for I.

For every 3-dimensional algebraic type $[\mathfrak{g}]$ in [36], we shall provide a matrix realization, that is, we will consider every G and $\mathfrak{g}$ respectively a Lie subgroup of $\mathrm{GL}(\mathbb{R}, 3)$ and a Lie subalgebra of $\mathfrak{gl}(\mathbb{R},3)$. In this way we will provide a canonical local coframe on M. Indeed, because locally $M = G \times \mathbb{R}^2$ we can consider the action of G on M as exerted by matrix multiplication. More precisely, we denote the right action of G by

$$\rho : (\mathrm{U}; \mathrm{A}, u, v) \in G \times M \to \rho_{\mathrm{U}}(\mathrm{A}, u, v) = (\mathrm{AU}, u, v) \in M,$$

and the left action by

$$\lambda : (\mathrm{U}; \mathrm{A}, u, v) \in G \times M \to \lambda_{\mathrm{U}}(\mathrm{UA}, u, v) = (\mathrm{AU}, u, v) \in M.$$

We will use the same notation for the actions of G on itself. In this case, we let $\mathfrak{g}_\rho$ and $\mathfrak{g}_\lambda$ be respectively the algebras of the right-invariant vector fields and of the left-invariant vector fields (which are isomorphic).

Now, let (a, b, c) be a coordinate system on the Lie group G. Then a generic 3×3 matrix $\mathrm{A} \in G$ depends on the three parameters a, b, c. We easily check that the matrix of 1-forms $\Omega_{\mathrm{LMC}} = \mathrm{A}^{-1}\, d\mathrm{A}$ has left-invariant 1-forms as entries. Indeed take a $\mathrm{U} \in G$ (for which the coordinates are given), then we have

$$\begin{aligned}\lambda_{\mathrm{U}}^{*}\Omega_{\mathrm{LMC}} &= (\mathrm{UA})^{-1}\ d(\mathrm{UA}) = \mathrm{A}^{-1}\mathrm{U}^{-1}\ (d\mathrm{U}\,\mathrm{A} + \mathrm{U}\,d\mathrm{A}) = \mathrm{A}^{-1}\mathrm{U}^{-1}\mathrm{U}\,d\mathrm{A} \\ &= \mathrm{A}^{-1}\,d\mathrm{A} = \Omega_{\mathrm{LMC}}.\end{aligned}$$

It follows that the entries of Ω_{LMC} must be linear combinations of three left-invariant 1-forms $\omega^1, \omega^2, \omega^3 \in \Omega^*(G)$, which are called ***Maurer-Cartan forms***. These forms satisfy the structure equations (*equations of Maurer-Cartan*)

$$d\omega^i = -\frac{1}{2} c^i_{j,k} \omega^j \wedge \omega^k,$$

where the $c^i_{j,k}$ are the structure constants of $\mathfrak{g}$. Let R_1, R_2, R_3 be the vector fields dual the Maurer-Cartan forms, that is $R_i \rightharpoonup \omega^j = \delta^i_j$, then the R_i are infinitesimal generators of the right action of G on itself, that is, $\langle R_1, R_2, R_3 \rangle = \mathfrak{g}_\lambda$. Let I_3 be the unit matrix in G and let X be a vector field on G. Then $\exp(tX) = \mathrm{Fl}^X_t(\mathrm{I}_3)$ is a 1-parameter subgroup of G (see [34, Proposition 1.48]). In particular

G is generated by the three subgroups $\mathrm{U}_i = \mathrm{U}_i(t_i) = \exp(t_i\, R_i)$, for $i = 1, 2, 3$ and $t_i \in \mathbb{R}$, and right-action of the R_i on G is exerted by the multiplication on the right by U_i. In particular t_1, t_2 and t_3 constitute a coordinate system on G.

If we set $\mathrm{V}_i = \mathrm{V}_i(t_i) = \mathrm{U}_i(-t_i) = \mathrm{U}_i^{-1}$, then the multiplication on the left by the V_i defines the left action canonically associated to the right action of the R_i. The infinitesimal generators of this left action are vectors fields L_1, L_2, L_3 on G such that $\exp(t_i\, L_i) = \mathrm{V}_i(t_i)$ and

$$[R_j, R_k] = c^i_{j,k}\, R_i,$$

$$[L_j, L_k] = c^i_{j,k}\, L_i,$$

$$[L_i, R_j] = 0,$$

for $i, j, k \in \{1, 2, 3\}$. In particular, the vector fields L_i are right-invariant and thus $\mathfrak{g} \simeq \mathfrak{g}_\rho = \langle L_1, L_2, L_3\rangle$. Most important, we have $\mathcal{L}_{L_i}\omega^j = 0$, that is $\mathfrak{g}$ is a symmetry algebra of the Maurer-Cartan forms.

At this point we go back to M and use the Maurer-Cartan forms ω^1, ω^2, ω^3.

Remark 5.1.2. On M we can define a local coframe θ^1, θ^2, θ^3, du, dv adapted to I by setting

$$\theta^i = T^i_j(u, v)\, \omega^j + U^i(u, v)\, du + V^i(u, v)\, dv, \quad i = 1, 2, 3. \tag{5.4}$$

The Lie algebra of right-invariant vector fields $\mathfrak{g} = \langle L_1, L_2, L_3\rangle$ is a 3-dimensional transverse symmetry algebra of I which acts freely on M. ■

In the following part of this section we focus on two points. First, the possible choices of the functions $T^i_j(u, v)$, $U^i(u, v)$ and $V^i(u, v)$ which gives rise to a GR_3D_5 Pfaffian system, that is, for which conditions (5.1) are satisfied. Second, how the action of G is going to change the forms (5.4).

The first point consists of two problems, that are, the construction of the adapted basis (5.4) and the analysis of the inequivalent (under the G-action) Pfaffian systems generated by (5.4). We shall start considering this latter problem.

According to Remark 5.1.2 two Pfaffian systems I and J with bases as in (5.4) are equivalent if and only if there is $\mathrm{U} \in G$ such that $\rho^*_{\mathrm{U}} J = I$. Because we assume I and J to be GR_3D_5 Pfaffian systems, the derived systems I' a J' are rank-2 invariant subsystems (see Section 3.4). This means that I and J are equivalent if and only if there is $\mathrm{U} \in G$ such that $\rho_{\mathrm{U}}^*(J') = I'$, that is, if and only if $\rho_{\mathrm{U}*}(J')^{\perp} = (I')^{\perp}$. Because $\dim I' = 2$, $(I')^{\perp}$ is a 1-parameter subalgebra of $\mathfrak{g}$. We can conclude that I and J are equivalent if and only if $(I')^{\perp}$ and $(J')^{\perp}$ are conjugated. Finally, then, we can say that inequivalent GR_3D_5 Pfaffian systems are associated to non-conjugate 1-parameter subalgebras of $\mathfrak{g}$. We denote by $O = \{E_1, \ldots, E_n\}$ an ***optimal list of 1-dimensional subalgebras*** of $\mathfrak{g}$, that is, O is a maximal set of representative of conjugation-classes of vectors.

The next task is then to check which vectors $E_i \in O$ give rise to GR_3D_5 Pfaffian systems I such that $E_i \in (I')^{\perp}$. Now, the condition $E_i \in (I')^{\perp}$ applied to (5.4) is equivalent to a choice of functions $T^i_j(u,v)$. The derived flag conditions (5.1) will provide constraints on the functions $U^i(u,v)$ and $V^i(u,v)$.

Now we are going to derive (5.3). The idea is to consider those 1-parameter subgroups which fix the conjugation class $E_i \in (I')^{\perp}$. We recall that the conjugation map defined by $\mathrm{U} \in G$ is the automorphism

$$\mathrm{conj}_{\mathrm{U}} : \mathrm{A} \in G \to \mathrm{UAU}^{-1} \in G.$$

Consequently $\mathrm{conj}_{\mathrm{U}}(\mathrm{A}) = \lambda_{\mathrm{U}}\rho_{\mathrm{U}^{-1}}(\mathrm{A}) = \rho_{\mathrm{U}^{-1}}\lambda_{\mathrm{U}}(\mathrm{A})$, and thus the conjugation class of a left-invariant vector field E_i is fixed only by $\mathrm{U} \in G$ such that $\rho_{\mathrm{U}^{-1}*}X = X$. That is way we now consider the right-action of G.

The matrix Ω_{LMC} must have at least three linearly independent entries $\Omega^{i,j}_{\mathrm{LMC}}$, with $(i,j) \in J_0 = [(i_1,j_1),(i_2,j_2),(i_3,j_3)]$ a list of three ordered pairs of indices, such that the ω^i are linear combinations of the $\Omega^{i,j}_{\mathrm{LMC}}$ (see [35, page 73]). Therefore we can build the 3×3 matrix of 1-forms Ω whose entries are zero 1-forms except for the J_0 entries, for which we define $\Omega^{i_k,j_k} = \omega^k$. In this way we can say that for some $\mathrm{A}_0, \mathrm{B}_0 \in \mathrm{GL}(\mathbb{R},3)$ we have $\Omega_{\mathrm{LMC}} = \mathrm{A}_0\Omega\mathrm{B}_0$. Let $\mathrm{U} = \mathrm{U}_i(t_i)$ be a 1-parameter subgroup of G, then $dU \neq 0$ and

$$\begin{aligned}\rho_{\mathrm{U}}^*\Omega_{\mathrm{LMC}} &= (\mathrm{AU})^{-1}\, d(\mathrm{AU}) = \mathrm{U}^{-1}\mathrm{A}^{-1}\ (d\,\mathrm{A}\,\mathrm{U} + \mathrm{A}\,d\mathrm{U})\\ &= \mathrm{U}^{-1}\,\Omega_{\mathrm{LMC}}\,\mathrm{U} + \mathrm{U}^{-1}\,d\mathrm{U}.\end{aligned}$$

Consequently we have

$$\begin{aligned} \rho_{\mathrm{U}}^{*}\Omega &= \rho_{\mathrm{U}}{}^{*}\left(\mathrm{A}_0^{-1}\Omega_{\mathrm{LMC}}\mathrm{B}_0^{-1}\right) = \mathrm{A}_0^{-1}\left(\rho_{\mathrm{U}}{}^{*}\Omega_{\mathrm{LMC}}\right)\mathrm{B}_0^{-1} = \mathrm{A}_0^{-1}\left(\mathrm{U}^{-1}\,\Omega_{\mathrm{LMC}}\,\mathrm{U} + \mathrm{U}^{-1}\,d\mathrm{U}\right)\mathrm{B}_0^{-1} \\ &= \mathrm{A}_0^{-1}\mathrm{U}^{-1}\,\Omega_{\mathrm{LMC}}\,\mathrm{U}\mathrm{B}_0^{-1} + \mathrm{A}_0^{-1}\mathrm{U}^{-1}\,d\mathrm{U}\,\mathrm{B}_0^{-1}. \end{aligned} \tag{5.5}$$

From (5.5), setting $\mathrm{A}_1 = \mathrm{A}_0^{-1}\mathrm{U}^{-1}$, $\mathrm{B}_1 = \mathrm{U}\mathrm{B}_0^{-1}$ and $\mathrm{K} = \mathrm{A}_1\,d\mathrm{U}\,\mathrm{U}^{-1}\mathrm{B}_1$, we see that Ω is changed by ρ_{U}^{*} as subject to an "affine transformation" $\rho_{\mathrm{U}}^{*}\Omega = \mathrm{A}_1\Omega\mathrm{B}_1 + \mathrm{K}$. We can look at the J_0 entries of both Ω and $\tilde{\Omega} = \rho_{\mathrm{U}}^{*}\Omega$ and define the column vectors of 1-forms $\vec{\Omega} = \left(\omega^1,\omega^2,\omega^3\right)^T$ and $\vec{\tilde{\Omega}}$ such that $\vec{\Omega}^k = \Omega^{i_k j_k} = \omega^k$ and $\vec{\tilde{\Omega}}^k = \tilde{\Omega}^{i_k j_k}$. The J_0 entries of the matrix $\bar{\Omega} = \mathrm{A}_1\Omega\mathrm{B}_1$ are combinations of $\omega^1,\omega^2,\omega^3$ and we could represent these entries as a column vector $\vec{\bar{\Omega}}$ of 1-forms $\vec{\bar{\Omega}}^k = \bar{\Omega}^{i_k j_k}$, in such a way that $\vec{\bar{\Omega}} = \mathrm{N}\vec{\Omega}$ for some $\mathrm{N} \in \mathrm{GL}(\mathbb{R},3)$ depending on the 1-parameter subgroup U and the matrices $\mathrm{A}_0, \mathrm{B}_0$. The J_0 entries of the matrix $\mathrm{K} = \mathrm{A}_0^{-1}\mathrm{U}^{-1}\,d\mathrm{U}\,\mathrm{B}_0^{-1}$ are 1-forms (depending on the 1-parameter subgroup U and the matrices $\mathrm{A}_0, \mathrm{B}_0$), which can be represented by a column vector $\vec{\mathrm{K}} = \left(\tau^1,\tau^2,\tau^3\right)^T$ such that $\tau^k = \mathrm{K}^{i_k j_k}$. Therefore we can describe the action of ρ_{U}^{*} on Ω as the vectorial mapping

$$g_U : \vec{\Omega} \to \vec{\tilde{\Omega}} = \vec{\bar{\Omega}} + \vec{\mathrm{K}} = \mathrm{N}\vec{\Omega} + \vec{\mathrm{K}} \tag{5.6}$$

for which we will drop the arrow notation when this does not lead to confusion.

We considered an optimal list 1-dimensional subalgebras of $\mathfrak{g}$, $O = \{E_1,\ldots,E_n\}$. Then for any $E \in O$ we have $E = a^j\,R_j$ for some (column) vector $\vec{a} = \left(a^1,a^2,a^3\right)^T \in \mathbb{R}^3 - \{0\}$. We can now consider the annihilator of E on G. This is a 2-dimensional space of forms on G, say $E^{\perp} = \langle\theta^1,\theta^2\rangle$, such that $E \lrcorner\, \theta^j = 0$ $(j = 1,2)$. We can complete θ^1,θ^2 to a coframe on G by choosing a 1-form θ^3 such that $E \lrcorner\, \theta^3 = 1$. Since the Maurer-Cartan forms are another coframe on G, then there is a matrix $\mathrm{M} = \left(T^i_j\right) \in \mathrm{GL}\left(\mathbb{R},3\right)$ such that $\theta^i = T^i_j\,\omega^j$. Introducing the vectorial notation $\Theta_V = \left(\theta^1,\theta^2,\theta^3\right)^T$ and $\Omega = \left(\omega^1,\omega^2,\omega^3\right)^T$ then $\Theta_V = \mathrm{M}\Omega$. Because $E \lrcorner\, \theta^i = a^h\,R_h \lrcorner\, m^i_k\,\omega^k = a^h m^i_k\,R_h \lrcorner\, \omega^k = a^h m^i_k \delta^k_h = m^i_h a^h$, the definition of the θ^i implies $M\vec{a} = (0,0,1)^T$.

To obtain our connection (5.2), we have to add to Θ_V an horizontal component $\mathrm{H} = \left(\alpha^1,\alpha^2,\alpha^3\right)^T$ and consider $T^i_j = T^i_j(u,v)$. Then (5.2) has the vectorial expression $\Theta = \Theta_V + \mathrm{H} = \mathrm{M}\Omega + \mathrm{H}$. A gauge transformation (5.6) will then act on our connection as

$$\Theta \to \tilde{\Theta} = \mathrm{M}\tilde{\Omega} + \mathrm{H} = \mathrm{M}\left(\mathrm{N}\Omega + \mathrm{K}\right) + \mathrm{H} = \mathrm{MN}\Omega + \left(\mathrm{M\,K} + \mathrm{H}\right)$$

We can then normalize $\tilde{\Theta}$ in order to get just a transformation of the horizontal component, that is, we can write this transformation as $\Theta \to \bar{\Theta} = M\Omega + \bar{\Gamma}$. Indeed, we take

$$\begin{aligned}\bar{\Theta} &= \mathrm{M}\left(\mathrm{MN}\right)^{-1}\tilde{\Theta} = \mathrm{M}\Omega + \mathrm{M}\left(\mathrm{MN}\right)^{-1}\left(\mathrm{M\,K} + \mathrm{H}\right) = \mathrm{M}\Omega + \mathrm{MN}^{-1}\mathrm{M}^{-1}\left(\mathrm{M\,K} + \mathrm{H}\right)\\ &= \mathrm{M}\Omega + \mathrm{MN}^{-1}\left(\mathrm{K} + \mathrm{M}^{-1}\mathrm{H}\right)\end{aligned}$$

which defines (5.3).

5.2 The list of the 3-dimensional real Lie algebras

We used the classification of the 3-dimensional real Lie algebras given in [36]. In this article, Patera and Winternitz labeled a basis by e_1, e_2, e_3 and reported the structure equations. In this dissertation we preferred to change basis to some E_1, E_2, E_3, for which the structure equation simplified our computations. In this section we provide the change of basis (if any) we applied and then we compare the optimal list of 1-dimensional subalgebras O_0 of [36] with the "specialized" one O we are going to use. The parameters used here are ϕ, h, t and s, with $0 \leq \phi < \pi$, $h = \pm 1$ and $t, s \in \mathbb{R}$, while in [36] they use $h = \epsilon$, $t = x$ and $s = y$.

Finally, there are two continuous families of algebras depending on a parameter a in [36], namely $A^a_{3,5}$ and $A^a_{3,7}$. In this dissertation we set $\epsilon = a$ and write these families as $A^\epsilon_{3,5}$ and $A^\epsilon_{3,7}$.

[$3A_1$] This is the abelian Lie algebra. There is none change of basis, $E_i = e_i$, and the optimal lists are

$$O_0 = \left[E_3, E_2 + t\,E_3, E_1 + t\,E_2 + s\,E_3\right], \quad t, s \in \mathbb{R},$$

$$O = \left[E_1, E_2, E_3, E_1 + t\,E_2, E_1 + t\,E_3, E_2 + t\,E_3, E_1 + t\,E_2 + s\,E_3,\right], \quad t \neq 0, s \neq 0.$$

$[A_1 \oplus A_2]$ This is the decomposable solvable Lie algebra. We perform the change of basis is $E_1 = e_2$, $E_2 = -e_1$, $E_3 = e_3$ and the optimal lists are

$$O_0 = [E_1, E_1 + h\, E_3, -\cos\phi\, E_2 + \sin\phi\, E_3]\,,$$

$$O = [E_3, E_1 + t\, E_3, E_2 + t\, E_3]\,.$$

$[A_{3,1}]$ The nilpotent Lie algebra (Heisenberg). There is not a change of basis, $E_i = e_i$, and the optimal lists are

$$O_0 = [E_1, \cos\phi\, E_2 + \sin\phi\, E_3]\,,$$

$$O = [E_1, E_2, E_3, E_2 + t\, E_3]$$

$[A_{3,2}]$ This is the parameter-free solvable Lie algebra. We did not change basis, $E_i = e_i$, and the optimal lists are

$$O_0 = O = [E_1, E_2, E_3]$$

$[A^{\epsilon}_{3,5}]$ This is a 1-parameter family of solvable Lie algebras. While in [36] they set $0 < |\epsilon| < 1$, here we let be $\epsilon = \pm 1$, so that $A^1_{3,5} = A_{3,3}$ and $A^{-1}_{3,5} = A_{3,4}$. We did this because in each case the connections were the same. There is not a change of basis, $E_i = e_i$, and the optimal lists are

$$O^1_0 = [E_3, \cos\phi\, E_1 + \sin\phi\, E_2]\,,$$

$$O^{-1}_0 = O^{\epsilon}_0 = [E_1, E_2, E_3, E_1 + h\, E_2]\,,$$

$$O = [E_1, E_2, E_2 + t\, E_1, E_3]\,, \quad t \neq 0.$$

Note that in O^1_0 the vectors E_1, E_2 and $\cos\phi\, E_1 + \sin\phi\, E_2$ are conjugated.

$[A^{\epsilon}_{3,7}]$ This is the other 1-parameter family of solvable Lie algebras. While in [36] they let $\epsilon > 0$, here we allow $\epsilon = 0$, so that $A^0_{3,7} = A_{3,6}$. We did not change basis, $E_i = e_i$, and the optimal lists are

$$O^{\epsilon}_0 = O = [E_1, E_3]$$

$[A_{3,8}]$ This is the semisimple Lie algebra $\mathfrak{sl}_2$. The change of basis is $E_1 = -2\,e_2$, $E_2 = e_1$, and $E_3 = e_3$. The optimal lists are

$$O_0 = [E_2, E_1, E_2 + E_3]\,,$$

$$O = [E_1, E_2, E_2 - E_3]\,.$$

$[A_{3,9}]$ This is the semisimple Lie algebra $\mathfrak{so}_3$. We did not change basis, $E_i = e_i$, and the optimal lists are

$$O_0 = [E_1]\,, \quad O = [E_3]\,.$$

5.3 $\mathbf{3A_1}$: the abelian algebra

5.3.1 Step 1

$3A_1$ is the Abelian Lie algebra with structure equations

$$[E_1, E_2] = [E_1, E_3] = [E_2, E_3] = 0.$$

We realize this algebra as associated to the group $G = D(3)$ of the diagonal 3×3 matrices

$$A = \begin{pmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \\ 0 & 0 & e^c \end{pmatrix}. \tag{5.7}$$

The (left invariant) Maurer-Cartan forms are

$$\omega^1 = d\,a, \quad \omega^2 = d\,b, \quad \omega^3 = d\,c. \tag{5.8}$$

Their structure equations are

$$d\omega^1 = d\omega^2 = d\omega^3 = 0. \tag{5.9}$$

The left-invariant vector fields are

$$R_1 = \partial_a, \quad R_2 = \partial_b, \quad R_3 = \partial_c.$$

The generators of the associated left action are

$$L_1 = -\partial_a, \quad L_2 = -\partial_b, \quad L_3 = -\partial_c. \tag{5.10}$$

5.3.2 Step 2

Proposition 5.3.1. An adapted basis for I can only be given by

$$I = \begin{cases} \theta^1 = t\,\omega^1 - \omega^2 + \alpha^1, \\ \theta^2 = s\,\omega^1 - \omega^3 + \alpha^2, \\ \theta^3 = \omega^1 + \alpha^3, \end{cases} \tag{5.11}$$

where $t = t(u, v) \neq 0$, $s = s(u, v) \neq 0$ and

$$d\alpha^1 = dt \wedge \alpha^3 \tag{5.12a}$$

$$d\alpha^2 = ds \wedge \alpha^3 \tag{5.12b}$$

$$d\alpha^3 \neq 0 \tag{5.12c}$$

$$dt \wedge ds \neq 0 \tag{5.12d}$$

Proof. An optimal list of 1-dimensional subalgebras is

$$O = [E_1, E_2, E_3, E_1 + t\,E_2, E_1 + t\,E_3, E_2 + t\,E_3, E_1 + t\,E_2 + s\,E_3,], \quad t \neq 0, \quad s \neq 0.$$

Considering $R_i = E_i$, the choices of $E_i = (I')^{\perp}$ produce the following seven cases.

$$\begin{aligned}
&\text{Case 1:} \quad \theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.\\
&\text{Case 2:} \quad \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.\\
&\text{Case 3:} \quad \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^2 + \alpha^2, \quad \theta^3 = \omega^3 + \alpha^3.\\
&\text{Case 4:} \quad \theta^1 = \omega^3 + \alpha^1, \quad \theta^2 = t\,\omega^1 - \omega^2 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.\\
&\text{Case 5:} \quad \theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = t\,\omega^1 - \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.\\
&\text{Case 6:} \quad \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = t\,\omega^2 - \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.\\
&\text{Case 7:} \quad \theta^1 = t\,\omega^1 - \omega^2 + \alpha^1, \quad \theta^2 = s\,\omega^1 - \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.
\end{aligned}$$

Here $\alpha^i = U^i(u,v)\,du + V^i(u,v)\,dv$, $t = t(u,v) \neq 0$ and $s = s(u,v) \neq 0$.

In Cases 1 to 6 we have $I'' \neq 0$, that is condition (5.1d) is not satisfied. Concerning Case 7, by the definition of the optimal list we have $t = t(u,v) \neq 0$ and $s = s(u,v) \neq 0$. Moreover, the first three derived flag conditions in (5.1) are respectively equivalent to $d\alpha^1 = dt \wedge \alpha^3$, $d\alpha^2 = ds \wedge \alpha^3$ and $d\alpha^3 \neq 0$. Finally, condition (5.1d), because we have $d\alpha^3 \neq 0$, is equivalent to $dt \wedge ds \neq 0$. Hence we have (5.12). $\square$

5.3.3 Step 3

Proposition 5.3.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $3A_1$. Then about each point of M there are local coordinates (a,b,c,u,v) such that $\Gamma = \langle -\partial_a,\, -\partial_b,\, -\partial_c \rangle$ and

$$I = \begin{cases} \theta^1 = u\,da - db, \\ \theta^2 = v\,da - dc + F\,du, \\ \theta^3 = da + F_v\,du, \end{cases} \tag{5.13}$$

where $F = F(u,v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.14}$$

Proof. Any gauge transformation is admissible since the adjoint action is trivial. Setting $U = U_3U_2U_1$ with $U_i = \exp(f_i(u,v)\,R_i)$, we have

$$U = \begin{pmatrix} e^{f_1(u,v)} & 0 & 0 \\ 0 & e^{f_2(u,v)} & 0 \\ 0 & 0 & e^{f_3(u,v)} \end{pmatrix}. \tag{5.15}$$

From (5.3), the effect on $\alpha^1, \alpha^2, \alpha^3$ is

$$\left[\begin{aligned} &\alpha^1 \to \bar{\alpha}^1 = \alpha^1 + t\,d\,f_1 - d\,f_2, \\ &\alpha^2 \to \bar{\alpha}^2 = \alpha^2 + s\,d\,f_1 - d\,f_3, \\ &\alpha^3 \to \bar{\alpha}^3 = \alpha^3 + d\,f_1. \end{aligned}\right. \tag{5.16}$$

The whole (5.12) is easily seen to be unchanged by (5.16). Now we just have to use (5.16) in order to simplify the expressions of α^1, α^2, and α^3.

Because of (5.12d), that is, $d\,t \wedge d\,s \neq 0$, there is a change of coordinates $(a,b,c,u,v) \to (a,b,c,\tilde{u},\tilde{v})$ such that $t(u,v) = \tilde{u}$ and $s(u,v) = \tilde{v}$. Rewriting (5.11) and (5.12) in terms of these new coordinates and dropping the tildes, we have

$$I = \begin{cases} \theta^1 = u\,\omega^1 - \omega^2 + \alpha^1, \\ \theta^2 = v\,\omega^1 - \omega^3 + \alpha^2, \\ \theta^3 = \omega^1 + \alpha^3. \end{cases} \tag{5.17}$$

where the α^i satisfy the following relations (we used (5.12d))

$$d\alpha^1 = d\,u \wedge \alpha^3, \tag{5.18a}$$

$$d\alpha^2 = d\,v \wedge \alpha^3, \tag{5.18b}$$

$$d\alpha^3 \neq 0. \tag{5.18c}$$

These are preserved by any gauge transformation (5.16) with mappings

$$\left[\begin{array}{l} \alpha^1 \to \bar{\alpha}^1 = \alpha^1 + u\,d\,f_1 - d\,f_2, \\ \alpha^2 \to \bar{\alpha}^2 = \alpha^2 + v\,d\,f_1 - d\,f_3, \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + d\,f_1. \end{array}\right. \tag{5.19}$$

Using (5.19), we can choose $f_1(u,v)$, $f_2(u,v)$ and $f_3(u,v)$ such that $\bar{\alpha}^1 = 0$, $\bar{\alpha}^2 = F(u,v)\,d\,u$ and $\bar{\alpha}^3 = C_1(u,v)\,d\,u + C_2(u,v)\,d\,v$, for arbitrary functions $F(u,v)$, $C_1(u,v)$ and $C_2(u,v)$ subject to the constraints given by (5.18).

From (5.18b) we find $C_1 = F_v$ and from (5.18a) we get $C_2 = 0$. Consequently (5.18c) is equivalent to $F_{vv} \neq 0$.

Finally, the GR_3D_5 Pfaffian system I can be written as

$$I = \begin{cases} \theta^1 = u\,\omega^1 - \omega^2 \\ \theta^2 = v\,\omega^1 - \omega^3 + F\,du \\ \theta^3 = \omega^1 + F_v\,du \end{cases} \tag{5.20}$$

with $F = F(u,v)$ subject to the constraint $F_{vv} \neq 0$. We express (5.20) in the local coordinates used at (5.8) and thus we obtain (5.13). $\square$

In this case we can consider the residual freedom in gauge. Using (5.7) and (5.15), the gauge transformations $\tau : M \to M$ are defined by

$$\bar{a} = a + f_1(u,v), \quad \bar{b} = b + f_2(u,v), \quad \bar{c} = c + f_3(u,v), \quad \bar{u} = u, \quad \bar{v} = v.$$

Considering (5.19), we want to preserve our normal form by setting

$$\left[\begin{array}{r} \tau^*\left(0\,d\bar{u}\right) = u\,df_1 - df_2 \\ \tau^*\left(\bar{G}\,d\bar{u}\right) = F\,du + v\,df_1 - df_3 \\ \tau^*\left(\bar{G}_{\bar{v}}\,d\bar{u}\right) = F_v\,du + df_1 \end{array}\right.$$

that is f_1, f_2 and f_3 must be solutions of the system

$$\left[\begin{aligned} 0 &= u\,df_1 - df_2 \\ G\,du &= F\,du + v\,df_1 - df_3 \\ G_v\,du &= F_v\,du + df_1 \end{aligned}\right. \tag{5.21}$$

where $G = \bar{G} \circ \tau$. Therefore there are two arbitrary functions in one variable $f(u)$ and $g(u)$ such that

$$\left[\begin{aligned} & f_1 = f' \\ & f_2 = uf' - f \\ & f_3 = g \\ & G = F + vf'' - g' \end{aligned}\right. \tag{5.22}$$

5.3.4 Step 4

Using the Monge Algorithm 3.5.8, we prove the following.

Proposition 5.3.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely. Denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $3A_1$. Then $\Gamma = \langle \partial_X, \partial_Y, \partial_Z \rangle$ and

$$Z_1 = h\left(Y_1, Y_2\right), \text{ with } h_{Y_2 Y_2} \neq 0. \tag{5.23}$$

Proof. By Theorem 3.5.7 the general Monge equation $Z_1 = H(X, Y, Z, Y_1, Y_2)$ defines a GR_3D_5 Pfaffian system I on the manifold N with local coordinates (X, Y, Z, Y_1, Y_2). By Proposition 5.3.2 we have local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle -\partial_a, -\partial_b, -\partial_c \rangle$ and

$$I = \begin{cases} \eta^1 = u\,da - db, \\ \eta^2 = v\,da - dc + F\,du, \\ \eta^3 = da + F_v\,du, \end{cases} \tag{5.24}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.25}$$

Now we just have to apply steps **[0]** to **[4]** of the Monge Algorithm 3.5.8 to obtain (5.23).

[0] By construction, (5.24) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] In I' we must find a 1-form θ^1 with Engel-rank $\operatorname{Eng}\big(\theta^1\big) = 1$ and then provide local coordinates $(y^1, \ldots, y^5)$ such that $\theta^1 = dy^2 - y^3\, dy^1$. With respect to the change of coordinates $\tau_1 : N \to N$ defined by

$$y^1 = a, \quad y^2 = b, \quad y^3 = u, \quad y^4 = c, \quad y^5 = v,$$

we have ${\theta^1}^* \theta^1 = db - u\, da = -\eta^1$, and in particular

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\, dy^1, \\ \\ \eta^2 = y^5\, dy^1 - dy^4 + F\, dy^3, \end{cases} \tag{5.26}$$

where we set $F = F(y^3, y^5) = F(u, v)$, and thus $F_{y^5 y^5} \neq 0$.

[2] In view of Proposition 3.3.3 and Remark 3.3.4, in this step we want to construct a 1-form $\theta^2 = dU - V^3\, dy^3 - V^1\, dy^1 = W\, \theta^1 + A\, \eta^2 \in I'$. The dy^4 and dy^5 components of η^2 are respectively $Y^4 = -1$ and $Y^5 = 0$. The PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = -A, \\ \frac{\partial U}{\partial y^5} &= AY^5 = 0. \end{aligned} \tag{5.27}$$

We see that for $A = -1$ we have the solution $U = y^4$. Remembering the notation $\eta^2 = Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5$ used in Proposition 3.3.3, the expressions (3.48) become

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = 0 + F = F, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3(U_2 - AY_0^2) - AY_0^1 = y^5. \end{aligned} \tag{5.28}$$

Using U and V^3 we define a new change of coordinates $\tau_2 : N \to N$

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = y^4, \quad x^5 = F.$$

Note that the determinant of the Jacobian matrix of τ_2, that is, $|\mathrm{J}_{\tau_2}| = F_{y^5} = F_v$, is non-vanishing by (5.14). The diffeomorphism τ_2 defines the mapping

$$x_3 = y^3, \quad x_5 = F(y^3, y^5),$$

and by the inverse function theorem we can defined $f = f\,(x_3, x_5)$ on N such that

$$f \circ \tau_2 = -y^5.$$

Consequently we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \end{cases} \tag{5.29}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5 x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\,dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases}$$

[4] Here we obtain the Monge normal form. In view of (3.84), we set

$$\tilde{H} = x^5 f_{x^5} - f = \tilde{H}\,(x_3, x_5)\,,$$

and define the change of coordinates $\tau_4 : N \to N$ by

$$X = x_1, \quad Y = x_2, \quad Z = x_4, \quad Y_1 = x_3, \quad Y_2 = f_{x_5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Y_1, \quad x_4 = Z, \quad x_5 = \tilde{f}\left(Y_1, Y_2\right),$$

where $\tilde{f} \circ \tau_4 = f_{x_5}$. Our Monge equation is then given by

$$Z_1 = H = \tilde{H} \circ \tau_4^{-1} = h\left(Y_1, Y_2\right). \tag{5.30}$$

The condition $h_{Y_2 Y_2} \neq 0$ is satisfied a priori, but one can derive it by this construction and the constraint $F_{vv} \neq 0$.

Finally, we have constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = a, \quad Y = b, \quad Z = c, \quad Y_1 = u, \quad Y_2 = -\frac{1}{F_v},$$

and with inverse τ^{-1} given by

$$a = X, \quad b = Y, \quad c = Z, \quad u = Y_1, \quad v = \tilde{F}\left(Y_1, Y_2\right).$$

According to these, $\Gamma = \langle -\partial_a, -\partial_b, -\partial_c \rangle$ is τ-related to $\langle -\partial_X, -\partial_Y, -\partial_Z \rangle = \langle \partial_X, \partial_Y, \partial_Z \rangle$. $\square$

5.4 $\mathbf{A_1 \oplus A_2}$: the decomposable solvable algebra

5.4.1 Step 1

We take the Lie algebra $A_1 \oplus A_2$ with structure equations

$$[E_1, E_3] = [E_2, E_3] = 0,$$

$$[E_1, E_2] = E_1.$$

This is associated to the group G of the 3×3 matrices

$$A = \begin{pmatrix} e^c & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & e^b \end{pmatrix}.$$

G is indeed a group, because the identity matrix has coordinates $(a, b, c) = (0, 0, 0)$ and

$$A^{-1} = \begin{pmatrix} e^{-c} & 0 & 0 \\ -ae^{-c} & 1 & 0 \\ 0 & 0 & e^{-b} \end{pmatrix} \in G.$$

The Maurer-Cartan forms are

$$\omega^1 = d\,a - a\,d\,c, \quad \omega^2 = d\,c, \quad \omega^3 = d\,b, \tag{5.31}$$

and their structure equations are

$$\begin{aligned} d\omega^1 &= -\omega^1 \wedge \omega^2, \\ d\omega^2 &= d\omega^3 = 0. \end{aligned} \tag{5.32}$$

The dual to (5.31) are

$$R_1 = \partial_a, \quad R_2 = a\,\partial_a + \partial_c, \quad R_3 = \partial_b.$$

The associated right invariant vector fields are

$$L_1 = -e^c\,\partial_a, \quad L_2 = -\partial_c, \quad L_3 = -\partial_b.$$

5.4.2 Step 2

Proposition 5.4.1. An adapted basis for I can only be given by

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = t\,\omega^2 - \omega^3 + \alpha^2, \\ \theta^3 = \omega^2 + \alpha^3 \end{cases} \tag{5.33}$$

where $t = t(u, v)$, and

$$d\alpha^1 = \alpha^1 \wedge \alpha^3, \tag{5.34a}$$

$$d\alpha^2 = dt \wedge \alpha^3, \tag{5.34b}$$

$$d\alpha^3 \neq 0, \tag{5.34c}$$

$$\alpha^1 \wedge dt \neq 0. \tag{5.34d}$$

Proof. An optimal list of 1-dimensional subalgebras is given by

$$O = [E_3, E_1 + t\, E_3, E_2 + t\, E_3], \quad t = t(u, v),$$

and thus the $E_i^{\perp}$ produce the following three cases.

$$\begin{aligned}
&\text{Case 1:} \quad \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^2 + \alpha^2, \quad \theta^3 = \omega^3 + \alpha^3.\\
&\text{Case 2:} \quad \theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = t\,\omega^1 - \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.\\
&\text{Case 3:} \quad \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = t\,\omega^2 - \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.
\end{aligned}$$

In Cases 1 and 2 we have $I'' \neq 0$. Concerning Case 3, condition (5.1a) is written as $d\alpha^1 = \alpha^1 \wedge \alpha^3$, while condition (5.1b) is $d\alpha^2 = dt \wedge \alpha^3$. Finally, conditions (5.1c) and (5.1d) are respectively equivalent to $d\alpha^3 \neq 0$ and $\alpha^1 \wedge dt \neq 0$. Hence (5.33) is a GR_3D_5 Pfaffian system provided that conditions (5.34) are satisfied. □

5.4.3 Step 3

Proposition 5.4.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_1 \oplus A_2$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle -e^c\,\partial_a,\, -\partial_c,\, -\partial_b \rangle$

and

$$I = \begin{cases} \theta^1 = da - a\,dc + F\,du + dv, \\ \theta^2 = db - u\,dc, \\ \theta^3 = dc + F_v\,du, \end{cases} \tag{5.35}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.36}$$

Proof. The admissible gauge transformations in this case are those which fix $\langle E_2 + t\,E_3 \rangle$. Because the right action on G is exerted by the multiplication on the right by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} e^{t_2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{t_3} \end{pmatrix}, \tag{5.37}$$

we have to consider only the multiplication by U_2 and U_3. Setting $t_i = f_i(u, v)$ and applying (5.3), the 1-forms α^i change as follows

$$\left[\begin{array}{l} \alpha^1 \to \bar{\alpha}^1 = e^{-f_2}\,\alpha^1, \\ \alpha^2 \to \bar{\alpha}^2 = \alpha^2 + t\,d\,f_2 - d\,f_3, \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + d\,f_2. \end{array} \right. \tag{5.38}$$

One can check that conditions (5.34) are unchanged by (5.38). Now we use (5.38) in order to simplify the α^i.

First condition (5.34d) implies $dt \neq 0$, and thus there is a change of coordinates $(u, v) \to (\tilde{u}, \tilde{v})$ such that $t(u, v) = \tilde{u}$. Neglecting the tildes, we can rewrite (5.33), (5.34), and (5.38), respectively, as

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = u\,\omega^2 - \omega^3 + \alpha^2, \\ \theta^3 = \omega^2 + \alpha^3, \end{cases} \tag{5.39}$$

$$d\alpha^1 = \alpha^1 \wedge \alpha^3, \tag{5.40a}$$

$$d\alpha^2 = du \wedge \alpha^3, \tag{5.40b}$$

$$\alpha^1 \wedge du \neq 0, \tag{5.40c}$$

$$d\alpha^3 \neq 0, \tag{5.40d}$$

$$\left[\begin{aligned} &\alpha^1 \to \bar{\alpha}^1 = e^{-f_2}\,\alpha^1, \\ &\alpha^2 \to \bar{\alpha}^2 = \alpha^2 + u\,df_2 - df_3, \\ &\alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df_2. \end{aligned}\right. \tag{5.41}$$

From the second in (5.41), solving the equation $\bar{\alpha}^2 = 0$, we find functions F, G, C^1, and C^2 of (u, v) such that (5.41) becomes

$$\begin{aligned} &\bar{\alpha}^1 = F\,du + dG, \\ &\bar{\alpha}^2 = 0, \\ &\bar{\alpha}^3 = C^1\,du + C^2\,dv. \end{aligned} \tag{5.42}$$

Now we plug (5.42) in (5.40). From (5.40b) we obtain $C^2 = 0$, while (5.40c) implies $G_v \neq 0$. At this point may as well consider new coordinates $(a, b, c, u, \tilde{v})$ where $\tilde{v} = G$. Dropping the tilde, we can then rewrite (5.42) as

$$\begin{aligned} &\bar{\alpha}^1 = F\,du + dv, \\ &\bar{\alpha}^2 = 0, \\ &\bar{\alpha}^3 = C\,du. \end{aligned} \tag{5.43}$$

This time, from (5.40a), we obtain $C = F_v$, and thus (5.40d) is equivalent to $F_{vv} \neq 0$.

In summary, the Pfaffian system (5.39) can be written as

$$I = \begin{cases} \theta^1 = \omega^1 + F\,du + dv, \\ \theta^2 = u\,\omega^2 - \omega^3, \\ \theta^3 = \omega^2 + F_v\,du, \end{cases} \tag{5.44}$$

with constraint

$$F_{vv} \neq 0. \tag{5.45}$$

Using (5.31), (5.44) becomes (5.33). $\square$

5.4.4 Step 4

Here we proceed, as in section 5.3.4, to prove the following.

Proposition 5.4.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A_1 \oplus A_2$. Then $\Gamma = \langle \partial_Z,\, Z\,\partial_Z - X\,\partial_X,\, \partial_Y \rangle$ and

$$Z_1 = X^{-2}\, h\big(XY_1, X^2 Y_2\big), \text{ with } D_2^2 h \neq 0. \tag{5.46}$$

Proof. We will follow the same steps as in the proof of Proposition 5.3.3, which are mainly dictated by the Monge Algorithm 3.5.8.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N with local coordinates (X, Y, Z, Y_1, Y_2). By Proposition 5.4.2 we have local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle -e^c\,\partial_a,\, -\partial_c,\, -\partial_b \rangle$ and

$$I = \begin{cases} \eta^1 = d\,b - u\,d\,c, \\[2ex] \eta^2 = d\,a - a\,d\,c + F\,d\,u + d\,v, \\[2ex] \eta^3 = d\,c + F_v\,d\,u, \end{cases} \tag{5.47}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.48}$$

[0] By construction, (5.47) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = c, \quad y^2 = b, \quad y^3 = u, \quad y^4 = a, \quad y^5 = v,$$

according to which

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\, dy^1, \\ \eta^2 = dy^4 - y^4\, dy^1 + F\, dy^3 + dy^5, \end{cases} \tag{5.49}$$

where we set $F = F(y^3, y^5) = F(u, v)$, with $F_{y^5 y^5} \neq 0$.

[2] We seek for $\theta^2 = dU - V^3\, dy^3 - V^1\, dy^1 = W\,\theta^1 + A\,\eta^2 \in I'$. The dy^4 and dy^5 components of η^2 are respectively $Y^4 = 1$ and $Y^5 = 1$. The PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = A, \\ \frac{\partial U}{\partial y^5} &= AY^5 = A, \end{aligned} \tag{5.50}$$

We see that for $A = 1$ we have the solution $U = y^4 + y^5$. Recalling the notation $\eta^2 = Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5$ used in Proposition 3.3.3, the expressions (3.48) become

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = 0 - F = -F, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3\left(\frac{\partial U}{\partial y^2} - AY_0^2\right) - AY_0^1 = y^4. \end{aligned} \tag{5.51}$$

Using U and V^3 we define a new change of coordinates $\tau_2 : N \to N$

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = y^4 + y^5, \quad x^5 = -F.$$

The Jacobian matrix of τ_2 is

$$\mathrm{J}_{\tau_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -F_{y^3} & 0 & -F_{y^5} \end{bmatrix}$$

Note that the determinant of the Jacobian matrix of τ_2, that is, $|\mathrm{J}_{\tau_2}| = -F_{y^5} = -F_v$, is non-vanishing by (5.48). The diffeomorphism τ_2 defines the mapping

$$x_3 = y^3, \quad x^4 = y^4 + y^5, \quad x_5 = F(y^3, y^5),$$

and by the inverse function theorem there are defined two functions $\tilde{F} = \tilde{F}\left(x^3, x^5\right)$ and $f = f\left(x^3, x^4, x^5\right) = -x^4 + \tilde{F}$ on N such that

$$f \circ \tau_2 = -y^4, \quad \tilde{F} \circ \tau_2 = y^5.$$

The inverse of τ_2 can thus be defined by

$$\tau_2^{-1}: \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = -f, \quad y^5 = x^4 + f = \tilde{F}.$$

Consequently we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \theta^2 = \eta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \end{cases} \tag{5.52}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5x^5} \neq 0$. By direct computation, we have $f_{x^5} = \tilde{F}_{x^5}$ and thus $f_{x^5} \circ \tau_2 = -\frac{1}{F_{y^5}}$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\, dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \theta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \\ \theta^3 = dx^3 - f_{x^5}\, dx^1. \end{cases}$$

Let's do some computations, to check that the definitions of f and $\tilde{F}$ were well posed.

$$\tau_2^* \theta^3 = dy^3 + \frac{1}{F_{y^5}}\, dx^4 = du + \frac{1}{F_v}\, dc = \frac{1}{F_v}\, \eta^3 \in I,$$

as expected.

[4] Here we obtain the Monge normal form. In view of (3.84), we set

$$\tilde{H} = x^5 f_{x^5} - f = x^5 \tilde{F}_{x^5} + x^4 - \tilde{F} = x^4 + \tilde{h}\,(x_3, x_5)\,.$$

Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x_1, \quad Y = x_2, \quad Z = x_4, \quad Y_1 = x_3, \quad Y_2 = f_{x_5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Y_1, \quad x_4 = Z, \quad x_5 = \tilde{f}\,(Y_1, Y_2)\,,$$

where $\tilde{f} \circ \tau_4 = f_{x_5}$. Our Monge equation is then given by

$$Z_1 = H = \tilde{H} \circ \tau_4^{-1} = Z + h\,(Y_1, Y_2)\,. \tag{5.53}$$

Moreover, we have constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = c, \quad Y = b, \quad Z = a, \quad Y_1 = u, \quad Y_2 = \frac{v}{F_v},$$

and with inverse τ^{-1} given by

$$a = X, \quad b = Y, \quad c = Z, \quad u = Y_1, \quad v = \bar{F}(Y_1, Y_2).$$

According to these, $\Gamma = \langle -e^c\,\partial_a,\, -\partial_c,\, -\partial_b \rangle$ is τ-related to $\left\langle -e^X\,\partial_Z, -\partial_X, -\partial_Y \right\rangle$.

[extra] We did not arrive at the claimed normal form yet. We have to add another step, which is merely the definition of a contact transformation, which will lead us to (5.46). Define the change of coordinates $\phi : N \to N$

$$\bar{X} = e^X, \quad \bar{Y} = Y, \quad \bar{Z} = e^{-X} Z, \quad \bar{Y}_1 = -e^{-X} Y_1, \quad \bar{Y}_2 = e^{-2X}(Y_2 - Y_1),$$

which is a local contact transformation such that $\bar{Z}_1 = e^{-2X}(Z_1 - Z)$. Dropping the bars, ϕ transforms (5.53) to

$$Z_1 = X^{-2} h\left(XY_1, X^2 Y_2\right),$$

and relates Γ to $\langle \partial_Z, X\,\partial_X - Z\,\partial_Z - Y_1\,\partial_{Y_1} - 2Y_2\,\partial_{Y_2}, \partial_Y \rangle$, which is the prolongation of $\langle \partial_Z, X\,\partial_X - Z\,\partial_Z, \partial_Y \rangle$. This completes our proof. □

5.5 $\mathbf{A_{3,1}}$: the Heisenberg (nilpotent) algebra

5.5.1 Step 1

We take the unique nilpotent Lie algebra $A_{3,1}$ with structure equations

$$\begin{aligned} &[E_1, E_2] = [E_1, E_3] = 0, \\ &[E_2, E_3] = E_1. \end{aligned} \tag{5.54}$$

This is associated to the group of upper-triangular 3×3 matrices

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

The (left invariant) Maurer-Cartan forms are then

$$\begin{aligned} \omega^1 &= db - a\,dc, \\ \omega^2 &= da, \\ \omega^3 &= dc, \end{aligned} \tag{5.55}$$

with structure equations

$$\begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega^3, \\ d\omega^2 &= d\omega^3 = 0. \end{aligned} \tag{5.56}$$

The dual to the (5.55) are

$$\begin{aligned} R_1 &= \partial_b, \\ R_2 &= \partial_a, \\ R_3 &= a\,\partial_b + \partial_c. \end{aligned}$$

The right action on G is exerted by the multiplication on the right by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_2 = \begin{pmatrix} 1 & t_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_3 \\ 0 & 0 & 1 \end{pmatrix}$$

while the associated right-invariant vector fields are

$$\begin{aligned} L_1 &= -\partial_b, \\ L_2 &= -\partial_a - c\,\partial_b, \\ L_3 &= -\partial_c. \end{aligned}$$

5.5.2 Step 2

Proposition 5.5.1. An adapted basis for I can only be given by

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = t\,\omega^2 - \omega^3 + \alpha^2, \\ \theta^3 = \omega^2 + \alpha^3, \end{cases} \tag{5.57}$$

where $t = t(u, v) \neq 0$ and

$$d\alpha^1 = \alpha^2 \wedge \alpha^3, \tag{5.58a}$$

$$d\alpha^2 = dt \wedge \alpha^3, \tag{5.58b}$$

$$dt \wedge \alpha^2 \neq 0, \tag{5.58c}$$

$$d\alpha^3 \neq 0. \tag{5.58d}$$

Proof. An optimal list of 1-dimensional subalgebras is given by

$$O = [E_1, E_2, E_3, E_2 + t\,E_3], \quad t \neq 0,$$

and thus the $E_i \in I'^{\perp}$ produce the following four cases:

Case 1: $\theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.$

Case 2: $\theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.$

Case 3: $\theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^2 + \alpha^2, \quad \theta^3 = \omega^3 + \alpha^3.$

Case 4: $\theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = t\,\omega^2 - \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.$

Here $\alpha^i = U^i du + V^i dv$, U^i, V^i and t are functions of (u, v) and $t \neq 0$.

Cases 1, 2 and 3 lead to $I'' \neq 0$. Case 4 gives the right structure equations. Indeed, conditions (5.1a) and (5.1b) are equivalent to $d\alpha^1 = \alpha^2 \wedge \alpha^3$ and $d\alpha^2 = dt \wedge \alpha^3$, while conditions (5.1c) and (5.1d) give $d\alpha^3 \neq 0$ and $dt \wedge \alpha^2 \neq 0$. Hence, conditions (5.58) hold. $\square$

5.5.3 Step 3

Proposition 5.5.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,1}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle -\partial_b,\, -\partial_a - c\,\partial_b,\, -\partial_c \rangle$

and

$$I = \begin{cases} \theta^1 = db - a\,dc + F\,du, \\ \\ \theta^2 = dc - u\,da - dv, \\ \\ \theta^3 = da + F_v\,du, \end{cases} \tag{5.59}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.60}$$

Proof. The admissible gauge transformations in this case are those which fix $E_4 = E_2 + t\,E_3$, thus those generated by E_1 and E_4. Setting $U_4(t_4) = \exp(t_4 E_4)$, then we are looking for the action of

$$U = U_1(g(u, v))\,U_4(f(u, v)) = \begin{pmatrix} 1 & f & \frac{1}{2}tf^2 + g \\ 0 & 1 & tf \\ 0 & 0 & 1 \end{pmatrix}.$$

From (5.3), the 1-forms α^i change as follows

$$\left[\begin{array}{l} \alpha^1 \to \bar{\alpha}^1 = \alpha^1 + dg - f\,\alpha^2 + \dfrac{1}{2} f^2\,dt, \\ \\ \alpha^2 \to \bar{\alpha}^2 = \alpha^2 - f\,dt, \\ \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df. \end{array}\right. \tag{5.61}$$

Conditions (5.58) are easily seen to be unchanged by (5.61). Let's use (5.61) to simplify the expressions of α^1, α^2, and α^3.

Because (5.58c), $dt \neq 0$ then there is a change of coordinates $(a, b, c, u, v) \to (a, b, c, \tilde{u}, v)$ such that $t(u, v) = \tilde{u}$. Writing the results of Proposition 5.5.1 in these new coordinates (dropping the tildes), we have

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \\ \theta^2 = u\,\omega^2 - \omega^3 + \alpha^2, \\ \\ \theta^3 = \omega^2 + \alpha^3, \end{cases}$$

$$d\alpha^1 = \alpha^2 \wedge \alpha^3, \tag{5.62a}$$

$$d\alpha^2 = du \wedge \alpha^3, \tag{5.62b}$$

$$du \wedge \alpha^2 \neq 0, \tag{5.62c}$$

$$d\alpha^3 \neq 0. \tag{5.62d}$$

The gauge transformation (5.61) now is

$$\begin{cases} \alpha^1 \to \bar{\alpha}^1 = \alpha^1 + dg - f\,\alpha^2 + \dfrac{1}{2}f^2\,du, \\ \alpha^2 \to \bar{\alpha}^2 = \alpha^2 - f\,du, \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df. \end{cases} \tag{5.63}$$

From this, we can define functions f and g such that

$$\begin{aligned} \bar{\alpha}^1 &= U^1\,du, \\ \bar{\alpha}^2 &= U^2\,du + {V^2}_v\,dv, \\ \bar{\alpha}^3 &= U^3\,du, \end{aligned} \tag{5.64}$$

where U^i and V^2 are arbitrary functions (u, v), subject to the constraints (5.62). In particular (5.62c) implies ${V^2}_v \neq 0$. We can change the last coordinate from v to $\tilde{v} = V^2$ and rewrite (5.64) (without tildes and bars) as

$$\begin{aligned} \alpha^1 &= U^1\,du, \\ \alpha^2 &= U^2\,du + dv, \\ \alpha^3 &= U^3\,du, \end{aligned} \tag{5.65}$$

for U^i arbitrary functions of (u, v), subject to the constraints (5.62). Consequently, (5.62b) implies $0 = du \wedge \alpha^3 = d\alpha^2 = -U^2_v\,du \wedge dv$, and thus $U^2 = U^2(u)$. From (5.62a) we have $-U^1_v\,du \wedge dv = d\alpha^1 = \alpha^2 \wedge \alpha^3 = -U^3\,du \wedge dv$, and thus $U_3 = U^1_v$. Let's now use this results and (5.65) to rewrite

(5.66) as

$$\begin{cases} \alpha^1 \to \bar{\alpha}^1 = (U^1 - fU^2 + \frac{1}{2}f^2)\,du + dg - f\,dv, \\ \alpha^2 \to \bar{\alpha}^2 = U^2\,du + dv - f\,du, \\ \alpha^3 \to \bar{\alpha}^3 = U^1_v\,du + df. \end{cases} \tag{5.66}$$

If we choose $f = U^2(u)$ and $g = U^2(u)v$, then

$$\begin{aligned} &\bar{\alpha}^1 = (U^1 - \frac{1}{2}{U^2}^2)\,du + d(U^2(u)v) - U^2\,dv = (U^1 - \frac{1}{2}{U^2}^2 + vU^2_u)\,du = F\,du, \\ &\bar{\alpha}^2 = dv, \\ &\bar{\alpha}^3 = (U^1_v + U^2_u)\,du = F_v\,du. \end{aligned} \tag{5.67}$$

With the same computations as above, we see that conditions (5.62a) to (5.62c) are satisfied for any $F = F(u,v)$. Condition (5.62d) then is equivalent to the constraint

$$F_{vv} \neq 0$$

. Finally, using the local expression (5.55) together with (5.67), we obtain (5.59). $\square$

5.5.4 Step 4

Here we proceed, as in section 5.3.4, to prove the following.

Proposition 5.5.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A_{3,1}$. Then locally $\Gamma = \langle \partial_Z,\, \partial_X,\, \partial_Y + X\,\partial_Z \rangle$ and

$$Z_1 = Y + h\,(Y_1, Y_2)\,, \text{ with } h_{Y_2Y_2} \neq 0. \tag{5.68}$$

Proof. We will follow the same steps as in the proof of Proposition 5.4.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N with local coordinates (X, Y, Z, Y_1, Y_2). By Proposition 5.5.2, there are local coordinates (a, b, c, u, v)

on N such that $\Gamma = \langle -\partial_b,\, -\partial_a - c\,\partial_b,\, -\partial_c \rangle$ and

$$I = \begin{cases} \eta^1 = dc - u\,da - dv, \\ \eta^2 = db - a\,dc + F\,du, \\ \eta^3 = da + F_v\,du, \end{cases} \tag{5.69}$$

where $F = F(u, v)$ is a differentiable function such that $F_{vv} \neq 0$.

We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.69) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = -a, \quad y^2 = -c + v, \quad y^3 = u, \quad y^4 = b, \quad y^5 = -c,$$

The inverse is τ_1^{-1} given by

$$a = -y^1, \quad b = y^4, \quad c = -y^5, \quad u = y^3, \quad v = y^2 - y^5.$$

We see that $\theta^1 = \tau_1^{-1\,*}\eta^1 = dy^2 - y^3\,dy^1$, as wanted. Define $\tilde{F} = F \circ \tau_1^{-1} = \tilde{F}(y^3; y^2 - y^5) = \tilde{F}(y^3, \bar{y})$. Accordingly, η^2 in these new coordinates is

$$\eta_0^2 = \tau_1^{-1\,*}\eta^2 = \tilde{F}\,dy^3 + dy^4 - y^1\,dy^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\,dy^1, \\ \eta_0^2 = \tilde{F}\,dy^3 + dy^4 - y^1\,dy^5. \end{cases} \tag{5.70}$$

[2] We seek for $\theta^2 = dU - V^3\,dy^3 - V^1\,dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. Using (5.70) and the notation in Proposition 3.3.3, we write

$$\begin{aligned} \eta_0^2 &= Y_0^1\,dy^1 + Y_0^2\,dy^2 + Y^3\,dy^3 + Y^4\,dy^4 + Y^5\,dy^5 \\ &= \tilde{F}\,dy^3 + dy^4 - y^1\,dy^5. \end{aligned}$$

Accordingly, the PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = A, \\ \frac{\partial U}{\partial y^5} &= AY^5 = -Ay^1, \end{aligned} \tag{5.71}$$

Take $A = -1$ and the solution of (5.71) is $U = -y^4 + y^5 y^1$. We can then rewrite the expressions (3.48) as

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = 0 + \tilde{F} = \tilde{F}, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 W - AY_0^1 = y^5. \end{aligned} \tag{5.72}$$

In particular, we can now write

$$\theta^2 = dU - \tilde{F}\, dy^3 - y^5\, dy^1.$$

From (5.72) define

$$\hat{f} = -V^1 = -y^5, \tag{5.73}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U = y^5 y^1 - y^4, \quad x^5 = V^3 = \tilde{F}.$$

The local diffeomorphism τ_2 has inverse

$$\tau_2^{-1} : \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = -x^4 + G, \quad y^5 = G,$$

for some function $G = G(x^3; x^2, x^5)$ such that $G \circ \tau^2 = x^5$. In particular from (5.73) we define

$$f = \hat{f} \circ \tau_2^{-1}. \tag{5.74}$$

Finally we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \theta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \end{cases} \tag{5.75}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\,dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases}$$

From (5.74), we compute

$$f_{x^5} = \frac{1}{\tilde{F}_{\bar{y}}} \circ \tau^2 = \hat{T}(x^3; x^2, x^5). \tag{5.76}$$

[4] Here we obtain the Monge normal form. Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x_1, \quad Y = x_2, \quad Z = x_4, \quad Y_1 = x_3, \quad Y_2 = f_{x_5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Y_1, \quad x_4 = Z, \quad x_5 = \tilde{T}(Y, Y_1, Y_2),$$

where $\tilde{T} \circ \tau_4 = \hat{T}$.

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = -a, \quad Y = v - c, \quad Z = ca - b, \quad Y_1 = u, \quad Y_2 = \frac{1}{F_v},$$

and with inverse τ^{-1} given by

$$a = -X, \quad b = -Z + X\bar{T}, \quad c = -\bar{T}, \quad u = Y_1, \quad v = Y - \bar{T}.$$

According to these, $\Gamma = \langle -\partial_b, -\partial_a - c\,\partial_b, -\partial_c\rangle$ is τ-related to $\langle \partial_Z, \partial_X, \partial_Y + X\,\partial_Z\rangle$. Consequently we have $H = Y + h\,(Y_1, Y_2)$. $\square$

5.6 $\mathbf{A_{3,2}}$: the solvable algebra not belonging to a family of algebras

5.6.1 Step 1

We consider the Lie algebra $A_{3,2}$ with structure equations

$$\begin{aligned} &[E_1, E_2] = 0, \\ &[E_1, E_3] = E_1, \\ &[E_2, E_3] = E_1 + E_2, \end{aligned} \tag{5.77}$$

and the matrix group representation of G that we use is given by

$$A = \begin{pmatrix} e^{-c} & -ce^{-c} & e^{-c}\,(a + b - ac) \\ 0 & e^{-c} & ae^{-c} \\ 0 & 0 & 1 \end{pmatrix}.$$

The (left invariant) Maurer-Cartan forms are then

$$\begin{aligned} &\omega^1 = db - (a + b)\; dc, \\ &\omega^2 = da - a\,dc, \\ &\omega^3 = dc, \end{aligned} \tag{5.78}$$

with structure equations

$$\begin{aligned} &d\omega^1 = -\omega^1 \wedge \omega^3 - \omega^2 \wedge \omega^3, \\ &d\omega^2 = -\omega^2 \wedge \omega^3, \\ &d\omega^3 = 0. \end{aligned} \tag{5.79}$$

The dual to the (5.78) are

$$\begin{aligned} &R_1 = \partial_b, \\ &R_2 = \partial_a, \\ &R_3 = a\,\partial_a + (a + b)\;\partial_b + \partial_c. \end{aligned}$$

The right action on G is exerted by the multiplication on the right by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_2 = \begin{pmatrix} 1 & 0 & t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}, U_3 = \begin{pmatrix} e^{-t_3} & -t_3 e^{-t_3} & 0 \\ 0 & e^{-t_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The associated right-invariant vector fields are

$$\begin{aligned} L_1 &= -e^{c}\,\partial_b, \\ L_2 &= -e^{\epsilon c}\,\partial_a - c e^{\epsilon c}\,\partial_b, \\ L_3 &= -\partial_c. \end{aligned}$$

5.6.2 Step 2

Proposition 5.6.1. An adapted basis for I can only be given by

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = \omega^2 + \alpha^2, \\ \theta^3 = \omega^3 + \alpha^3, \end{cases} \tag{5.80}$$

where

$$d\alpha^1 = \alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^3, \tag{5.81a}$$

$$d\alpha^2 = \alpha^2 \wedge \alpha^3, \tag{5.81b}$$

$$d\alpha^3 \neq 0, \tag{5.81c}$$

$$\alpha^1 \wedge \alpha^2 \neq 0. \tag{5.81d}$$

Proof. An optimal list of 1-dimensional subalgebras is

$$O = [E_1, E_2, E_3]\,.$$

Consequently $E_i \in I'^{\perp}$ produces the following three cases:

$$\text{Case 1: } \theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.$$

$$\text{Case 2: } \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.$$

$$\text{Case 3: } \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^2 + \alpha^2, \quad \theta^3 = \omega^3 + \alpha^3.$$

Here $\alpha^i = U^i(u,v)\,du + V^i(u,v)\,dv$. Imposing conditions (5.1a) and (5.1b) in Cases 1 and 2 lead to $I'' \neq 0$.

In Case 3, conditions (5.1a) and (5.1b) are respectively equivalent to $d\alpha^1 = \alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^3$ and $d\alpha^2 = \alpha^2 \wedge \alpha^3$. Conditions (5.1c) and (5.1d) are respectively equivalent to $d\alpha^3 \neq 0$ and $\alpha^1 \wedge \alpha^2 \neq 0$. Hence, the derived flag conditions are (5.81). $\square$

5.6.3 Step 3

Proposition 5.6.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,2}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle -e^c\,\partial_b,\ -e^c\,\partial_a - ce^c\,\partial_b,\ -\partial_c \rangle$ and

$$I = \begin{cases} \theta^1 = db - (a+b)\ dc + F\,du + dv, \\ \theta^2 = da - a\,dc + du, \\ \theta^3 = dc + F_v\,du, \end{cases} \tag{5.82}$$

where $F = F(u,v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.83}$$

Proof. The admissible gauge transformations in this case are those which fix E_3, thus only the action of $U_3(f(u,v))$. From (5.3), the 1-forms α^i change by

$$\left[\begin{aligned} &\alpha^1 \to \bar{\alpha}^1 = e^{-f}\,\left(\alpha^1 - f\,\alpha^2\right), \\ &\alpha^2 \to \bar{\alpha}^2 = e^{-f}\,\alpha^2, \\ &\alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df. \end{aligned}\right. \tag{5.84}$$

From (5.81d) we see that $\alpha^1 \neq 0$ and we can find f such that e^{-f} is an integrating factor of α^2, that is, $\bar{\alpha}^2 = e^{-f}\,\alpha^2 = d\tilde{u}$ for some function $\tilde{u}(u,v)$. Because of (5.81d), we have $d\tilde{u} \neq 0$ and we can define the change of coordinates $(a,b,c,u,v) \to (a,b,c,\tilde{u},v)$. Dropping the tilde and the bars, we can write the right-hand sides of (5.84) as

$$\begin{aligned} \alpha^1 &= U^1\,du + G_v\,dv, \\ \alpha^2 &= du, \\ \alpha^3 &= U^3\,du + V^3\,dV, \end{aligned} \tag{5.85}$$

for some functions U^i, V^3 and G of (u,v). Using (5.81d) again, we find $G_v \neq 0$. Another change of coordinates produces

$$\begin{aligned} \alpha^1 &= F\,du + dv, \\ \alpha^2 &= du, \\ \alpha^3 &= U^3\,du + V^3\,dV, \end{aligned} \tag{5.86}$$

for some functions U^3, V^3 and F of (u,v). From (5.81b) we get $0 = d\alpha^2 = \alpha^2 \wedge \alpha^3 = du \wedge \alpha^3$, thus $V^3 = 0$. Consequently (5.81a) becomes

$$\begin{aligned} -F_v\,du \wedge dv = d\alpha^1 &= \alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^3 \\ &= -U^3\,du \wedge dv + 0, \end{aligned}$$

which gives $U^3 = F_v$. Finally, we can write (5.86) as

$$\begin{aligned} \alpha^1 &= F\,du + dv, \\ \alpha^2 &= du, \\ \alpha^3 &= F_v\,du, \end{aligned} \tag{5.87}$$

thus (5.81c) is equivalent to $F_{vv} \neq 0$ and (5.80) becomes

$$I = \begin{cases} \theta^1 = \omega^1 + F\,du + dv, \\ \theta^2 = \omega^2 + du, \\ \theta^3 = \omega^3 + F_v\,du. \end{cases} \tag{5.88}$$

By means of (5.78), we write (5.88) as (5.82). □

5.6.4 Step 4

Here we proceed, as in section 5.3.4, to prove the following.

Proposition 5.6.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A_{3,2}$. Then locally $\Gamma = \langle \partial_Z,\, \ln X\, \partial_Z + X\, \partial_Y,\, Z\, \partial_Z - X\, \partial_X \rangle$ and

$$Z_1 = X^{-2} Y + X^{-2}\, h\big(Y - XY_1, X^2 Y_2\big), \text{ with } D^2_{[2]} h \neq 0. \tag{5.89}$$

Proof. We will follow the same steps as in the proof of Proposition 5.4.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N with local coordinates (X, Y, Z, Y_1, Y_2). By Proposition 5.6.2, there are local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle -e^c\, \partial_b,\, -e^c\, \partial_a - ce^c\, \partial_b,\, -\partial_c \rangle$, and

$$I = \begin{cases} \eta^1 = da - a\,dc + du, \\ \eta^3 = db - (a+b)\; dc + F\,du + dv, \\ \eta^3 = dc + F_v\,du, \end{cases} \tag{5.90}$$

where $F = F(u, v)$ is a differentiable function such that $F_{vv} \neq 0$.

We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.90) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = c, \quad y^2 = u + a, \quad y^3 = a, \quad y^4 = b, \quad y^5 = v,$$

The inverse is τ_1^{-1} given by

$$a = y^3, \quad b = y^4, \quad c = y^1, \quad u = y^2 - y^3, \quad v = y^5.$$

We see that $\theta^1 = \tau_1^{-1^*}\eta^1 = dy^2 - y^3\, dy^1$, as wanted. Define $\tilde{F} = F \circ \tau_1^{-1} = \tilde{F}(y^2 - y^3, y^5) = \tilde{F}(\bar{y}, y^5)$. Accordingly, η^2 in these new coordinates is

$$\eta_0^2 = \tau_1^{-1^*}\eta^2 = -(y^3 + y^4)\, dy^1 + \tilde{F}\, dy^2 - \tilde{F}\, dy^3 + dy^4 + dy^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\, dy^1, \\ \eta_0^2 = -(y^3 + y^4)\, dy^1 + \tilde{F}\, dy^2 - \tilde{F}\, dy^3 + dy^4 + dy^5. \end{cases} \tag{5.91}$$

[2] We seek for $\theta^2 = dU - V^3\, dy^3 - V^1\, dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. Using (5.91) and the notation in Proposition 3.3.3, we write

$$\begin{aligned} \eta_0^2 &= Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5 \\ &= -(y^3 + y^4)\, dy^1 + \tilde{F}\, dy^2 - \tilde{F}\, dy^3 + dy^4 + dy^5. \end{aligned}$$

Accordingly, the PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = A, \\ \frac{\partial U}{\partial y^5} &= AY^5 = A, \end{aligned} \tag{5.92}$$

Take $A = 1$ and the solution of (5.92) is $U = y^4 + y^5$. We can then rewrite the expressions (3.48) as

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0 - \tilde{F} = -\tilde{F}, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = 0 + \tilde{F} = \tilde{F}, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 W - AY_0^1 = 0 - y^3\tilde{F} + (y^3 + y^4) = y^4 + y^3(1 - \tilde{F}). \end{aligned} \tag{5.93}$$

In particular, we can now write

$$\theta^2 = dU - \tilde{F}\, dy^3 - [y^4 + y^3(1 - \tilde{F})]\, dy^1.$$

From (5.93) define

$$\hat{f} = -V^1 = -y^4 + y^3(\tilde{F} - 1), \tag{5.94}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U = y^4 + y^5, \quad x^5 = V^3 = \tilde{F}.$$

The local diffeomorphism τ_2 has inverse

$$\tau_2^{-1} : \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = x^4 - G, \quad y^5 = G,$$

for some function $G = G(x^2 - x^3; x^5)$ such that $G \circ \tau^2 = x^5$. In particular from (5.94) we define

$$f = \hat{f} \circ \tau_2^{-1}. \tag{5.95}$$

Finally we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \theta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \end{cases} \tag{5.96}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5 x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\, dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases}$$

From (5.95), we compute

$$f_{x^5} = x^3 + \hat{T}(x^2, x^3, x^5), \tag{5.97}$$

where $\hat{T} \circ \tau_2^{-1} \circ \tau_1^{-1} = \frac{1}{F_v}$.

[4] Here we obtain the Monge normal form. Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x_1, \quad Y = x_2, \quad Z = x_4, \quad Y_1 = x_3, \quad Y_2 = f_{x_5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Y_1, \quad x_4 = Z, \quad x_5 = \tilde{T}(Y, Y_1, Y_2),$$

where $\tilde{T} \circ \tau_4 = \hat{T} + x^3$.

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = c, \quad Y = u + a, \quad Z = b + v, \quad Y_1 = a, \quad Y_2 = a + \frac{1}{F_v},$$

and with inverse τ^{-1} given by

$$a = Y_1, \quad b = Z - \bar{T}, \quad c = X, \quad u = Y - Y_1, \quad v = \bar{T}.$$

According to these, $\Gamma = \langle -e^c\,\partial_b,\, -e^c\,\partial_a - ce^c\,\partial_b,\, -\partial_c\rangle$ is τ-related to $\langle -e^X\,\partial_Z,\, -e^X\,(\partial_Y + X\,\partial_Z + \partial_{Y_1} + \partial_{Y_2}),\, -\partial_X\rangle$. Consequently we have $H = Z + Y + h(Y_1 - Y, Y_2 - Y)$.

[extra] Define the (contact) transformation on N

$$\tau_0 : \quad \bar{X} = e^X, \quad \bar{Y} = Y, \quad \bar{Z} = e^{-X} Z, \quad \bar{Y}_1 = e^{-X} Y_1, \quad \bar{Y}_2 = e^{-2X}\left(Y_2 - Y_1\right),$$

according to which we have $\bar{Z}_1 = e^{-2X}\,(Z_1 - Z)$. Define $\bar{\tau} = \tau_0 \circ \tau$. Then, dropping the bars, Γ is $\bar{\tau}$-related to $\langle \partial_Z, X\,\partial_Y + \ln X\,\partial_Z + \partial_{Y_1}, X\,\partial_X - Z\,\partial_Z - Y_1\,\partial_{Y_1} - 2Y_2\,\partial_{Y_2}\rangle$, which is the prolongation of $\langle \partial_Z, X\,\partial_Y + \ln X\,\partial_Z, X\,\partial_X - Z\,\partial_Z\rangle$. Finally, our general Monge equation has expression $Z_1 = \dfrac{1}{X^2}\,(Y + h\,(Y - XY_1, Y_2 - XY_1))$. $\square$

5.7 $\mathbf{A^{\epsilon}_{3,5}}$: the one parameter family of solvable algebras with $\epsilon \neq 0$

5.7.1 Step 1

The Lie algebra $A^{\epsilon}_{3,5}$ has structure equations

$$\begin{aligned} &[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \\ &[E_2, E_3] = \epsilon\,E_2, \quad \epsilon \neq 0. \end{aligned} \tag{5.98}$$

The matrix group representation of G that we use is given by

$$A = \begin{pmatrix} e^{-c} & 0 & e^{-c}a \\ 0 & e^{-\epsilon c} & \epsilon b e^{-\epsilon c} \\ 0 & 0 & 1 \end{pmatrix}.$$

The (left invariant) Maurer-Cartan forms are then

$$\begin{aligned} \omega^1 &= da - a\,dc, \\ \omega^2 &= db - \epsilon b\,dc, \\ \omega^3 &= dc, \end{aligned} \tag{5.99}$$

with structure equations

$$\begin{aligned} d\omega^1 &= -\omega^1 \wedge \omega^3, \\ d\omega^2 &= -\epsilon\,\omega^2 \wedge \omega^3, \\ d\omega^3 &= 0. \end{aligned} \tag{5.100}$$

The dual to the (5.99) are

$$R_1 = \partial_a,$$

$$R_2 = \partial_b,$$

$$R_3 = a\,\partial_a + \epsilon b\,\partial_b + \partial_c.$$

The right action on G is exerted by the multiplication on the right by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \epsilon t_2 \\ 0 & 0 & 1 \end{pmatrix}, U_3 = \begin{pmatrix} e^{-t_3} & 1 & 0 \\ 0 & e^{-\epsilon t_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The associated right-invariant vector fields are

$$L_1 = -e^{c}\,\partial_a,$$

$$L_2 = -e^{\epsilon c}\,\partial_b,$$

$$L_3 = -\partial_c.$$

5.7.2 Step 2

Proposition 5.7.1. An adapted basis for I can only be given by

$$I = \begin{cases} \theta^1 = \omega^1 - t\,\omega^2 + \alpha^1, \\ \theta^2 = \omega^3 + \alpha^2, \\ \theta^3 = \omega^2 + \alpha^3, \end{cases} \tag{5.101}$$

where $t = t(u, v) \neq 0$ and

$$d\alpha^1 = \alpha^1 \wedge \alpha^3, \tag{5.102a}$$

$$d\alpha^2 = \epsilon\,\alpha^2 \wedge \alpha^3, \tag{5.102b}$$

$$d\alpha^3 \neq 0, \tag{5.102c}$$

$$\alpha^1 \wedge \alpha^2 \neq 0. \tag{5.102d}$$

Proof. An optimal list of 1-dimensional subalgebras is given by

$$O = [E_1, E_2, E_2 + t\,E_1, E_3]\,, \quad t \in \mathbb{R} - \{0\}\,,$$

thus the condition $E_i \in (I')^{\perp}$ produces the following four cases.

Case 1: $\theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.$

Case 2: $\theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.$

Case 3: $\theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^2 + \alpha^2, \quad \theta^3 = \omega^3 + \alpha^3.$

Case 4: $\theta^1 = \omega^1 - t\,\omega^2 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^2 + \alpha^3.$

Here $\alpha^i = U^i(u,v)\,du + V^i(u,v)\,dv$ and $t = t(u,v) \neq 0$. Cases 1 to 3 lead to $I'' \neq 0$. In Case 4, conditions (5.1a) and (5.1b) are respectively equivalent to $d\alpha^1 = \alpha^1 \wedge \alpha^3$ and $d\alpha^2 = \epsilon\,\alpha^2 \wedge \alpha^3$. According to these, conditions (5.1c) and (5.1d) are respectively equivalent to $d\alpha^3 \neq 0$ and $\alpha^1 \wedge \alpha^2 \neq 0$. $a set of infinitesimal \mathbf{DT}(I) = [3,2,0]$ if and only if (5.101) and (5.102) hold. □

5.7.3 Step 3

Proposition 5.7.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,5}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle -e^c\,\partial_a,\ -e^{\epsilon c}\,\partial_b,\ -\partial_c \rangle$ and

$$I = \begin{cases} \theta^1 = da - a\,dc + du, \\ \theta^2 = db - \epsilon b\,dc + F\,du + dv, \\ \theta^3 = \epsilon\,dc + F_v\,du, \end{cases} \tag{5.103}$$

where $F = F(u,v)$ is a differentiable function subject to the constraint

$$F_{vv} \neq 0. \tag{5.104}$$

Proof. The admissible gauge transformations in this case are those which fix E_3, and thus only the action of $U_3(f(u,v))$. From (5.3), the 1-forms α^i change as follows

$$\begin{cases} \alpha^1 \to \bar{\alpha}^1 = e^{-f}\,\alpha^1, \\ \alpha^2 \to \bar{\alpha}^2 = e^{-\epsilon f}\,\alpha^2, \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df. \end{cases} \tag{5.105}$$

From (5.102d) we see that $\alpha^1 \neq 0$ and $\alpha^2 \neq 0$. We can find f such that e^{-f} is an integrating factor of α^1, that is, $\bar{\alpha}^1 = e^{-f}\,\alpha^1 = d\tilde{u}$ for some function $\tilde{u} = \tilde{u}(u,v)$. Because of (5.102d), we have $d\tilde{u} \neq 0$ and we can define the change of coordinates $(a,b,c,u,v) \to (a,b,c,\tilde{u},v)$. Dropping the tilde and the bars, we can write the right-hand sides of (5.105) as

$$\begin{aligned} \alpha^1 &= du, \\ \alpha^2 &= U^1\,du + G_v\,dv, \\ \alpha^3 &= U^3\,du + V^3\,dV, \end{aligned} \tag{5.106}$$

for some functions U^i, V^3 and G of (u,v). Using (5.102d) again, we find $G_v \neq 0$. Another change of coordinates produces

$$\begin{aligned} \alpha^1 &= du, \\ \alpha^2 &= F\,du + dv, \\ \alpha^3 &= U^3\,du + V^3\,dV, \end{aligned} \tag{5.107}$$

for some functions U^3, V^3 and F of (u,v). From (5.102a) we get $0 = d\alpha^1 = \alpha^1 \wedge \alpha^3 = du \wedge \alpha^3$, thus $V^3 = 0$. Consequently (5.102b) becomes

$$-F_v\,du \wedge dv = d\alpha^2 = \epsilon\,\alpha^2 \wedge \alpha^3 = -\epsilon U^3\,du \wedge dv + 0,$$

which gives $U^3 = \frac{F_v}{\epsilon}$, and thus we can write (5.107) as

$$\begin{aligned} \alpha^1 &= du, \\ \alpha^2 &= F\,du + dv, \\ \alpha^3 &= \frac{F_v}{\epsilon}\,du, \end{aligned} \tag{5.108}$$

thus (5.102c) is equivalent to $F_{vv} \neq 0$. Finally, (5.101) becomes

$$I = \begin{cases} \theta^1 = \omega^1 + du, \\ \theta^2 = \omega^2 + F\,du + dv, \\ \theta^3_0 = \omega^3 + \dfrac{F_v}{\epsilon}\,du, \end{cases} \tag{5.109}$$

which, setting $\theta^3 = \epsilon\theta^3_0$ and using the local expressions (5.99), produces (5.103). □

5.7.4 Step 4

Here we proceed, as in section 5.3.4, to prove the following.

Proposition 5.7.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A^{\epsilon}_{3,5}$. Then $\Gamma = \langle X^{1/\epsilon}\,\partial_Y,\, \partial_Z,\, \epsilon(Z\,\partial_Z - X\,\partial_X)\rangle$ and

$$Z_1 = \frac{1}{\epsilon} X^{-2}\, h\big(\epsilon X Y_1 - Y, \epsilon^2 X^2 Y_2 + \epsilon^2 X Y_1 - Y\big), \text{ with } D^2_{[2]} h \neq 0. \tag{5.110}$$

Proof. We will follow the same steps as in the proof of Proposition 5.6.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N. By Proposition 5.6.2, there are local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle -e^c\,\partial_b,\, -e^c\,\partial_a - ce^c\,\partial_b,\, -\partial_c\rangle$ and

$$I = \begin{cases} \eta^1 = da - a\,dc + du, \\ \eta^3 = db - \epsilon b\,dc + F\,du + dv, \\ \eta^3 = \epsilon dc + F_v\,du, \end{cases} \tag{5.111}$$

where $F = F(u,v)$ is a differentiable function such that $F_{vv} \neq 0$. We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.111) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = c, \quad y^2 = u + a, \quad y^3 = a, \quad y^4 = b, \quad y^5 = v.$$

The inverse τ_1^{-1} is given by

$$a = y^3, \quad b = y^4, \quad c = y^1, \quad u = y^2 - y^3, \quad v = y^5.$$

We see that $\theta^1 = \tau_1^{-1\,*}\eta^1 = dy^2 - y^3\,dy^1$, as wanted. Define $\tilde{F} = F \circ \tau_1^{-1} = \tilde{F}(y^2 - y^3, y^5) = \tilde{F}(\bar{y}, y^5)$. Accordingly, η^2 in these new coordinates is

$$\eta_0^2 = \tau_1^{-1\,*}\eta^2 = -\epsilon y^4\,dy^1 + \tilde{F}\,dy^2 - \tilde{F}\,dy^3 + dy^4 + dy^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\,dy^1, \\ \eta_0^2 = -\epsilon y^4\,dy^1 + \tilde{F}\,dy^2 - \tilde{F}\,dy^3 + dy^4 + dy^5. \end{cases} \tag{5.112}$$

[2] We seek for $\theta^2 = dU - V^3\,dy^3 - V^1\,dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. Using (5.112) and the notation in Proposition 3.3.3, we write

$$\begin{aligned} \eta_0^2 &= Y_0^1\,dy^1 + Y_0^2\,dy^2 + Y^3\,dy^3 + Y^4\,dy^4 + Y^5\,dy^5 \\ &= -\epsilon y^4\,dy^1 + \tilde{F}\,dy^2 - \tilde{F}\,dy^3 + dy^4 + dy^5. \end{aligned}$$

Accordingly, the PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = A, \\ \frac{\partial U}{\partial y^5} &= AY^5 = A, \end{aligned} \tag{5.113}$$

Take $A = 1$ and the solution of (5.113) is $U = y^4 + y^5$. We can then rewrite the expressions (3.48) as

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0 - \tilde{F} = -\tilde{F}, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = 0 + \tilde{F} = \tilde{F}, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 W - AY_0^1 = \epsilon y^4 - y^3 \tilde{F}. \end{aligned} \tag{5.114}$$

In particular, we can now write

$$\theta^2 = dU - \tilde{F}\, dy^3 - (\epsilon y^4 - y^3 \tilde{F})\, dy^1.$$

From (5.114) define

$$\hat{f} = -V^1 = -\epsilon y^4 + y^3 \tilde{F}, \tag{5.115}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U = y^4 + y^5, \quad x^5 = V^3 = \tilde{F}.$$

The local diffeomorphism τ_2 has inverse

$$\tau_2^{-1} : \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = x^4 - G, \quad y^5 = G,$$

for some function $G = G(x^2, x^3, x^5)$ such that $G \circ \tau^2 = x^5$. In particular from (5.115) we define

$$f = \hat{f} \circ \tau_2^{-1} = -\epsilon x^4 + x^3 x^5 + \epsilon \tilde{f}(x^2, x^3, \tilde{x}^5). \tag{5.116}$$

Finally we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \theta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \end{cases} \tag{5.117}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5 x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\, dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \theta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \\ \theta^3 = dx^3 - f_{x^5}\, dx^1. \end{cases}$$

From (5.116), we compute

$$f_{x^5} = x^3 + \hat{T}(x^2, x^3, x^5), \tag{5.118}$$

where $\hat{T} \circ \tau_2^{-1} \circ \tau_1^{-1} = \frac{\epsilon}{F_v}$.

[4] Here we obtain the Monge normal form. Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x_1, \quad Y = x_2, \quad Z = x_4, \quad Y_1 = x_3, \quad Y_2 = f_{x_5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Y_1, \quad x_4 = Z, \quad x_5 = \tilde{T}(Y, Y_1, Y_2),$$

where $\tilde{T} \circ \tau_4 = \hat{T} + x^3$.

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = c, \quad Y = u + a, \quad Z = b + v, \quad Y_1 = a, \quad Y_2 = a + \frac{\epsilon}{F_v},$$

and with inverse τ^{-1} given by

$$a = Y_1, \quad b = Z - \bar{T}, \quad c = X, \quad u = Y - Y_1, \quad v = \bar{T}.$$

According to these, $\Gamma = \langle -e^c\, \partial_a, -e^{\epsilon c}\, \partial_b, -\partial_c \rangle$ is τ-related to $\langle -e^X\, (\partial_Y + \partial_{Y_1} + \partial_{Y_2}), -e^{\epsilon X}\, \partial_Z, -\partial_X \rangle$. Consequently we have $H = \epsilon Z + h(Y - Y_1, Y_2 - Y_1)$.

[extra] Define the (contact) transformation on N

$$\tau_0 : \quad \bar{X} = e^{X/\epsilon}, \quad \bar{Y} = \frac{Y}{\epsilon}, \quad \bar{Z} = \frac{e^{-\epsilon X} Z}{\epsilon}, \quad \bar{Y}_1 = -e^{-X/\epsilon} Y_1, \quad \bar{Y}_2 = e^{-2X/\epsilon}(\epsilon Y_2 - Y_1),$$

according to which we have $\bar{Z}_1 = e^{-(\epsilon+1/\epsilon)X}(Z_1 - \epsilon Z)$. Define $\bar{\tau} = \tau_0 \circ \tau$. Then, dropping the bars, Γ is $\bar{\tau}$-related to $\langle X^{1/\epsilon}[\partial_Y + \frac{1}{\epsilon X}\,\partial_{Y_1} + \frac{1-\epsilon}{\epsilon^2 X^2}\,\partial_{Y_2}],\, \partial_Z, -\epsilon X\,\partial_X + \epsilon Z\,\partial_Z + \epsilon Y_1\,\partial_{Y_1} + 2\epsilon Y_2\,\partial_{Y_2}\rangle$, which is the prolongation of $\langle X^{1/\epsilon}\partial_Y,\, \partial_Z, -\epsilon X\,\partial_X + \epsilon Z\,\partial_Z\rangle$. Finally, our general Monge equation has expression $Z_1 = \dfrac{1}{\epsilon X^2}\,(Y + h\,(Y - XY_1, Y_2 - XY_1))$. Note that the factor $\frac{1}{\epsilon}$ can be "absorbed" by h, but we use it for computational purposes in Section 6.5. □

5.8 $\mathbf{A}^{\epsilon}_{\mathbf{3,7}}$: the one parameter family of solvable algebras with $\epsilon \geq 0$

5.8.1 Step 1

The Lie algebra $A^{\epsilon}_{3,7}$ has structure equations

$$
\begin{aligned}
&[E_1, E_2] = 0, \quad [E_1, E_3] = \epsilon\, E_1 - E_2, \\
&[E_2, E_3] = E_1 + \epsilon\, E_2, \quad \epsilon \geq 0.
\end{aligned}
\tag{5.119}
$$

The matrix group representation of G that we use is given by

$$
A = e^{-\epsilon c}\begin{pmatrix} \cos c & -\sin c & \epsilon a\cos c - \epsilon b \sin c + a\sin c + b\cos c \\ \sin c & \cos c & \epsilon a \sin c + \epsilon b\cos c - a\cos c + b\sin c \\ 0 & 0 & e^{\epsilon c} \end{pmatrix}.
$$

The (left invariant) Maurer-Cartan forms are then

$$
\begin{aligned}
&\omega^1 = da - \epsilon\, db + \left(\epsilon^2 b - b - 2\epsilon a\right)\, dc, \\
&\omega^2 = \epsilon\, da + db + \left(a - \epsilon^2 a - 2\epsilon b\right)\, dc, \\
&\omega^3 = dc,
\end{aligned}
\tag{5.120}
$$

with structure equations

$$
\begin{aligned}
&d\omega^1 = -\epsilon\,\omega^1 \wedge \omega^3 - \omega^2 \wedge \omega^3, \\
&d\omega^2 = \omega^1 \wedge \omega^3 - \epsilon\,\omega^2 \wedge \omega^3, \\
&d\omega^3 = 0.
\end{aligned}
\tag{5.121}
$$

The dual to the (5.120) are

$$\begin{aligned}
R_1 &= \frac{1}{1+\epsilon^2}\,\partial_a - \frac{\epsilon}{1+\epsilon^2}\,\partial_b,\\
R_2 &= \frac{\epsilon}{1+\epsilon^2}\,\partial_a + \frac{1}{1+\epsilon^2}\,\partial_b,\\
R_3 &= (\epsilon a + b)\;\partial_a + (\epsilon b - a)\;\partial_b + \partial_c.
\end{aligned}$$

The right action on G is exerted by the multiplication on the right by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t_1 \\ 0 & 0 & 1 \end{pmatrix},\ U_2 = \begin{pmatrix} 1 & 0 & t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ U_3 = e^{-\epsilon t_3}\begin{pmatrix} \cos t_3 & -\sin t_3 & 0 \\ \sin t_3 & \cos t_3 & 0 \\ 0 & 0 & e^{\epsilon t_3} \end{pmatrix}.$$

The associated right-invariant vector fields are

$$\begin{aligned}
L_1 &= \frac{e^{\epsilon c}\,(\epsilon \sin c - \cos c)}{1+\epsilon^2}\,\partial_a + \frac{e^{\epsilon c}\,(\epsilon \cos c + \sin c)}{1+\epsilon^2}\,\partial_b,\\
L_2 &= -\frac{e^{\epsilon c}\,(\epsilon \cos c + \sin c)}{1+\epsilon^2}\,\partial_a + \frac{e^{\epsilon c}\,(\epsilon \sin c - \cos c)}{1+\epsilon^2}\,\partial_b,\\
L_3 &= -\partial_c.
\end{aligned}$$

5.8.2 Step 2

Proposition 5.8.1. An adapted basis for I can only be given by

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = \omega^2 + \alpha^2, \\ \theta^3 = \omega^3 + \alpha^3, \end{cases} \tag{5.122}$$

where

$$d\alpha^1 = \left(\epsilon\,\alpha^1 + \alpha^2\right) \wedge \alpha^3, \tag{5.123a}$$

$$d\alpha^2 = \left(\epsilon\,\alpha^2 - \alpha^1\right) \wedge \alpha^3, \tag{5.123b}$$

$$d\alpha^3 \neq 0, \tag{5.123c}$$

$$\alpha^1 \wedge \alpha^2 \neq 0. \tag{5.123d}$$

Proof. An optimal list of 1-dimensional subalgebras is $O = [E_1, E_3]$, and thus the $E_i \in (I')^{\perp}$ produces the following two cases:

$$\text{Case 1: } \theta^1 = \omega^2 + \alpha^1, \quad \theta^2 = \omega^3 + \alpha^2, \quad \theta^3 = \omega^1 + \alpha^3.$$

$$\text{Case 2: } \theta^1 = \omega^1 + \alpha^1, \quad \theta^2 = \omega^2 + \alpha^2, \quad \theta^3 = \omega^3 + \alpha^3.$$

Here $\alpha^i = U^i(u, v)\, du + V^i(u, v)\, dv$. Case 1 leads to $I'' \neq 0$. In Case 2, conditions (5.1a) and (5.1b) are respectively equivalent to $d\alpha^1 = (\epsilon\, \alpha^1 + \alpha^2) \wedge \alpha^3$ and $d\alpha^2 = (\epsilon\, \alpha^2 - \alpha^1) \wedge \alpha^3$. Consequently, conditions (5.1c) and (5.1d) are respectively equivalent to $d\alpha^3 \neq 0$ and $\alpha^1 \wedge \alpha^2 \neq 0$. These are (5.123). □

5.8.3 Step 3

Proposition 5.8.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A^{\epsilon}_{3,7}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle A\, \partial_a + B\, \partial_b,\ A\, \partial_b - B\, \partial_a,\ -\partial_c \rangle$, where $A = \dfrac{e^{\epsilon c}\,(\epsilon \sin c - \cos c)}{1 + \epsilon^2}$ and $B = \dfrac{e^{\epsilon c}\,(\epsilon \cos c + \sin c)}{1 + \epsilon^2}$, and

$$I = \begin{cases} \theta^1 = da - \epsilon\, db + \left(\epsilon^2 b - b - 2\epsilon a\right)\, dc + F\, du + dv, \\[2ex] \theta^2 = \epsilon\, da + db + \left(a - \epsilon^2 a - 2\epsilon b\right)\, dc + du, \\[2ex] \theta^3 = dc - \dfrac{F_v}{\epsilon^2 + 1}\left((F - \epsilon)\, du + dv\right), \end{cases} \tag{5.124}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint

$$D_v[(F - \epsilon)\, F_v - F_u] \neq 0. \tag{5.125}$$

Proof. The admissible gauge transformations in this case are those which fix E_3, thus those generated by $U_3(f(u,v))$. From (5.3), the 1-forms α^i change by

$$\left[\begin{array}{l} \alpha^1 \to \bar{\alpha}^1 = e^{-\epsilon f}\left(\cos f\, \alpha^1 - \sin f\, \alpha^2\right) \\ \alpha^2 \to \bar{\alpha}^2 = e^{-\epsilon f}\left(\sin f\, \alpha^1 + \cos f\, \alpha^2\right) \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df \end{array}\right. \tag{5.126}$$

Looking at the second of (5.126), we can find f such that $\bar{V}^2 = 0$. In particular we can write the right-hand sides of (5.126) as

$$\begin{aligned} \bar{\alpha}^1 &= U^1\, du + d\tilde{V}, \\ \bar{\alpha}^2 &= \tilde{U}_u\, du, \\ \bar{\alpha}^3 &= U^3\, du + V^3\, dv, \end{aligned} \tag{5.127}$$

for some functions $\tilde{U}$,$\tilde{V}$, U^i and V^3 of (u,v). From (5.123d) we see that $\tilde{U}_u\tilde{V}_v \neq 0$, therefore we can change coordinates from (a,b,c,u,v) to $(a,b,c,\tilde{U},\tilde{V})$. Dropping the tilde and the bars we write (5.127) as

$$\begin{aligned} \alpha^1 &= F\, du + dv, \\ \alpha^2 &= du, \\ \alpha^3 &= U^3\, du + V^3\, dv, \end{aligned} \tag{5.128}$$

From (5.123b) we have

$$\begin{aligned} 0 = d\alpha^2 = (\epsilon\, \alpha^2 - \alpha^1) \wedge \alpha^3 &= [(\epsilon - F)\, du - dv)] \wedge (U^3\, du + V^3\, dv) \\ &= [V^3(\epsilon - F) + U^3]\, du \wedge dv \end{aligned}$$

and from (5.123a) we obtain

$$\begin{aligned} -F_v\, du \wedge dv = d\alpha^1 = (\epsilon\, \alpha^1 + \alpha^2) \wedge \alpha^3 &= [(1 + \epsilon F)\, du + \epsilon\, dv] \wedge (U^3\, du + V^3\, dv) \\ &= [V^3(1 + \epsilon F) - \epsilon U^3]\, du \wedge dv. \end{aligned}$$

Consequently we have the system

$$V^3(\epsilon - F) + U^3 = 0,$$
$$V^3(1 + \epsilon F) - \epsilon U^3 = -F_v,$$

that is, $U^3 = \frac{1}{\epsilon^2+1}(\epsilon - F)F_v$ and $V^3 = -\frac{1}{\epsilon^2+1}F_v$. Consequently we can write (5.128) as

$$\begin{aligned} \alpha^1 &= F\,du + dv, \\ \alpha^2 &= du, \\ \alpha^3 &= -\frac{1}{\epsilon^2+1}F_v[(F-\epsilon)\,du + dv]. \end{aligned} \tag{5.129}$$

Accordingly (5.123c) is equivalent to $d(F_v[(F-\epsilon)\,du + dv]) \neq 0$, that is,

$$0 \neq D_v[F_v(F-\epsilon)] - D_u[F_v] = D_v[F_v(F-\epsilon) - F_u],$$

which is equivalent to (5.125).

Finally, using (5.120) and (5.129), the GR_3D_5 Pfaffian system (5.122) has the local expression (5.124). □

5.8.4 Step 4

Here we proceed, as in section 5.3.4, to prove the following.

Proposition 5.8.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A^{\epsilon}_{3,7}$. Then $\Gamma = \langle e^{\epsilon X}\sin X\,\partial_Y,\ -e^{\epsilon X}\cos X\,\partial_Y,\ -\partial_X\rangle$ and

$$Z_1 = h\big(Z, Y_2 - 2\epsilon Y_1 + \epsilon^2 Y + Y\big), \text{ with } D^2_{[2]}h \neq 0. \tag{5.130}$$

Proof. We will follow the same steps as in the proof of Proposition 5.4.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N. By Proposition 5.8.2, there are local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle A\,\partial_a + B\,\partial_b,\ A\,\partial_b -$

$B\,\partial_a,\ -\partial_c\rangle$, where $A = \dfrac{e^{\epsilon c}(\epsilon \sin c - \cos c)}{1+\epsilon^2}$ and $B = \dfrac{e^{\epsilon c}(\epsilon \cos c + \sin c)}{1+\epsilon^2}$, and

$$I = \begin{cases} \eta^1 = \epsilon\, da + db + \left(a - \epsilon^2 a - 2\epsilon b\right)\, dc + du, \\ \eta^2 = da - \epsilon\, db + \left(\epsilon^2 b - b - 2\epsilon a\right)\, dc + F\, du + dv, \\ \eta^3 = dc - \dfrac{F_v}{\epsilon^2+1}\left((F-\epsilon)\, du + dv\right), \end{cases} \tag{5.131}$$

where $F = F(u,v)$ is a differentiable function subject to the constraint $D_v[(F-\epsilon)F_v - F_u] \neq 0$. We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.131) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Immediately we see that $\mathrm{Eng}\left(\eta^1\right) = 1$. Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = c, \quad y^2 = u + \epsilon a + b, \quad y^3 = -a + \epsilon^2 a + 2\epsilon b, \quad y^4 = u, \quad y^5 = v,$$

The inverse is τ_1^{-1} given by

$$\begin{aligned} &a = \frac{-y^3 + 2\epsilon y^2 - 2\epsilon y^4}{\epsilon^2+1}, \quad b = \frac{-y^2\epsilon^2 + y^2 + y^4\epsilon^2 - y^4 + \epsilon y^3}{\epsilon^2+1}, \\ &c = y^1, \quad u = y^4, \quad v = y^5. \end{aligned}$$

Accordingly, we see that $\theta^1 = {\tau_1^{-1}}^*\eta^1 = dy^2 - y^3\, dy^1$, as wanted. Now, we write η^2 in these new coordinates. Set $F = F(y^4, y^5) = F(u,v)$, then

$$\eta^2_0 = {\tau_1^{-1}}^*\eta^2 = (-y^2\epsilon^2 + y^4\epsilon^2 + \epsilon y^3 - y^2 + y^4)\, dy^1 + \epsilon\, dy^2 - dy^3 + (F-\epsilon)\, dy^4 + dy^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\, dy^1, \\ \eta^2_0 = (-y^2\epsilon^2 + y^4\epsilon^2 + \epsilon y^3 - y^2 + y^4)\, dy^1 + \epsilon\, dy^2 - dy^3 + (F-\epsilon)\, dy^4 + dy^5. \end{cases} \tag{5.132}$$

[2] We seek for $\theta^2 = dU - V^3\, dy^3 - V^1\, dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. Using (5.132) and the notation in Proposition 3.3.3, we write

$$\begin{aligned}\eta_0^2 &= Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5\\ &= (-y^2\epsilon^2 + y^4\epsilon^2 + \epsilon y^3 - y^2 + y^4)\, dy^1 + \epsilon\, dy^2 - dy^3 + (F-\epsilon)\, dy^4 + dy^5.\end{aligned}$$

Accordingly, the PDE system (3.47) becomes

$$\begin{aligned}\frac{\partial U}{\partial y^4} &= AY^4 = A(F-\epsilon),\\ \frac{\partial U}{\partial y^5} &= AY^5 = A.\end{aligned} \tag{5.133}$$

Without loss in generality, because $\frac{\partial(FF_{y^5} - \epsilon F_{y^5} - F_{y^4})}{\partial y^5} \neq 0$, we can set $F = \frac{G_{y^4} + \epsilon G_{y^5}}{G_{y^5}}$ for a generic function G. Then take $A = G_{y^5}$ and (5.133) becomes

$$\begin{aligned}\frac{\partial U}{\partial y^4} &= G_{y^5}\left(\frac{G_{y^4} + \epsilon G_{y^5}}{G_{y^5}} - \epsilon\right) = G_{y^5}\frac{G_{y^4}}{G_{y^5}} = G_{y^4},\\ \frac{\partial U}{\partial y^5} &= G_{y^5},\end{aligned} \tag{5.134}$$

and the solution is $U = G(y^4, y^5)$, thus $F = \frac{U_{y^4} + \epsilon U_{y^5}}{U_{y^5}}$ and $A = U_{y^5}$. Consequently, the expressions (3.48) become

$$\begin{aligned}W &= \frac{\partial U}{\partial y^2} - AY_0^2 = -\epsilon U_{y^5},\\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = U_{y^5},\\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 W - AY_0^1 = -U_{y^5}(-y^2\epsilon^2 + y^4\epsilon^2 + \epsilon y^3 - y^2 + y^4).\end{aligned} \tag{5.135}$$

From (5.135) we define

$$\hat{f} = -V^1 = U_{y^5}(-y^2\epsilon^2 + y^4\epsilon^2 + \epsilon y^3 - y^2 + y^4), \tag{5.136}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U, \quad x^5 = U_{y^5}.$$

The inverse of τ_2 can be defined by

$$\tau_2^{-1}: \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = \tilde{U}, \quad y^5 = \tilde{V}.$$

In particular from (5.136) we compute

$$f = \hat{f} \circ \tau_2^{-1} = x^5[(\epsilon^2 + 1)(\tilde{U} - x^2) + \epsilon x^3]. \tag{5.137}$$

Consequently we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \\ \theta^2 = \eta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \end{cases} \tag{5.138}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5 x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\, dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\, dx^1, \\ \\ \theta^2 = dx^4 - x^5\, dx^3 + f\, dx^1, \\ \\ \theta^3 = dx^3 - f_{x^5}\, dx^1. \end{cases}$$

From (5.194), we compute

$$f_{x^5} = x^5 \hat{T}(x^4, x^5) - x^2(\epsilon^2 + 1) + \epsilon x^3. \tag{5.139}$$

[4] Here we obtain the Monge normal form. Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = f_{x^5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x^1 = X, \quad x^2 = Y, \quad x^3 = Y_1, \quad x^4 = Z, \quad x^5 = \tilde{T}(Y, Z, Y_1, Y_2),$$

where $\tilde{T} \circ \tau_4 = f_{x^5}$.

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = c, \quad Y = u + \epsilon a + b, \quad Z = U_v, \quad Y_1 = 2\epsilon b + (\epsilon^2 - 1)a,$$
$$Y_2 = (\epsilon^2 - 1)b - 2\epsilon a - (\epsilon^2 + 1)(u + \bar{U}_4)$$

and with inverse τ^{-1} given by

$$a = \frac{2\epsilon Y - Y_1 - 2\epsilon \hat{U}}{\epsilon^2 + 1}, \quad b = \frac{(1 - \epsilon^2)Y + \epsilon Y_1 + (\epsilon^2 - 1)\hat{U}}{\epsilon^2 + 1}, \quad c = X, \quad u = \hat{U}, \quad v = \hat{V}.$$

According to these,

$$\Gamma = \langle \frac{e^{\epsilon c}(\epsilon \sin c - \cos c)}{1 + \epsilon^2}\, \partial_a + \frac{e^{\epsilon c}(\epsilon \cos c + \sin c)}{1 + \epsilon^2}\, \partial_b,$$
$$\frac{e^{\epsilon c}(\epsilon \sin c - \cos c)}{1 + \epsilon^2}\, \partial_b - \frac{e^{\epsilon c}(\epsilon \cos c + \sin c)}{1 + \epsilon^2}\, \partial_a,\, -\partial_c \rangle$$

is τ-related to

$$\langle e^{\epsilon X} \sin X \partial_Y + (\epsilon \sin X + \cos X)e^{\epsilon X} \partial_{Y_1} + (\epsilon \cos X - \sin X)e^{\epsilon X} \partial_{Y_2},$$
$$- e^{\epsilon X} \cos X \partial_Y - (\epsilon \cos X - \sin X)e^{\epsilon X} \partial_{Y_1} + (\epsilon \sin X + \cos X)e^{\epsilon X} \partial_{Y_2},\, -\partial_X \rangle,$$

which is the prolongation of $\langle e^{\epsilon X} \sin X \partial_Y,\, -e^{\epsilon X} \cos X \partial_Y,\, -\partial_X \rangle$. Consequently we have $Z_1 = h(Z, Y_2 - 2\epsilon Y_1 + \epsilon^2 Y + Y)$. □

5.9 $\mathbf{A_{3,8}}$: the special linear algebra

5.9.1 Step 1

This Lie algebra has structure equations

$$[E_1, E_2] = 2\, E_2,$$
$$[E_1, E_3] = -2\, E_3, \tag{5.140}$$
$$[E_2, E_3] = E_1.$$

The matrix group representation of G that we use is given by

$$A = \begin{pmatrix} 1+2cb & -ce^{2a} & b\,(1+cb)\,e^{-2a} \\ -2b & e^{2a} & -b^2e^{-2a} \\ 2c\,(1+cb) & -c^2e^{2a} & (1+cb)^2\,e^{-2a} \end{pmatrix}$$

The (left-invariant) Maurer-Cartan forms are then

$$\begin{aligned} \omega^1 &= d\,a - b\,d\,c, \\ \omega^2 &= e^{-2a}\,\left(d\,b - b^2\,d\,c\right), \\ \omega^3 &= e^{2a}\,d\,c, \end{aligned} \tag{5.141}$$

with structure equations

$$\begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega^3, \\ d\omega^2 &= -2\,\omega^1 \wedge \omega^2, \\ d\omega^3 &= 2\,\omega^1 \wedge \omega^3. \end{aligned} \tag{5.142}$$

The dual to (5.141) are

$$\begin{aligned} R_1 &= \partial_a, \\ R_2 &= e^{2a}\,\partial_b, \\ R_3 &= be^{-2a}\,\partial_a + b^2e^{-2a}\,\partial_b + e^{-2a}\,\partial_c. \end{aligned}$$

The associated 1-parameter subgroups of G are

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2t_1} & 0 \\ 0 & 0 & e^{-2t_1} \end{pmatrix},\ U_2 = \begin{pmatrix} 1 & 0 & t_2 \\ -2t_2 & 1 & -t_2^2 \\ 0 & 0 & 1 \end{pmatrix},\ U_3 = \begin{pmatrix} 1 & -t_3 & 0 \\ 0 & 1 & 0 \\ 2t_3 & -t_3^2 & 1 \end{pmatrix}$$

while the associated right-invariant vector fields are

$$\begin{aligned} L_1 &= -\partial_a - 2b\,\partial_b + 2c\,\partial_c, \\ L_2 &= c\,\partial_a + (1+2cb)\ \partial_b - c^2\,\partial_c, \\ L_3 &= \partial_c. \end{aligned}$$

An optimal list of 1-dimensional subalgebras is given by

$$O = \left[\frac{1}{2}(E_2 - E_3), E_1, E_2\right],$$

thus for $(I')^{\perp}$ we have the following three cases.

Case 1: $\theta^1 = \omega^1 + \alpha^1$, $\theta^2 = \omega^2 + \omega^3 + \alpha^2$, $\theta^3 = \omega^2 - \omega^3 + \alpha^3$.

Case 2: $\theta^1 = \omega^2 + \alpha^1$, $\theta^2 = \omega^3 + \alpha^2$, $\theta^3 = \omega^1 + \alpha^3$.

Case 3: $\theta^1 = \omega^1 + \alpha^1$, $\theta^2 = \omega^3 + \alpha^2$, $\theta^3 = \omega^2 + \alpha^3$.

In Case 3 we have $I'' \neq 0$. We shall see that the other two cases give rise to inequivalent GR_3D_5 Pfaffian systems, which we will consider separately.

5.9.2 Step 2.1

Proposition 5.9.1. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,8}$ and that $\langle E_1 \rangle$ is the Cartan subalgebra of Γ. If $E_1 \notin (I')^{\perp}$ then we have

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = \omega^2 + \omega^3 + \alpha^2, \\ \theta^3 = \omega^2 - \omega^3 + \alpha^3, \end{cases} \tag{5.143}$$

with derived flag conditions

$$d\alpha^1 = \alpha^2 \wedge \alpha^3, \tag{5.144a}$$

$$d\alpha^2 = 2\,\alpha^1 \wedge (\alpha^2 - 2\,\alpha^3), \tag{5.144b}$$

$$d\alpha^3 \neq -2\,\alpha^1 \wedge \alpha^3, \tag{5.144c}$$

$$\alpha^1 \wedge \alpha^2 \neq 0. \tag{5.144d}$$

Proof. From Section 5.9.1 and the hypothesis $E_1 \notin (I')^{\perp}$, we must consider Case 1. In this case, conditions (5.1a) and (5.1b) are respectively equivalent to $d\alpha^1 = \alpha^2 \wedge \alpha^3$ and $d\alpha^2 = 2\,\alpha^1 \wedge (\alpha^2 - 2\,\alpha^3)$. Conditions (5.1c) and (5.1d) are respectively equivalent to $d\alpha^3 \neq -2\,\alpha^1 \wedge \alpha^3$ and $\alpha^1 \wedge \alpha^2 \neq 0$. $\square$

5.9.3 Step 3.1

Proposition 5.9.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,8}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle -\partial_a - 2b\,\partial_b + 2c\,\partial_c,\ c\,\partial_a + (1 + 2cb)\ \partial_b - c^2\,\partial_c,\ \partial_c \rangle$. Moreover, if $\partial_a \notin (I')^\perp$ then

$$I = \begin{cases} \theta^1 = da - b\,dc + F\,du + dv, \\ \theta^2 = e^{-2a}\,db - b^2 e^{-2a}\,dc + du, \\ \theta^3 = e^{2a}\,dc - \left(\dfrac{1}{2} - FF_v\right)\,du - F_v\,dv, \end{cases} \tag{5.145}$$

where $F = F(u, v)$ is a differentiable function such that

$$D_v[FF_v - v - F_u] \neq 0. \tag{5.146}$$

Proof. We start from the results of Proposition 5.9.1. The admissible gauge transformations are given by

$$U(f(u,v)) = \exp\left(\frac{f}{2}(R_2 - R_3)\right) = \begin{pmatrix} \cos f & \frac{1}{2}\sin f & \frac{1}{2}\sin f \\ -\sin f & \frac{1}{2}(1 + \cos f) & \frac{1}{2}(\cos f - 1) \\ -\sin f & \frac{1}{2}(\cos f - 1) & \frac{1}{2}(1 + \cos f) \end{pmatrix}.$$

From (5.3), the 1-forms α^i change by

$$\left[\begin{aligned} &\alpha^1 \to \bar{\alpha}^1 = \cos f\,\alpha^1 + \frac{1}{2}\sin f\,\alpha^2, \\ &\alpha^2 \to \bar{\alpha}^2 = -2\sin f\,\alpha^1 + \cos f\,\alpha^2, \\ &\alpha^3 \to \bar{\alpha}^3 = -\sin f\,\alpha^1 + \frac{1}{2}(\cos f - 1)\,\alpha^2 + \alpha^3 - \frac{1}{2}\,df. \end{aligned}\right. \tag{5.147}$$

From (5.144d), that is, $\alpha^1 \wedge \alpha^2 \neq 0$, we see that $\alpha^1 \neq 0$ and $\alpha^2 \neq 0$ and without loss in generality we can apply (5.147) to obtain $\bar{\alpha}^2 = U_u\, du$, for some function $U = U(u,v)$. Moreover, we can rewrite the right-hand side of (5.147) as

$$\bar{\alpha}^1 = F^1\, du + dV,$$

$$\bar{\alpha}^2 = U_u\, du,$$

$$\bar{\alpha}^3 = C^1\, du + C^2\, dv,$$

for generic functions F^1, V, C^1 and C^2 of (u,v) subject to the constraint (5.144). Now, condition (5.144d), implies $V_v\, U_u \neq 0$. We can thus change coordinates to $(a, b, c, \tilde{u} = U, \tilde{v} = V)$. Dropping the tilde and the bars we write

$$\alpha^1 = F\, du + dv,$$

$$\alpha^2 = du,$$

$$\alpha^3 = C^1\, du + C^2\, dv.$$

Condition (5.144d) is satisfied. Condition (5.144a) becomes

$$-F_v\, du \wedge dv = d\alpha^1 = \alpha^2 \wedge \alpha^3 = C^2\, du \wedge dv,$$

and thus $C^2 = -F_v$. Consequently, we can write (5.144b) as

$$0 = d\alpha^2 = 2\,\alpha^1 \wedge (\alpha^2 - 2\,\alpha^3) = 2(F\, du + dv) \wedge ((1-2C^1)\, du + 2F_v\, dv) = 2(1 - 2C^1 - 2FF_v)\, du \wedge dv,$$

and thus we have $C^1 = -\frac{1}{2} + FF_v$. We can now rewrite (5.143)

$$I = \begin{cases} \theta^1 = \omega^1 + F\, du + dv, \\ \theta^2 = \omega^2 + \omega^3 + du, \\ \theta^3 = \omega^2 - \omega^3 - \left(\dfrac{1}{2} - FF_v\right)\, du - F_v\, dv, \end{cases} \tag{5.148}$$

which is (5.145), using (5.141).

Finally, from (5.144c) we get

$$0 \neq d\alpha^3 + 2\,\alpha^1 \wedge \alpha^3 = \left(F_{uv} - F_v^2 - FF_{vv} + 2FF_v + 1 - 2FF_v\right)\, du \wedge dv = (F_{uv} - F_v^2 - FF_{vv} + 1)\, du \wedge dv,$$

that is, $F_{uv} - F_v^2 - FF_{vv} + 1 \neq 0$. This proves (5.146). $\square$

5.9.4 Step 4.1

Here we proceed, as in section 5.3.4, to prove the following proposition.

Proposition 5.9.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A_{3,8}$. Moreover, assume that I is the GR_3D_5 Pfaffian system generated by the given general Monge equation and that $(I')^\perp$ is not the Cartan subalgebra of Γ. Then $\Gamma = \langle 2X\,\partial_X - 2\partial_Y,\, 2X\,\partial_Y - X^2\,\partial_X,\, \partial_X \rangle$ and

$$Z_1 = e^Y\, h\left(Z, Y_2 e^{-2Y} - \frac{1}{2}{Y_1}^2 e^{-2Y}\right), \text{ with } D^2_{[2]} h \neq 0. \tag{5.149}$$

Proof. We will follow the same steps as in the proof of Proposition 5.4.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N with local coordinates (X, Y, Z, Y_1, Y_2). By Proposition 5.9.2, there are local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle -\partial_a - 2b\,\partial_b + 2c\,\partial_c,\, c\,\partial_a + (1 + 2cb)\,\partial_b - c^2\,\partial_c,\, \partial_c \rangle$, and

$$I = \begin{cases} \eta^1 = e^{-2a}\,db - b^2 e^{-2a}\,dc + du, \\ \eta^2 = da - b\,dc + F\,du + dv, \\ \eta^3 = e^{2a}\,dc - \left(\dfrac{1}{2} - FF_v\right)\,du - F_v\,dv, \end{cases} \tag{5.150}$$

where $F = F(u, v)$ is a differentiable function such that $\frac{\partial(FF_v - v - F_u)}{\partial v} \neq 0$. We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.150) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = c + \frac{1}{b + e^{2a}}, \quad y^2 = u + \ln\left(\frac{1}{2}(e^{2a} + b)^2\right) - 2a, \quad y^3 = b^2 e^{-2a} - e^{2a}, \quad y^4 = u, \quad y^5 = v,$$

The inverse is τ_1^{-1} given by

$$a = -\frac{3}{2}\ln 2 + \ln(2e^{y^2-y^4} - y^3) - \frac{1}{2}(y^2 - y^4), \quad b = \frac{1}{8}e^{y^4-y^2}(4e^{2y^2-2y^4} - y^{3^2}),$$
$$c = y^1 - \frac{2}{2e^{y^2-y^4} - y^3}, \quad u = y^4, \quad v = y^5.$$

Accordingly, we see that $\theta^1 = \tau_1^{-1^*}\eta^1 = dy^2 - y^3\, dy^1$, as wanted. Now, we write η^2 in these new coordinates. Set $F = F(y^4, y^5) = F(u, v)$, with $\frac{\partial(FF_{y^5} - y^5 - F_{y^4})}{\partial y^5} \neq 0$, then

$$\eta_0^2 = \tau_1^{-1^*}\eta^2 = -\frac{1}{8}e^{y^4-y^2}(4e^{2y^2-2y^4} - y^{3^2})\, dy^1 - \frac{1}{4}e^{y^4-y^2}\, dy^3 + F\, dy^4 + dy^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\, dy^1, \\ \eta_0^2 = -\frac{1}{8}e^{y^4-y^2}(4e^{2y^2-2y^4} - y^{3^2})\, dy^1 - \frac{1}{4}e^{y^4-y^2}\, dy^3 + F\, dy^4 + dy^5. \end{cases} \tag{5.151}$$

[2] We seek for $\theta^2 = dU - V^3\, dy^3 - V^1\, dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. The dy^4 and dy^5 components of η_0^2 are respectively $Y^4 = F$ and $Y^5 = 1$. The PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = AF, \\ \frac{\partial U}{\partial y^5} &= AY^5 = A, \end{aligned} \tag{5.152}$$

Without loss in generality, because $\frac{\partial(FF_{y^5} - y^5 - F_{y^4})}{\partial y^5} \neq 0$, we can set $F = \frac{G_{y^4}}{G_{y^5}}$ for a generic function G. Then take $A = G_{y^5}$ and (5.152) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= G_{y^4}, \\ \frac{\partial U}{\partial y^5} &= G_{y^5}, \end{aligned} \tag{5.153}$$

and the solution is $U = G(y^4, y^5)$, thus $F = \frac{U_{y^4}}{U_{y^5}}$ and $A = U_{y^5}$. Using (5.151) and the notation in Proposition 3.3.3, that is

$$\begin{aligned} \eta_0^2 &= Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5 \\ &= -\frac{1}{8}e^{y^4-y^2}(4e^{2y^2-2y^4} - y^{3^2})\, dy^1 - \frac{1}{4}e^{y^4-y^2}\, dy^3 + \frac{U_{y^4}}{U_{y^5}}\, dy^4 + dy^5, \end{aligned}$$

the expressions (3.48) become

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = \frac{1}{4}e^{y^4-y^2}U_{y^5}, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3(\frac{\partial U}{\partial y^2} - AY_0^2) - AY_0^1 = -\frac{1}{8}e^{y^4-y^2}(4e^{2y^2-2y^4} - {y^3}^2)U_{y^5}. \end{aligned} \tag{5.154}$$

In particular, we can now write

$$\theta^2 = U_{y^5}\,\eta_0^2 = \frac{1}{8}e^{y^4-y^2}(4e^{2y^2-2y^4} - {y^3}^2)U_{y^5}\,dy^1 - \frac{1}{4}e^{y^4-y^2}U_{y^5}\,dy^3 + dU.$$

From (5.154) let's define

$$\hat{f} = -V^1 = \frac{1}{2}({y^3}^2 - 4e^{2y^2-2y^4})V^3, \tag{5.155}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U, \quad x^5 = V^3 = \frac{1}{4}e^{y^4-y^2}U_{y^5}.$$

The local diffeomorphism τ_2 defines the mapping

$$x_2 = y^2, \quad x^4 = U, \quad x_5 = \frac{1}{4}e^{y^4-y^2}U_{y^5},$$

and by the inverse function theorem there are defined two functions $\tilde{U} = \tilde{U}\left(x^2, x^4, x^5\right)$ and $\tilde{V} = \tilde{V}\left(x^2, x^4, x^5\right)$ on N such that

$$\tilde{U} \circ \tau_2 = y^4, \quad \tilde{V} \circ \tau_2 = y^5.$$

The inverse of τ_2 can thus be defined by

$$\tau_2^{-1}: \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = \tilde{U}, \quad y^5 = \tilde{V}.$$

In particular from (5.155) we compute

$$f = \hat{f} \circ \tau_2^{-1} = \frac{1}{2}({x^3}^2 - 4e^{2x^2-2\tilde{U}})x^5. \tag{5.156}$$

Consequently we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = \eta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \end{cases} \tag{5.157}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\,dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases}$$

[4] Here we obtain the Monge normal form. In view of (3.84), we set $\tilde{H} = x^5 f_{x^5} - f$ and from (5.156) we compute.

$$\tilde{H} = \frac{1}{2} x^{5^2} \tilde{U}_{x^5} e^{-x^2+\tilde{U}} (4e^{2x^2-2\tilde{U}} - x^{3^2}). \tag{5.158}$$

Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = f_{x^5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x^1 = X, \quad x^2 = Y, \quad x^3 = Y_1, \quad x^4 = Z, \quad x^5 = \tilde{T}\,(Z, Y, Y_1, Y_2)\,,$$

where $\tilde{T} \circ \tau_4 = f_{x^5}$. From (5.158) we know that the general Monge equation is now expressed by

$$Z_1 = H = \tilde{H} \circ \tau_4^{-1} = H(Z, Y, Y^1, Y^2). \tag{5.159}$$

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = c + \frac{1}{b + e^{2a}}, \quad Y = u + \ln\left(\frac{1}{2}(e^{2a} + b)^2\right) - 2a,$$
$$Y = b, \quad Z = U(u, v), \quad Y_1 = b^2 e^{-2a} - e^{2a}, \quad Y_2 = \check{T}\,(a, b, u, v)\,,$$

and with inverse τ^{-1} given by

$$a = -\frac{3}{2}\ln 2 + \ln\left(2e^{Y-\check{U}} - Y_1\right) - \frac{1}{2}(Y - \check{U}), \quad b = \frac{1}{8}e^{-Y+\check{U}}(4e^{2Y-2\check{U}} - {Y_1}^2),$$
$$c = X - \frac{2}{2e^{Y-\check{U}} - Y_1}, \quad u = \check{U}, \quad v = \check{V}.$$

According to these, Γ is τ-related to

$$\langle 2X\,\partial_X - 2\partial_Y - 2Y_1\partial_{Y_1} - 4Y_2\partial_{Y_2}, -X^2\,\partial_X + 2X\,\partial_Y + (2XY_1 + 2)\,\partial_{Y_1} + (4XY_2 + 2Y_1)\,\partial_{Y_2}, \partial_X\rangle,$$

which is the prolongation of $\langle 2X\,\partial_X - 2\partial_Y,\, 2X\,\partial_Y - X^2\,\partial_X,\, \partial_X\rangle$. From this we check that (5.159) has to be (5.149). □

5.9.5 Step 2.2

With the notations of Section 5.9.1, we have the following.

Proposition 5.9.4. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,8}$ and that $\langle E_1\rangle$ is the Cartan subalgebra of Γ. If $E_1 \in (I')^\perp$ then we have

$$I = \begin{cases} \theta^1 = \omega^2 + \alpha^1, \\ \theta^2 = \omega^3 + \alpha^2, \\ \theta^3 = \omega^1 + \alpha^3, \end{cases} \tag{5.160}$$

where the derived flag conditions are

$$d\alpha^1 = -2\,\alpha^1 \wedge \alpha^3, \tag{5.161a}$$

$$d\alpha^2 = 2\,\alpha^2 \wedge \alpha^3, \tag{5.161b}$$

$$d\alpha^3 \neq \alpha^1 \wedge \alpha^2, \tag{5.161c}$$

$$\alpha^1 \wedge \alpha^2 \neq 0. \tag{5.161d}$$

Proof. From Proposition 5.9.1 and the hypothesis $E_1 \in (I')^\perp$, we must consider Case 2. In this case, conditions (5.1a) and (5.1b) are respectively equivalent to $d\alpha^1 = -2\,\alpha^1 \wedge \alpha^3$ and $d\alpha^2 = 2\,\alpha^2 \wedge \alpha^3$. Conditions (5.1c) and (5.1d) can be written as $d\alpha^3 \neq \alpha^1 \wedge \alpha^2$ and $\alpha^1 \wedge \alpha^2 \neq 0$. These give (5.161). □

5.9.6 Step 3.2

Proposition 5.9.5. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,8}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that $\Gamma = \langle -\partial_a - 2b\,\partial_b + 2c\,\partial_c,\ c\,\partial_a + (1 + 2cb)\,\partial_b - c^2\,\partial_c,\ \partial_c\rangle$. Moreover, if $\partial_a \in (I')^\perp$ then

$$I = \begin{cases} \theta^1 = e^{-2a}\,db - b^2 e^{-2a}\,dc + F\,du + dv, \\ \theta^2 = e^{2a}\,dc + du, \\ \theta^3 = da - b\,dc - \dfrac{1}{2} F_v\,du, \end{cases} \tag{5.162}$$

where $F = F(u, v)$ is a differentiable function such that

$$F_{vv} \neq -2. \tag{5.163}$$

Proof. The admissible gauge transformations in this case are those which fix E_1, thus only the action of

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2t_1} & 0 \\ 0 & 0 & e^{-2t_1} \end{pmatrix}.$$

From (5.3), the 1-forms α^i change by

$$\left[\begin{aligned} &\alpha \to \bar{\alpha} = e^{2f}\,\alpha \\ &\beta \to \bar{\beta} = e^{-2f}\,\beta \\ &\gamma \to \bar{\gamma} = \gamma + df \end{aligned}\right. \tag{5.164}$$

From (5.161d)), that is, $\alpha^1 \wedge \alpha^2 \neq 0$, we see that $\alpha^1 \neq 0$ and $\alpha^2 \neq 0$ and without loss in generality we can apply (5.164) to obtain $\bar{\alpha}^2 = dU$, for some function $U = U(u,v)$. Moreover, we can rewrite the right-hand side of (5.147) as

$$\begin{aligned} \bar{\alpha}^1 &= F\,du + G_v\,dv, \\ \bar{\alpha}^2 &= dU, \\ \bar{\alpha}^3 &= U^3\,du + V^3\,dv, \end{aligned}$$

for generic functions F, G, U^3 and V^3 of (u,v) subject to the constraint (5.161). Now, condition (5.161d), implies $G_v\,U_u \neq 0$. We can thus change coordinates to $(a,b,c,\tilde{u} = U, \tilde{v} = G)$. Dropping the tilde and the bars we write

$$\begin{aligned} \alpha^1 &= F\,du + dv, \\ \alpha^2 &= du, \\ \alpha^3 &= U^3\,du + V^3\,dv, \end{aligned}$$

Condition (5.161d) is satisfied. Condition (5.161b) becomes

$$0 = d\alpha^2 = 2\alpha^2 \wedge \alpha^3 = 2V^3\,du \wedge dv,$$

and thus $V^3 = 0$. Consequently, we can write (5.161a) as

$$-F_v\,du \wedge dv = d\alpha^1 = -2\,\alpha^1 \wedge \alpha^3 = 2U^3\,du \wedge dv,$$

and thus we have $U^3 = -\frac{1}{2}F_v$. We can now rewrite (5.160)

$$I = \begin{cases} \theta^1 = \omega^2 + F\,du + dv, \\ \theta^2 = \omega^3 + du, \\ \theta^3 = \omega^1 - \dfrac{1}{2}F_v\,du, \end{cases} \tag{5.165}$$

which is (5.162), using (5.141). Finally, condition (5.161c) is now

$$\frac{1}{2}F_{vv}\,du \wedge dv = d\alpha^3 \neq \alpha^1 \wedge \alpha^2 = -du \wedge dv,$$

from which we obtain (5.163). □

5.9.7 Step 4.2

Here we proceed, as in section 5.3.4, to prove the following proposition.

Proposition 5.9.6. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A_{3,8}$. Moreover, assume that I is the GR_3D_5 Pfaffian system generated by the given general Monge equation and that $(I')^\perp$ is the Cartan subalgebra of Γ. Then $\Gamma = \langle 2X\,\partial_X - 2Z\,\partial_Z,\, -X^2\,\partial_X + (1 + 2XZ)\,\partial_Z,\, \partial_X \rangle$ and

$$Z_1 = Z^2 + Y_1^2\,h\left(Y, \frac{Y_2 - 2Y_1 Z}{Y_1^2}\right), \text{ with } D^2_{[2]}h \neq 0. \tag{5.166}$$

Proof. We will follow the same steps as in the proof of Proposition 5.4.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N with local coordinates (X, Y, Z, Y_1, Y_2). By Proposition 5.9.2, there are local coordinates (a, b, c, u, v)

on N such that $\Gamma = \langle -\partial_a - 2b\,\partial_b + 2c\,\partial_c,\, c\,\partial_a + (1+2cb)\,\partial_b - c^2\,\partial_c,\, \partial_c\rangle$ and

$$I = \begin{cases} \eta^1 = e^{2a}\,dc + du, \\ \eta^2 = e^{-2a}\,db - b^2 e^{-2a}\,dc + F\,du + dv, \\ \eta^3 = da - b\,dc - \dfrac{1}{2}F_v\,du, \end{cases} \tag{5.167}$$

where $F = F(u,v)$ is a differentiable function such that $F_{vv} \neq -2$. We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.167) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = c, \quad y^2 = -u, \quad y^3 = e^{2a}, \quad y^4 = b, \quad y^5 = v,$$

The inverse is τ_1^{-1} given by

$$a = \frac{1}{2}\ln(y^3), \quad b = y^4, \quad c = y^1, \quad u = -y^2, \quad v = y^5.$$

Accordingly, we see that $\theta^1 = {\tau_1^{-1}}^*\eta^1 = dy^2 - y^3\,dy^1$, as wanted. Now, we write η^2 in these new coordinates. Set $F = F(-y^2, y^5) = F(u,v)$, with $F_{y^5y^5} \neq -2$, then

$$\eta_0^2 = {\tau_1^{-1}}^*\eta^2 = \frac{1}{y^3}\,dy^4 - \frac{{y^4}^2}{y^3}\,dy^1 - F\,dy^2 + dy^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\,dy^1, \\ \eta_0^2 = -\dfrac{{y^4}^2}{y^3}\,dy^1 - F\,dy^2 + \dfrac{1}{y^3}\,dy^4 + dy^5. \end{cases} \tag{5.168}$$

[2] We seek for $\theta^2 = dU - V^3\,dy^3 - V^1\,dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. The dy^4 and dy^5 components of η_0^2 are respectively $Y^4 = \frac{1}{y^3}$ and $Y^5 = 1$. The PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = \frac{A}{y^3}, \\ \frac{\partial U}{\partial y^5} &= AY^5 = A, \end{aligned} \tag{5.169}$$

Take $A = y^3$ and the solution of (5.169) is $U = y^4 + y^5 y^3$. Using (5.168) and the notation in Proposition 3.3.3, that is

$$\begin{aligned} \eta_0^2 &= Y_0^1\,dy^1 + Y_0^2\,dy^2 + Y^3\,dy^3 + Y^4\,dy^4 + Y^5\,dy^5 \\ &= -\frac{{y^4}^2}{y^3}\,dy^1 - F\,dy^2 + \frac{1}{y^3}\,dy^4 + dy^5, \end{aligned}$$

the expressions (3.48) become

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0 + y^3 F = y^3 F, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = y^5 - 0 = y^5, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 W - AY_0^1 = 0 + y^3 F + {y^4}^2 = y^3 F + {y^4}^2. \end{aligned} \tag{5.170}$$

In particular, we can now write

$$\theta^2 = dU - y^5\,dy^3 - (y^3 F + {y^4}^2)\,dy^1.$$

From (5.170) define

$$\hat{f} = -V^1 = -y^3 F - {y^4}^2, \tag{5.171}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U = y^4 + y^5 y^3, \quad x^5 = V^3 = y^5.$$

The local diffeomorphism τ_2 has inverse

$$\tau_2^{-1}: \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = x^4 - x^5 x^3, \quad y^5 = x^5.$$

In particular, writing $F = F(-x^2, x^5) =$, from (5.171) we compute

$$f = \hat{f} \circ \tau_2^{-1} = -{x^4}^2 + 2x^4x^5x^3 - {x^5}^2{x^3}^2 - {x^3}^2F. \tag{5.172}$$

According to this algorithm we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \end{cases} \tag{5.173}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\,dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases}$$

At this point, from (5.172), we compute

$$\begin{aligned} f_{x^5} &= 2x^4x^3 - 2x^5{x^3}^2 - {x^3}^2F_{x^5} = x^3(2x^4 - 2x^5x^3 - x^3F_{x^5}), \\ f_{x^5x^5} &= -{x^3}^2(2 - F_{x^5x^5}) \neq 0. \end{aligned} \tag{5.174}$$

[4] Here we obtain the Monge normal form. In view of (3.84), we set $\tilde{H} = x^5f_{x^5} - f$ and from (5.174) we compute.

$$\tilde{H} = {x^4}^2 + {x^3}^2(F - x^5F_{x^5} - x^5). \tag{5.175}$$

Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x_1, \quad Y = x_2, \quad Z = x_4, \quad Y_1 = x_3, \quad Y_2 = f_{x_5} = x^3(2x^4 - 2x^5x^3 - x^3F_{x^5}).$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x_1 = X, \quad x_2 = Y, \quad x_3 = Y_1, \quad x_4 = Z, \quad x_5 = \tilde{T}\,(Y; Z, Y_1, Y_2)\,,$$

where $\tilde{T} \circ \tau_4 = x^5$. From (5.175) we know that the general Monge equation is now expressed by

$$Z_1 = H = \tilde{H} \circ \tau_4^{-1} = Z^2 + Y^2\, h(Y; Z, Y_1, Y_2). \tag{5.176}$$

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = c, \quad Y = -u, \quad Z = b + ve^{2a}, \quad Y_1 = e^{2a}, \quad Y_2 = e^{2a}(2b - e^{2a}F_v),$$

and with inverse τ^{-1} given by

$$a = \ln Y_1, \quad b = Z - Y_1\check{T}, \quad c = X, \quad u = -Y, \quad v = \check{T}.$$

According to these, Γ is τ-related to

$$\langle 2\,(X\,\partial_X - Z\,\partial_Z - Y_1\,\partial_{Y_1} - 2Y_2\,\partial_{Y_2}), -X^2\,\partial_X + (1 + 2XZ)\;\partial_Z + 2XY_1\,\partial_{Y_1} + 2\,(Y_1 + 2XY_2)\;\partial_X\rangle,$$

which is the prolongation of $\langle 2X\,\partial_X - 2Z\,\partial_Z,\, -X^2\,\partial_X + (1 + 2XZ)\;\partial_Z,\, \partial_X\rangle$. From this we check that the expression h in (5.176) has to be as in (5.166). $\square$

5.10 $\mathbf{A_{3,9}}$: the special orthogonal algebra

5.10.1 Step 1

We take the Lie algebra $A_{3,9}$ with structure equations

$$\begin{aligned}[E_1, E_2] &= E_3,\\ [E_1, E_3] &= -E_2,\\ [E_2, E_3] &= E_1.\end{aligned} \tag{5.177}$$

The matrix group representation of G that we use is given by

$$A = \begin{pmatrix} \cos c \cos b & -\sin c \cos a + \cos c \sin b \sin a & \sin c \sin a + \cos c \sin b \cos a \\ \sin c \cos b & \cos c \cos a + \sin c \sin b \sin a & -\cos c \sin a + \sin c \sin b \cos a \\ -\sin b & \cos b \sin a & \cos b \cos a \end{pmatrix}.$$

The (left invariant) Maurer-Cartan forms are then

$$\begin{aligned} \omega^1 &= da - \sin b\, dc, \\ \omega^2 &= \cos a\, db + \cos b \sin a\, dc, \\ \omega^3 &= -\sin a\, db + \cos b \cos a\, dc, \end{aligned} \tag{5.178}$$

with structure equations

$$\begin{aligned} d\omega^1 &= -\omega^2 \wedge \omega^3, \\ d\omega^2 &= \omega^1 \wedge \omega^3, \\ d\omega^3 &= -\omega^1 \wedge \omega^2. \end{aligned} \tag{5.179}$$

The dual to the (5.178) are

$$\begin{aligned} R_1 &= \partial_a, \\ R_2 &= \frac{\sin a \sin b}{\cos b}\, \partial_a + \cos a\, \partial_b + \frac{\sin a}{\cos b}\, \partial_c, \\ R_3 &= \frac{\cos a \sin b}{\cos b}\, \partial_a - \sin a\, \partial_b + \frac{\cos a}{\cos b}\, \partial_c. \end{aligned}$$

The right action is exerted by the multiplication on the right by the matrices

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t_1 & -\sin t_1 \\ 0 & \sin t_1 & \cos t_1 \end{pmatrix}, U_2 = \begin{pmatrix} \cos t_2 & 0 & \sin t_2 \\ 0 & 1 & 0 \\ -\sin t_2 & 0 & \cos t_2 \end{pmatrix}, U_3 = \begin{pmatrix} \cos t_2 & -\sin t_2 & 0 \\ \sin t_2 & \cos t_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The associated right-invariant vector fields are

$$\begin{aligned} L_1 &= -\frac{\cos c}{\cos b}\, \partial_a + \sin c\, \partial_b - \frac{\cos c \sin b}{\cos b}\, \partial_c, \\ L_2 &= -\frac{\sin c}{\cos b}\, \partial_a - \cos c\, \partial_b - \frac{\sin c \sin b}{\cos b}\, \partial_c, \\ L_3 &= -\partial_c. \end{aligned}$$

5.10.2 Step 2

Proposition 5.10.1. An adapted basis for I is given by

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = \omega^2 + \alpha^2, \\ \theta^3 = \omega^3 + \alpha^3, \end{cases} \tag{5.180}$$

where

$$d\alpha^1 = \alpha^2 \wedge \alpha^3, \tag{5.181a}$$

$$d\alpha^2 = -\alpha^1 \wedge \alpha^3, \tag{5.181b}$$

$$d\alpha^3 \neq \alpha^1 \wedge \alpha^2, \tag{5.181c}$$

$$\alpha^1 \wedge \alpha^2 \neq 0. \tag{5.181d}$$

Proof. In this case all 1-dimensional subalgebras are conjugate and an optimal list is given by $O = [E_3]$, and thus we have only one case:

$$I = \begin{cases} \theta^1 = \omega^1 + \alpha^1, \\ \theta^2 = \omega^2 + \alpha^2, \\ \theta^3 = \omega^3 + \alpha^3, \end{cases}$$

where $\alpha^i = U^i(u,v)\,du + V^i(u,v)\,dv$. Conditions (5.1a) and (5.1b) are respectively equivalent to $d\alpha^1 = \alpha^2 \wedge \alpha^3$ and $d\alpha^2 = -\alpha^1 \wedge \alpha^3$. Conditions (5.1c) and (5.1d) are respectively equivalent to $d\alpha^3 \neq \alpha^1 \wedge \alpha^2$ and $\alpha^1 \wedge \alpha^2 \neq 0$. Hence, the derived flag conditions give (5.181). □

5.10.3 Step 3

Proposition 5.10.2. Let I be a GR_3D_5 Pfaffian system on a 5-manifold M. Assume that I admits a 3-dimensional symmetry group G which acts freely and transversely on M and denote by Γ a set of infinitesimal generators of the action of G on M. Assume that Γ has algebraic type $A_{3,9}$. Then about each point of M there are local coordinates (a, b, c, u, v) such that

$$\Gamma = \langle A_c\,\partial_a + B_c\,\partial_b + C_c\,\partial_c, \quad A\,\partial_a + B\,\partial_b + C\,\partial_c, \quad -\partial_c\rangle,$$

where $A = \dfrac{-\sin c}{\cos b}$ and $B = -\cos c$, $C = A\sin b$, and

$$I = \begin{cases} \theta^1 = d\,a - \sin b\,d\,c + d\,u, \\ \theta^2 = \cos a\,d\,b + \cos b \sin a\,d\,c + F\,d\,u + d\,v, \\ \theta^3 = \cos b \cos a\,d\,c - \sin a\,d\,b + FF_v\,d\,u + F_v\,d\,v, \end{cases} \tag{5.182}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint

$$D_v(FF_v + v - F_u) \neq 0. \tag{5.183}$$

Proof. The admissible gauge transformations in this case are those which fix E_3, thus that generated by $U_3\,(f(u, v))$. From (5.3), the 1-forms α^i change by

$$\left[\begin{array}{l} \alpha^1 \to \bar{\alpha}^1 = \cos f\,\alpha^1 - \sin f\,\alpha^2, \\ \alpha^2 \to \bar{\alpha}^2 = \sin f\,\alpha^1 + \cos f\,\alpha^2, \\ \alpha^3 \to \bar{\alpha}^3 = \alpha^3 + df. \end{array}\right. \tag{5.184}$$

From (5.181d) we see that $\alpha^1 \neq 0$ and $\alpha^2 \neq 0$. Without loss in generality we can assume $V^2 \neq 0$, thus a gauge transformation (5.184) with $f = \arctan \frac{V^1}{V^2}$ gives

$$\begin{aligned} \bar{\alpha}^1 &= \tilde{U}_u\,d\,u, \\ \bar{\alpha}^2 &= U^2\,d\,u + \tilde{V}_v\,d\,v, \\ \bar{\alpha}^3 &= U^3\,d\,u + U^4\,d\,v, \end{aligned} \tag{5.185}$$

for some functions U^i, V^i, $\tilde{U}$ and $\tilde{V}$ of (u, v). From (5.181d) we see that $\tilde{U}_u \tilde{V}_v \neq 0$. We can change coordinates from (a, b, c, u, v) to $(a, b, c, \tilde{U}, \tilde{U})$. Dropping the tildes and the bars, we write (5.186) as

$$\begin{aligned} \alpha^1 &= du, \\ \alpha^2 &= F\,du + dv, \\ \alpha^3 &= U^3\,du + U^4\,dv. \end{aligned} \tag{5.186}$$

From (5.181b) we get

$$-F_v\,du \wedge dv = d\alpha^2 = -\alpha^1 \wedge \alpha^3 = -U^4\,du \wedge dv,$$

that is, $U^4 = F_v$. Consequently from (5.181a) we obtain

$$0 = d\alpha^1 = \alpha^2 \wedge \alpha^3 = (FU^4 - U^3)\,du \wedge dv = (FF_v - U^3)\,du \wedge dv$$

and thus $U^3 = FF_v$. Accordingly (5.180) becomes

$$I = \begin{cases} \theta^1 = \omega^1 + du, \\ \theta^2 = \omega^2 + F\,du + dv, \\ \theta^3 = \omega^3 + FF_v\,du + F_v\,dv, \end{cases}$$

which, using (5.178), gives (5.182). To conclude, condition (5.181c) is

$$0 \neq d\alpha^3 - \alpha^1 \wedge \alpha^2 = (D_u[F_v] - D_v[FF_v] - 1)\,du \wedge dv = D_v[F_u - FF_v - v]\,du \wedge dv,$$

which is (5.183). □

5.10.4 Step 4

Here we proceed, as in section 5.3.4, to prove the following.

Proposition 5.10.3. Let $Z_1 = H(X, Y, Z, Y_1, Y_2)$, be a second-order Monge equation such that $\dfrac{\partial^2 H}{\partial {Y_2}^2} \neq 0$. Assume that this equation admits a 3-dimensional symmetry group G which acts freely and transversely, denote by Γ a set of infinitesimal generators of this action and assume that Γ is of algebraic type $A_{3,9}$. Then $\Gamma = \langle \frac{\cos X}{\sqrt{1-{Y_1}^2}}(Y_1\,\partial_X + \partial_Y),\ \frac{-\sin X}{\sqrt{1-{Y_1}^2}}(Y_1\,\partial_X + \partial_Y),\ \partial_X\rangle$ and

$$Z_1 = \sqrt{1 - {Y_1}^2 + \frac{{Y_2}^2}{1 - {Y_1}^2}}\; h\Big(Z, Y + \arctan\frac{Y_2}{1 - {Y_1}^2}\Big),\ \text{with } h \neq 0. \tag{5.187}$$

Proof. We will follow the same steps as in the proof of Proposition 5.4.3.

Let I be the GR_3D_5 Pfaffian system defined by the given Monge equation on the manifold N. By Proposition 5.10.2, there are local coordinates (a, b, c, u, v) on N such that $\Gamma = \langle -\frac{\cos c}{\cos b}\,\partial_a + \sin c\,\partial_b - \cos c\tan b\,\partial_c,\quad -\frac{\sin c}{\cos b}\,\partial_a - \cos c\,\partial_b - \sin c\tan b\,\partial_c,\quad -\partial_c\rangle$ and

$$I = \begin{cases} \eta^1 = d\,a - \sin b\,d\,c + d\,u, \\ \eta^2 = \cos a\,d\,b + \cos b\sin a\,d\,c + F\,d\,u + d\,v, \\ \eta^3 = \cos b\cos a\,d\,c - \sin a\,d\,b + FF_v\,d\,u + F_v\,d\,v, \end{cases} \tag{5.188}$$

where $F = F(u, v)$ is a differentiable function subject to the constraint $D_v(FF_v + v - F_u) \neq 0$. We now start the Monge Algorithm 3.5.8.

[0] By construction, (5.188) is an adapted basis of I, that is, $I' = \{\eta^1, \eta^2\}$.

[1] Immediately we see that $\mathrm{Eng}\big(\eta^1\big) = 1$. Define the change of coordinates $\tau_1 : N \to N$ by

$$y^1 = -c, \quad y^2 = -a - u, \quad y^3 = \sin b, \quad y^4 = u, \quad y^5 = v.$$

The inverse τ_1^{-1} is given by

$$a = -y^2 - y^4, \quad b = \arcsin y^3, \quad c = -y^1, \quad u = y^4, \quad v = y^5.$$

Accordingly, we see that $\theta^1 = -{\tau_1^{-1}}^*\eta^1 = d\,y^2 - y^3\,d\,y^1$, as wanted. Now, we write η^2 in these new coordinates. Set $F = F(y^4, y^5) = F(u, v)$, then

$$\eta_0^2 = {\tau_1^{-1}}^*\eta^2 = \sqrt{1 - {y^3}^2}\,\sin\left(y^2 + y^4\right)d\,y^1 + \frac{\cos\left(y^2 + y^4\right)}{\sqrt{1 - {y^3}^2}}d\,y^3 + F\,d\,y^4 + d\,y^5,$$

and thus the local expression of I' in these new coordinates is

$$I' = \begin{cases} \theta^1 = dy^2 - y^3\, dy^1, \\ \eta_0^2 = \sqrt{1 - {y^3}^2}\sin(y^2 + y^4)\, dy^1 + \dfrac{\cos(y^2 + y^4)}{\sqrt{1 - {y^3}^2}} dy^3 + F\, dy^4 + dy^5. \end{cases} \tag{5.189}$$

[2] We seek for $\theta^2 = dU - V^3\, dy^3 - V^1\, dy^1 = W\,\theta^1 + A\,\eta_0^2 \in I'$. Using (5.189) and the notation in Proposition 3.3.3, we write

$$\begin{aligned} \eta_0^2 &= Y_0^1\, dy^1 + Y_0^2\, dy^2 + Y^3\, dy^3 + Y^4\, dy^4 + Y^5\, dy^5 \\ &= \sqrt{1 - {y^3}^2}\sin(y^2 + y^4)\, dy^1 + \frac{\cos(y^2 + y^4)}{\sqrt{1 - {y^3}^2}} dy^3 + F\, dy^4 + dy^5. \end{aligned}$$

Accordingly, the PDE system (3.47) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= AY^4 = AF, \\ \frac{\partial U}{\partial y^5} &= AY^5 = A, \end{aligned} \tag{5.190}$$

Without loss in generality, we can set $F = \frac{G_{y^4}}{G_{y^5}}$ for a generic function G. Then take $A = G_{y^5}$ and (5.190) becomes

$$\begin{aligned} \frac{\partial U}{\partial y^4} &= G_{y^4}, \\ \frac{\partial U}{\partial y^5} &= G_{y^5}, \end{aligned} \tag{5.191}$$

and the solution is $U = G(y^4, y^5)$, thus $F = \frac{U_{y^4}}{U_{y^5}}$ and $A = U_{y^5}$. Consequently, the expressions (3.48) become

$$\begin{aligned} W &= \frac{\partial U}{\partial y^2} - AY_0^2 = 0, \\ V^3 &= \frac{\partial U}{\partial y^3} - AY^3 = -U_{y^5}\frac{\cos(y^2 + y^4)}{\sqrt{1 - {y^3}^2}}, \\ V^1 &= \frac{\partial U}{\partial y^1} + y^3 W - AY_0^1 = -U_{y^5}\sqrt{1 - {y^3}^2}\sin(y^2 + y^4). \end{aligned} \tag{5.192}$$

From (5.192) we define

$$\hat{f} = -V^1 = -U_{y^5}\sqrt{1 - {y^3}^2}\sin(y^2 + y^4), \tag{5.193}$$

and a new change of coordinates $\tau_2 : N \to N$ such that

$$x^1 = y^1, \quad x^2 = y^2, \quad x^3 = y^3, \quad x^4 = U, \quad x^5 = V^3 = -U_{y^5} \frac{\cos(y^2 + y^4)}{\sqrt{1 - {y^3}^2}}.$$

The inverse of τ_2 can be defined by

$$\tau_2^{-1} : \quad y^1 = x^1, \quad y^2 = x^2, \quad y^3 = x^3, \quad y^4 = \tilde{U}, \quad y^5 = \tilde{V}.$$

In particular from (5.193) we compute

$$f = \hat{f} \circ \tau_2^{-1} = \frac{x^5(1 - {x^3}^2)\sin(x^2 + \bar{U})}{\cos(x^2 + \bar{U})}, \tag{5.194}$$

where $\bar{U} = \bar{U}(x^2, x^4, x^5)$. Consequently we have

$$I' = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = {\tau_2^{-1}}^* \eta_0^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \end{cases} \tag{5.195}$$

which is the general Goursat normal form of I'. By Theorem 3.4.4 we have $f_{x^5x^5} \neq 0$.

[3] We can complete I' to a basis of I by setting

$$\theta^3 = dx^3 - f_{x^5}\,dx^1.$$

Then I can be written in the general Goursat normal form

$$I = \begin{cases} \theta^1 = dx^2 - x^3\,dx^1, \\ \theta^2 = dx^4 - x^5\,dx^3 + f\,dx^1, \\ \theta^3 = dx^3 - f_{x^5}\,dx^1. \end{cases}$$

From (5.194), we compute

$$f_{x^5} = (1 - {x^3}^2)\hat{T}(x^2, x^4, x^5). \tag{5.196}$$

[4] Here we obtain the Monge normal form. Define the change of coordinates $\tau_4 : N \to N$ by

$$X = x^1, \quad Y = x^2, \quad Z = x^4, \quad Y_1 = x^3, \quad Y_2 = f_{x^5}.$$

The inverse $\tau_4^{-1} : N \to N$ is defined by

$$x^1 = X, \quad x^2 = Y, \quad x^3 = Y_1, \quad x^4 = Z, \quad x^5 = \tilde{T}(Y, Z, Y_1, Y_2),$$

where $\tilde{T} \circ \tau_4 = f_{x^5}$.

We have thus constructed a local change of coordinates $\tau = \tau_4 \circ \tau_2 \circ \tau_1 : N \to N$ given by

$$X = -c, \quad Y = -u - a, \quad Z = U, \quad Y_1 = \sin b, \quad Y_2 = \bar{T}(a, b, u, v)$$

and with inverse τ^{-1} given by

$$a = -Y - \hat{U}, \quad b = \arcsin Y_1, \quad c = -X, \quad u = \hat{U}(Y, Z, Y_1, Y_2), \quad v = \hat{V}(Y, Z, Y_1, Y_2).$$

According to these, Γ is τ-related to

$$\begin{aligned}
&\langle \frac{Y_1 \cos X}{\sqrt{1 - {Y_1}^2}}\, \partial_X + \frac{\cos X}{\sqrt{1 - {Y_1}^2}}\, \partial_Y - \sin X \sqrt{1 - {Y_1}^2}\, \partial_{Y_1} \\
&\quad - \frac{\cos X {Y_1}^4 + 2 \sin X {Y_1}^3 Y_2 - 2 \cos X {Y_1}^2 - 2 \sin X Y_1 Y_2 + \cos X + \cos X {Y_2}^2}{(1 - {Y_1}^2)^{3/2}}\, \partial_{Y_2}, \\
&\frac{-\sin X Y_1}{\sqrt{1 - {Y_1}^2}}\, \partial_X - \frac{\sin X}{\sqrt{1 - {Y_1}^2}}\, \partial_Y - \cos X \sqrt{1 - {Y_1}^2}\, \partial_{Y_1} \\
&\quad - \frac{-\sin X {Y_1}^4 + 2 \cos X {Y_1}^3 Y_2 + 2 \sin X {Y_1}^2 - 2 \cos X Y_1 Y_2 - \sin X - \sin X {Y_2}^2}{(1 - {Y_1}^2)^{3/2}}\, \partial_{Y_2}, \quad \partial_X \rangle,
\end{aligned}$$

which is the prolongation of $\langle \frac{Y_1 \cos X}{\sqrt{1 - {Y_1}^2}}\, \partial_X + \frac{\cos X}{\sqrt{1 - {Y_1}^2}}\, \partial_Y,\ \frac{-\sin X Y_1}{\sqrt{1 - {Y_1}^2}}\, \partial_X - \frac{\sin X}{\sqrt{1 - {Y_1}^2}}\, \partial_Y,\ \partial_X \rangle$. Consequently we have $Z_1 = \sqrt{1 - {Y_1}^2 + \frac{{Y_2}^2}{1 - {Y_1}^2}}\, h\Big(Z, Y + \arctan \frac{Y_2}{1 - {Y_1}^2}\Big)$. $\square$

CHAPTER 6

EXAMPLES

In this chapter we list examples of inequivalent general Monge equations, for each algebraic type $[\mathfrak{g}]$ considered in Chapter 5. We determine the root type of their corresponding Cartan 2-tensor using the program FiveVariables (see Chapter 8). The full symmetry algebra $\mathcal{S}ym$ of each of the equations considered is also calculated and, in some cases, we identify the corresponding nonlinear involutive system obtained by lifting.

In Table 3.1 (see page 63) we reported the normal forms obtained by Cartan. We recall that every general Monge equation of root type $[\infty]$ is equivalent to the Hilbert-Cartan equation $Z_1 = {Y_2}^2$ (see examples 4.1.14 and 4.3.2, where the corresponding nonlinear involutive system is given). The full symmetry algebra of this equation is the 14-dimensional exceptional simple Lie algebra $\mathfrak{g}_2$, see Section 7.4. For each algebraic type $[\mathfrak{g}]$ we were able to find a representative of the root type $[\infty]$, except for the algebraic type $A_{3,9}$.

Concerning the root type [4], Cartan obtained the general Monge normal form

$$Z_1 = -\frac{1}{2}\left({Y_2}^2 + \frac{10}{3}k{Y_1}^2 + \left(1 + k^2 - k''\right)Y^2\right),$$

where $k = k(X)$ is an invariant. If k is a constant then $\dim \mathcal{S}ym = 7$ and two equations are equivalent if and only if they have the same invariant. If k is not a constant then $\dim \mathcal{S}ym = 6$. In this case, assume I_1 and I_2 are GR_3D_5 Pfaffian systems with corresponding invariants k_1 and k_2 such that ${k_i}' = F_i(k_i)$, then I_1 and I_2 are equivalent if and only if F_1 and F_2 are the same expressions. We provide one example for which $\dim \mathcal{S}ym = 6$ and various for which $\dim \mathcal{S}ym = 7$, postponing to a future analysis a more detailed classification of this last ones.

According to Cartan, when the root type is $[2, 2]$ then $\dim \mathcal{S}ym = 5$ or $\dim \mathcal{S}ym = 6$. When $\dim \mathcal{S}ym = 6$ then $\mathcal{S}ym$ is the direct sum of two 3-dimensional Lie algebras or it is the algebra of Euclidean movement in the space. Cartan provides a couple of normal forms only for some special cases where $\dim \mathcal{S}ym = 6$. We present several examples of both cases, whose full classification will be shortly available.

For the root types $[3,1]$, $[2,1,1]$ and $[1,1,1,1]$ Cartan did not provide any normal forms, but he pointed out that in these cases $\dim \mathcal{S}ym \leq 5$. We produce a wide list of inequivalent general Monge equations with these root types. In particular, the list for the root type $[3,1]$ includes all inequivalent Monge equations.

For the convenience of the reader, we report our root-type lists in Section 6.10.

6.1 $3A_1$

In this section,

$$Z_1 = h(Y_1, Y_2), \quad h_{Y_2Y_2} \neq 0, \text{ and } \Gamma = \langle \partial_X, \partial_Y, \partial_Z \rangle.$$

The root type $[\infty]$ is represented by $Z_1 = {Y_2}^2$. For the other root types we have the following examples.

#	Root type	$\dim \mathcal{S}ym$	$h(Y_1, Y_2)$
6.1.1	$[4]$	7	${Y_2}^3$
6.1.2	$[4]$	7	${Y_1}^2 + {Y_2}^2$
6.1.3	$[3,1]$	4	${Y_1}^3 + {Y_2}^2$
6.1.4	$[2,2]$	5	${Y_1}^2 + {Y_2}^3$
6.1.5	$[2,2]$	4	${Y_1}^3 + {Y_2}^3$
6.1.6	$[2,1,1]$	5	$Y_1{Y_2}^2$
6.1.7	$[1,1,1,1]$	5	$Y_1{Y_2}^3$

#	Completion of Γ to $\mathcal{S}ym$
6.1.1	$X\,\partial_Y,\quad X\,\partial_X - 5Z\,\partial_Z,\quad Y\,\partial_Y + 3Z\,\partial_Z,\quad {Y_2}^2\,\partial_X + ({Y_2}^2 Y_1 - \frac{1}{3}Z)\,\partial_Y + \frac{2}{5}{Y_2}^5\,\partial_Z.$
6.1.2	$X\,\partial_Y + 2Y\,\partial_Z,\quad Y\,\partial_Y + 2Z\,\partial_Z,\quad e^X\,\partial_Y + 2Y_1 e^X\,\partial_Z,\quad e^{-X}\,\partial_Y + 2Y_1 e^{-X}\,\partial_Z.$
6.1.3	$X\,\partial_X - Y\,\partial_Y - 5Z\,\partial_Z.$
6.1.4	$X\,\partial_Y + 2Y\,\partial_Z,\quad X\,\partial_X + 4Y\,\partial_Y + 7Z\,\partial_Z.$
6.1.5	$Y\,\partial_Y + 3Z\,\partial_Z.$
6.1.6	$X\,\partial_X - 4Z\,\partial_Z,\quad Y\,\partial_Y + 3Z\,\partial_Z.$
6.1.7	$X\,\partial_X - 6Z\,\partial_Z,\quad Y\,\partial_Y + 4Z\,\partial_Z.$

From Example 6.1.6, the general Monge equation

$$Z' = Y' Y''^2 \tag{6.1}$$

is lifted (see Section 4.4) to the nonlinear involutive system

$$\begin{aligned} r &= \frac{t}{5x^2}(9t^4x^4 - 10t^2x^2y + 5y^2), \\ s &= \frac{t}{x}(t^2x^2 - y). \end{aligned} \tag{6.2}$$

The 2-dimensional integral manifold $\mathbf{s} : (\hat{x}, X) \in \mathbb{R}^2 \to J(\mathbb{R}^2, \mathbb{R})$ of (6.2) is

$$x = \hat{x}, \quad y = X\hat{x} + 3Y''^2, \quad z = \frac{12}{5\hat{x}}Y''^5 + 3Y'Y''^2 + \hat{x}Y - Z, \tag{6.3}$$

where Y and Z satisfy (6.1).

6.2 $\quad A_1 \oplus A_2$

In this section we have

$$Z_1 = X^{-2}\,h(V, W), \quad \text{where } V = XY_1, \quad W = X^2 Y_2, \quad h_{WW} \neq 0, \text{ and}$$

$$\Gamma = \langle \partial_Z,\, Z\,\partial_Z - X\,\partial_X,\, \partial_Y \rangle.$$

The root type $[\infty]$ is represented by $Z_1 = {Y_1}^4 {Y_2}^{-1}$, which is obtained by setting $h(XY_1, X^2Y_2) = h(V, W) = V^4W^{-1}$. For the other root types we have the following examples.

#	Root type	$\dim \mathcal{S}ym$	$h(V,W)$
6.2.1	$[4]$	7	W^2
6.2.2	$[3,1]$	3	$V^m + W^2$, $m \notin \{0,1,2\}$
6.2.3	$[3,1]$	4	$V^{-5}W^2$
6.2.4	$[2,2]$	5	$(VW)^{2/3}$
6.2.5	$[2,1,1]$	5	W^n, $n \in \{-1, \frac{1}{3}, 3\}$
6.2.6	$[1,1,1,1]$	5	W^n, $n \notin \{-1, 0, \frac{1}{3}, 1, 2, 3\}$
6.2.7	$[1,1,1,1]$	5	$V^{-n-3}W^n$, $n \neq 2$

#	Completion of Γ to $\mathcal{S}ym$
6.2.1	$X\,\partial_Y,\quad Y\,\partial_Y + 2Z\,\partial_Z,\quad \ln X\,\partial_Y - 2Y_1\,\partial_Z,\quad X\ln X\,\partial_Y - 2(Y - XY_1)\,\partial_Z.$
6.2.2	$0\partial_X.$
6.2.3	$Y\,\partial_Y - 3Z\,\partial_Z.$
6.2.4	$\partial_X,\quad Y\,\partial_Y + \frac{4}{3}Z\,\partial_Z.$
6.2.5	$X\,\partial_Y,\quad Y\,\partial_Y + nZ\,\partial_Z.$
6.2.6	$X\,\partial_Y,\quad Y\,\partial_Y + nZ\,\partial_Z.$
6.2.7	$X\,\partial_Y,\quad Y\,\partial_Y - 3Z\,\partial_Z.$

6.3 $A_{3,1}$

We start from

$$Z_1 = Y + h(Y_1, Y_2),\quad h_{Y_2Y_2} \neq 0,\quad \Gamma = \langle \partial_Z,\, \partial_X,\, \partial_Y + X\,\partial_Z \rangle.$$

The root type $[\infty]$ has the representative $Z_1 = Y + {Y_2}^2$. For the other root types we have

#	Root type	$\dim \mathcal{S}ym$	$h(Y_1, Y_2)$
6.3.1	$[4]$	7	${Y_2}^{(-1)}$
6.3.2	$[4]$	7	${Y_1}^2 + {Y_2}^2$
6.3.3	$[3, 1]$	3	${Y_1}^3 + {Y_2}^2$
6.3.4	$[2, 2]$	6	$\ln Y_2$
6.3.5	$[2, 1, 1]$	4	$Y_1{Y_2}^2$
6.3.6	$[1, 1, 1, 1]$	5	${Y_2}^3$

#	Completion of Γ to $\mathcal{S}ym$
6.3.1	$X\,\partial_Y + \frac{1}{2}X^2\,\partial_Z, \quad X\,\partial_X + Y\,\partial_Y + 2Z\,\partial_Z,$ $e^{Y_1/\sqrt{2}}\left[\partial_X + (Y_1 - \sqrt{2})\,\partial_Y + Y\,\partial_Z\right],$ $e^{-Y_1/\sqrt{2}}\left[\partial_X + (Y_1 + \sqrt{2})\,\partial_Y + Y\,\partial_Z\right].$
6.3.2	$e^X\,\partial_Y + (2Y_1 + 1)e^X\,\partial_Z, \quad e^{-X}\,\partial_Y + (2Y_1 - 1)e^{-X}\,\partial_Z,$ $(\frac{1}{4}X^2 - Y)\,\partial_Y + (YX - 2Z + \frac{1}{12}X^3 + Y_1)\,\partial_Z,$ $\frac{1}{2}X\,\partial_Y + (Y + \frac{1}{4}X^2)\,\partial_Z.$
6.3.3	$0\,\partial_X.$
6.3.4	$X\,\partial_Y + \frac{1}{2}X^2\,\partial_Z,$ $X\,\partial_X + 2\,\partial_Y + Z\,\partial_Z,$ $(\frac{X^2}{2} - \frac{2}{Y_2})\,\partial_X + (Z + 2X - \frac{Y_1}{Y_2})\,\partial_Y - (\frac{Y}{Y_2} - XZ + \frac{1+\ln Y2}{Y_2})\,\partial_Z.$
6.3.5	$X\,\partial_X + \frac{5}{2}Y\,\partial_Y + \frac{7}{2}Z\,\partial_Z.$
6.3.6	$X\,\partial_Y + \frac{1}{2}X^2\,\partial_Z, \quad X\,\partial_X + 3Y\,\partial_Y + 4Z\,\partial_Z.$

From Example 6.3.6, the general Monge equation

$$Z' = Y + Y''^3 \tag{6.4}$$

is lifted (see Section 4.4) to the nonlinear involutive system

$$\begin{aligned} r &= -\frac{1}{4x}(2p + 2q + y\sin 2t - 2yt), \\ s &= \sqrt{\frac{y}{x}}\cos t. \end{aligned} \tag{6.5}$$

The 2-dimensional integral manifold $\mathbf{s} : (\hat{x}, X) \in \mathbb{R}^2 \to J(\mathbb{R}^2, \mathbb{R})$ of (6.5) is

$$\begin{aligned} x &= \frac{1}{\sqrt[4]{96}}\hat{x}^2, \quad y = \frac{1}{\sqrt[4]{96}}(\hat{x}-X)^2 + \frac{6}{\sqrt[4]{96}}Y''^2, \\ z &= \frac{5}{4}(\hat{x}-X)\,Y''^3 - 3Y''^2Y' + \frac{1}{8}(\hat{x}-X)^3\,Y'' + (\hat{x}-X)\,Y + Z \\ &\quad + \frac{1}{8\sqrt{6}}\left[6Y''^2 + (\hat{x}-X)^2\right]^2 \arctan\left(\frac{\hat{x}-X}{6Y''}\right), \end{aligned} \tag{6.6}$$

where Y and Z satisfy (6.4).

6.4 $A_{3,2}$

In this section we have

$$Z_1 = \frac{Y + h(V,W)}{X^2}, \quad \text{where } V = Y - Y_1 X, \quad W = Y_2 X^2, \quad h_{WW} \neq 0, \text{ and}$$

$$\Gamma = \langle \partial_Z, \quad \ln X\, \partial_Z + X\, \partial_Y,\; Z\, \partial_Z - X\, \partial_X \rangle$$

One representatives of the root type $[\infty]$ is the parametric general Monge equation

$$Z_1 = \frac{Y + m(W^2 + 2nV^2)}{X^2},$$

for $n \in \{1, -\frac{1}{9}\}$, $m \neq 0$. We were not able to provide a representative of the root type $[2,2]$. For the other root types we have the following.

#	Root type	$\dim \mathcal{S}ym$	$h(V,W)$
6.4.1	$[4]$	7	$W^2 \pm V^m$, $m \in \{0,1,2\}$
6.4.2	$[3,1]$	3	$W^2 \pm V^m$, $m \notin \{0,1,2\}$
6.4.3	$[2,1,1]$	4	W^{-1}
6.4.4	$[2,1,1]$		$W^{1/3}$
6.4.5	$[1,1,1,1]$	4	W^3
6.4.6	$[1,1,1,1]$		$W^{2/3}$

#	Completion of Γ to $\mathcal{S}ym$
6.4.1	for $m=0$: $\partial_Y - \frac{1}{X}\,\partial_Z$, $\frac{\ln X + 2Y_1 X + 1}{X}\,\partial_Z - \ln X\,\partial_Y$, $\left[\frac{1}{2}\ln X(\ln X - 2) - 2Y + 2Y_1 X\right]\,\partial_Z - X(1-\ln X)\,\partial_Y$, $\left[\ln X(\ln X + 4) + 4Y\right]\partial_Y - X^{-1}\left[\ln X(\ln X + 4Y_1 X + 6) - 8ZX - 4Y + 4Y_1 X - 2\right]\,\partial_Z$.
6.4.2	$0\partial_X$.
6.4.3	$\partial_Y - \frac{1}{X}\,\partial_Z$.
6.4.4	at least that in 6.4.3, one more at most.
6.4.5	same as in 6.4.3.
6.4.6	at least that in 6.4.3, one more at most.

6.5 $A^{\epsilon}_{3,5}$

In this section

$$Z_1 = \frac{1}{\epsilon X^2} h(V,W), \quad \text{where } V = \epsilon X Y_1 - Y, \quad \epsilon^2 X^2 Y_2 + \epsilon^2 X Y_1 - Y, \quad h_{WW} \neq 0, \text{ and}$$

$$\Gamma = \langle X^{1/\epsilon}\,\partial_Y,\, \partial_Z,\, \epsilon(Z\,\partial_Z - X\,\partial_X)\rangle$$

The algebraic type considered in this section depends on the parameter $\epsilon \neq 0$. We show several examples of the root type $[\infty]$, according to different values of ϵ.

Root type	$h(V,W)$	ϵ
$[\infty]$	W^2	$\pm1, \pm4$
$[\infty]$	$V^{-5}W^2$	$1, 9$
$[\infty]$	W^{-1}	±5
$[\infty]$	$W^{1/3}$	±1

Other root types are represented by the following examples.

#	Root type	$\dim \mathcal{S}ym$	$h(V,W)$
6.5.1	$[4]$	7	W^2, $\epsilon \notin \{\pm 1, \pm 4\}$
6.5.2	$[3,1]$	3	$V^{-5}W^2$, $\epsilon \notin \{1, 9\}$
6.5.3	$[2,2]$	6	W^{-1}, $\epsilon = \pm 1$
6.5.4	$[2,1,1]$	5	W^{-1}, $\epsilon \notin \{\pm 1, \pm 5\}$
6.5.5	$[2,1,1]$	5	$W^{1/3}$, $\epsilon \neq \pm 1$
6.5.6	$[2,1,1]$	5	$W^{2/3}$, $\epsilon = \pm 1$
6.5.7	$[2,1,1]$	4	$V^m W^2$, $m \notin \{0, -5\}$
6.5.8	$[1,1,1,1]$	5	W^3
6.5.9	$[1,1,1,1]$	4	$V^m W^3$, $m \neq 0$

#	Completion of Γ to $\mathcal{S}ym$
6.5.1	$X^{-1/\epsilon}\,\partial_Y, \quad Y\,\partial_Y + 2Z\,\partial_Z,$ $X^{1-1/\epsilon}\,\partial_Y + 2(\epsilon-2)\epsilon X^{-1/\epsilon}(\epsilon X Y_1 + Y)\,\partial_Z,$ $X^{1+1/\epsilon}\,\partial_Y + 2(\epsilon+2)\epsilon X^{1/\epsilon}(\epsilon X Y_1 - Y)\,\partial_Z.$
6.5.2	for $\epsilon \notin \{1, 9\}$: $\quad Y\,\partial_Y - 3Z\,\partial_Z.$
6.5.3	$\frac{1}{X}\,\partial_Y, \quad Y\,\partial_Y - Z\,\partial_Z, \quad (XY_1 + Y)\,\partial_X + \frac{1}{2X}(X^2{Y_1}^2 - Y^2)\,\partial_Y - \frac{1}{X^2}\,\partial_Z.$
6.5.4	$X^{-1/\epsilon}\,\partial_Y, \quad Y\,\partial_Y - Z\,\partial_Z.$
6.5.5	$X^{-1/\epsilon}\,\partial_Y, \quad Y\,\partial_Y + \frac{1}{3}Z\,\partial_Z.$
6.5.6	for $\epsilon = \pm 1$: $\quad X^{-1/\epsilon}\,\partial_Y, \quad Y\,\partial_Y + \frac{2}{3}Z\,\partial_Z.$
6.5.7	for $m \notin \{0, -5\}$: $\quad Y\,\partial_Y + (m+2)Z\,\partial_Z.$
6.5.8	$X^{-1/\epsilon}\,\partial_Y, \quad Y\,\partial_Y + (m+2)Z\,\partial_Z.$
6.5.9	for $m \neq 0$: $\quad Y\,\partial_Y + (m+2)Z\,\partial_Z.$

6.6 $A_{3,7}^{\epsilon}$

This section deals with the general Monge equations associated to the last 1-parameter family of algebras, namely

$$Z_1 = h(Z, W), \quad \text{where } W = Y_2 - 2\epsilon Y_1 + (\epsilon^2+1)Y, \quad h_{WW} \neq 0, \quad , \epsilon \geq 0 \text{ and}$$
$$\Gamma = \langle e^{\epsilon X} \sin X\, \partial_Y,\, -e^{\epsilon X} \cos X\, \partial_Y,\, -\partial_X \rangle$$

The representatives of the root types $[\infty]$ we found are

Root type	$h(Z,W)$
$[\infty]$	$\frac{2}{3}\epsilon Z + W^{2/3}$.
$[\infty]$	$Z^m W^{2/3}$, $\epsilon = 0$.

We notice that Γ is the Lie algebra associated to the group of Euclidean transformations in the plane, when $\epsilon = 0$. In this case the previous list reduces to the representative $Z_1 = Z^m(Y_2 + Y)^{2/3}$, for any m.

We were not able to produce examples for the root types $[3,1]$ and $[2,2]$. Some examples of the other root types are the following.

#	Root type	$\dim \mathcal{S}ym$	$h(V,W)$
6.6.1	$[4]$	7	W^2
6.6.2	$[2,1,1]$	5	W^{-1}
6.6.3	$[1,1,1,1]$	5	W^3

#	Completion of Γ to $\mathcal{S}ym$
6.6.1	$Y\,\partial_Y + 2Z\,\partial_Z, \quad \partial_Z,$ $e^{-\epsilon X} \sin X\, \partial_Y - 8\epsilon e^{-\epsilon X}[Y_1 \cos X + \sin X(Y + Y\epsilon^2 - \epsilon Y_1)]\,\partial_Z,$ $e^{\epsilon X} \cos X\, \partial_Y + 8\epsilon e^{\epsilon X}[Y_1 \sin X - \cos X(Y + \epsilon^2 Y - \epsilon Y_1)]\,\partial_Z.$
6.6.2	$Y\,\partial_Y - Z\,\partial_Z, \quad \partial_Z.$
6.6.3	$Y\,\partial_Y + 3Z\,\partial_Z, \quad \partial_Z.$

6.7 $A_{3,8,1}$

In this section we have

$$Z_1 = e^Y\, h(Z, W), \quad \text{where } W = e^{-2Y} Y_2 - \frac{1}{2} e^{-2Y} {Y_1}^2, \quad h_{WW} \neq 0, \text{ and}$$

$$\Gamma = \langle 2X\, \partial_X - 2\partial_Y,\, 2X\, \partial_Y - X^2\, \partial_X,\, \partial_X \rangle$$

In the following lists of examples of root type, we assume the parameters m and n to be free if not otherwise specified.

The representatives of the root types $[\infty]$ that we found are

$$Z_1 = e^{(1-2n)Y}\, Z^m \left(Y_2 - \frac{1}{2} {Y_1}^2 \right)^n, \text{ for } n \in \left\{ -1, \frac{1}{3}, \frac{2}{3}, 2 \right\}.$$

In the following list we have examples of the other root types.

#	Root type	$\dim \mathcal{S}ym$	$h(Z, W)$
6.7.1	$[4]$	4	$1 + W^n$, $n \in \{-1, 2\}$
6.7.2	$[4]$	6	$1 + W^{2/3}$
6.7.3	$[3, 1]$	3	$Z + W^2$
6.7.4	$[2, 2]$	6	$Z^m W^n$, $n \notin \{-1, 0, 1, \frac{1}{3}, \frac{2}{3}, 2\}$
6.7.5	$[2, 1, 1]$	3	$Z^m + W^2$, $m \notin \{0, 1\}$
6.7.6	$[2, 1, 1]$	3	$Z^m + W^n$, $m \neq 0$, $n \in \{-1, \frac{1}{3}\}$
6.7.7	$[1, 1, 1, 1]$	3	$Z + W^n$, $n \notin \{-1, \frac{1}{3}, 2\}$

#	Completion of Γ to $\mathcal{S}ym$
6.7.1	∂_Z.
6.7.2	$\partial_Z, \quad e^{-Y/2}\,\partial_X - e^{Y/2}\,\partial_Z, \quad e^{-Y/2}X\,\partial_X - 2e^{-Y/2}\,\partial_Y - e^{Y/2}X\,\partial_Z$.
6.7.3	$0\,\partial_X$.
6.7.4	for $A = Y_2 - \frac{1}{2}{Y_1}^2$, $E = e^{(1-2n)Y}$, $n \neq \frac{1}{2}$, $m \neq 1$: $\quad \mathbf{v}_1 = \frac{2}{2n-1}\,\partial_Y + \frac{2}{m-1}Z\,\partial_Z$, $\mathbf{v}_2 = \frac{2n}{2n-1}\,Z^m\,\partial_Z, \quad \mathbf{v}_3 = EA^{n-1}\,\partial_X + \left(EY_1A^{n-1} + \frac{1}{n(m-1)}Z^{1-m}\right)\partial_Y$ $+ \left(\frac{(2n-1)^2}{2n(m-1)^2}Z^{2-m} + (n-1)A^{2n-1}E^2Z^m\right)\partial_Z$.
6.7.5	$0\,\partial_X$.
6.7.6	$0\,\partial_X$.
6.7.7	$0\,\partial_X$.

Here we notice that in the case 6.7.4, $\mathcal{S}ym = A_{3,8} \oplus A_{3,8}$ is the direct sum of two copies of $\mathfrak{sl}_2$, because $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ satisfies the structure equations

$$[\mathbf{v}_1, \mathbf{v}_2] = 2\mathbf{v}_2, \quad [\mathbf{v}_1, \mathbf{v}_3] = -2\mathbf{v}_3, \quad [\mathbf{v}_2, \mathbf{v}_3] = \mathbf{v}_1,$$

used in Section 5.9.

6.8 $A_{3,8,2}$

Here we have

$$Z_1 = Z^2 + Y_1^2 h(Y, W), \quad \text{where } W = \frac{Y_2 - 2Y_1Z}{Y_1^2}, \quad h_{WW} \neq 0, \text{ and}$$

$$\Gamma = \langle 2X\,\partial_X - 2Z\,\partial_Z,\, (1 + 2XZ)\,\partial_Z - X^2\,\partial_X,\, \partial_X \rangle$$

The parameters m and n are assumed free if not otherwise specified in the following lists.

The representatives of the root types $[\infty]$ that we found are

Root type	$h(Y, W)$
$[\infty]$	$mY^n + pW^2$, for $p \in \{-\frac{9}{32}, \frac{1}{32}\}$.

We were not able to provide a representative of the root types [4] and $[3,1]$. For the other root types we have the following examples.

#	Root type	$\dim \mathcal{S}ym$	$h(Y,W)$
6.8.1	$[2,2]$	6	$mY^n + pW^2$, for $p \notin \{-\frac{9}{32}, 0, \frac{1}{32}\}\ W^2$,
6.8.2	$[2,1,1]$	3	$mY^n + pW^{2/3}$, $mp \neq 0$,
6.8.3	$[2,1,1]$	3	mY^nW^2, $m \neq 0$,
6.8.4	$[1,1,1,1]$	3	mY^nW^3, $m \neq 0$, $n \notin \{0,1\}$.
6.8.5	$[1,1,1,1]$	4	mY^nW^3, $m \neq 0$, $n \in \{0,1\}$.

#	Completion of Γ to $\mathcal{S}ym$
6.8.1.a	for $m = n = p = 1$, the *Airy* wave functions are envolved.
6.8.1.b	for $p \neq -\frac{1}{4}$, $m = 0$: $2Y\,\partial_Y, \quad Y^2\,\partial_Y + \frac{4p}{4p+1} Y_1\,\partial_Z, \quad -\partial_Y.$
6.8.1.c	for $p = -\frac{1}{4}$, $m = 0$: $\mathbf{v}_1 = Y_1\,\partial_Z, \quad \mathbf{v}_2 = \partial_Y, \quad \mathbf{v}_3 = -Y\,\partial_Y.$
6.8.2	$0\,\partial_X.$
6.8.3	$0\,\partial_X.$
6.8.4	$0\,\partial_X.$
6.8.5	$Y^n\,\partial_Y.$

Here we notice that in the Example 6.8.1.b, $\mathcal{S}ym = A_{3,8} \oplus A_{3,8}$ is the direct sum of two copies of $\mathfrak{sl}_2$. On the other side, in Example 6.8.1.c we have $\mathcal{S}ym = A_{3,8} \oplus A_{3,5}^{-1}$, where $A_{3,5}^{-1} = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$ has structure equations

$$[\mathbf{v}_1, \mathbf{v}_3] = \mathbf{v}_1, \quad [\mathbf{v}_2, \mathbf{v}_3] = -\mathbf{v}_2,$$

as seen in Section 5.7.3.

6.9 $A_{3,9}$

This section deals with the general Monge equations associated to $\mathfrak{so}(3)$. These are expressed by

$$Z_1 = \sqrt{1 - {Y_1}^2 + \frac{{Y_2}^2}{1 - {Y_1}^2}}\, h(Z,W), \quad \text{where } W = Y + \arctan \frac{Y_2}{1 - {Y_1}^2}, \quad h \neq 0, \text{ and}$$

$$\Gamma = \langle \frac{\cos X}{\sqrt{1 - {Y_1}^2}} \left(Y_1\,\partial_X + \partial_Y \right), \frac{-\sin X}{\sqrt{1 - {Y_1}^2}} \left(Y_1\,\partial_X + \partial_Y \right), \partial_X \rangle$$

We were able to find only the following two root types, the symmetries are yet to be computed.

#	Root type	$\dim \mathcal{S}ym$	$h(Z,W)$
6.9.1	$[2,2]$		$f(Z)\,e^{AW}$
6.9.2	$[1,1,1,1]$		W
6.9.3	$[1,1,1,1]$		W^2

6.10 Summary of root-type lists

For the convenience of the reader, in this section we summarize the lists of general Monge equations obtained in this chapter, sorting them by root type. The dimension "dim" and the algebraic type $[\mathfrak{g}]$, according to [36], of the full symmetry $\mathcal{S}ym$ are provided. In particular we remark that all the Monge equations of root type $[3,1]$ here listed are inequivalent. We remind the reader that $A_{3,8,1}$ and $A_{3,8,2}$ are representatives of the two inequivalent actions of $\mathfrak{sl}(2)$ on the plane (see [35]).

Finally, the algebraic types were computed with the precious aid of the *LieAlgebras* routine of DifferentialGeometry (Maple 11 and later versions).

Root type $[4]$

dim	#
7	6.1.1
7	6.1.2
7	6.2.1
7	6.3.1
7	6.3.2
7	6.4.1
7	6.5.1
7	6.6.1
6	6.7.2
4	6.7.1

Root type $[3,1]$

dim	#	$[\mathfrak{g}]$
4	6.1.3	$A_{4,5}$
4	6.2.3	$A_{4,12}$
3	6.2.2	$A_1 \oplus A_2$
3	6.3.3	$A_{3,1}$
3	6.4.2	$A_{3,2}$
3	6.5.2	$A_{3,5}^{\epsilon}$, $\epsilon \notin \{1,9\}$
3	6.7.3	$A_{3,8,1}$

Root type $[2,2]$

dim	#	$[\mathfrak{g}]$
6	6.3.4	
6	6.5.3	
6	6.7.4	$A_{3,8} \oplus A_{3,8}$
6	6.8.1.a	
6	6.8.1.8	$A_{3,8} \oplus A_{3,8}$
6	6.8.1.c	$A_{3,8} \oplus A_{3,5}^{-1}$
5	6.1.4	
5	6.2.4	
4	6.1.5	
	6.9.1	

Root type $[2,1,1]$

dim	#	$[\mathfrak{g}]$
5	6.1.6	
5	6.2.5	
5	6.5.4	
5	6.5.5	
5	6.5.6	
5	6.6.2	
4	6.3.5	$A_{4,9}^{1/2}$
4	6.4.3	$A_{3,2} \oplus A_1$
4	6.5.7	$2\,A_2,\ \epsilon \neq m+2$
3	6.7.5	$A_{3,8,1}$
3	6.7.6	$A_{3,8,1}$
3	6.7.2	$A_{3,8,2}$
3	6.7.3	$A_{3,8,2}$
	6.4.4	

Root type $[1,1,1,1]$

dim	#	$[\mathfrak{g}]$
5	6.1.7	
5	6.2.6	
5	6.2.7	
5	6.3.6	
5	6.5.8	
5	6.6.3	
4	6.4.5	$A_{3,1} \oplus A_1$
4	6.5.9	$2\,A_2,\ \epsilon \neq m+3$
4	6.8.5	$A_{3,8} \oplus A_1$
3	6.7.7	$A_{3,8,1}$
3	6.8.4	$A_{3,8,2}$
	6.4.6	
	6.9.2	
	6.9.3	

CHAPTER 7

DARBOUX INTEGRABLE HYPERBOLIC PDE IN THE PLANE

In this Chapter we are concerned with the classification of second order hyperbolic scalar partial differential equations in two variables (PDE in the plane) which can be solved by the method of Darboux. These are shown to give rise to hyperbolic Pfaffian systems of rank-3 on a 7-manifold which are Darboux integrable.

The geometric definition of Darboux integrable hyperbolic PDE is given, followed by a characterizing theorem. Our symmetry normal forms from Chapter 5 are then used to provide a broad classification of such PDE.

Some examples will be provided at the end.

7.1 Basic definitions

Following [20] and [38], a rank-3 Pfaffian system I defined on a 7-dimensional manifold M_7 is said to be ***hyperbolic*** if there exists a local coframe α, α^1, α^2, π^1, σ^1, π^2, σ^2 on M_7 such that

[i] $I = \{\alpha, \alpha^1, \alpha^2\}$,

[ii] $\mathrm{Eng}(\alpha) = 2$ and the following structure equations are satisfied

$$\begin{aligned} d\alpha &\equiv 0 && \operatorname{mod} I,\\ d\alpha^1 &\equiv \pi^1\wedge\sigma^1 && \operatorname{mod} I,\\ d\alpha^2 &\equiv \pi^2\wedge\sigma^2 && \operatorname{mod} I. \end{aligned} \tag{7.1}$$

More precisely, see [20, Proposition 5.5], there exists an adapted coframe on M_7 such that

$$\begin{aligned} d\alpha &\equiv \alpha^1\wedge\pi^1+\alpha^2\wedge\pi^2 && \operatorname{mod}\alpha,\\ d\alpha^1 &\equiv N_2\,\alpha^2\wedge\sigma^2+\pi^1\wedge\sigma^1 && \operatorname{mod}(\alpha,\alpha^1),\\ d\alpha^2 &\equiv N_1\,\alpha^1\wedge\sigma^1+\pi^2\wedge\sigma^2 && \operatorname{mod}(\alpha,\alpha^2), \end{aligned} \tag{7.2}$$

for some functions $N_i \in C^\infty(M)$. The functions N_i are called the ***Monge-Ampère invariants*** of I. These structure equations imply that every hyperbolic Pfaffian system I can be only one of the following three types:

[I] I is ***Monge-Ampère*** if $N_i = 0$ for $i = 1,\, 2$;

[II] I is ***semi-Monge-Ampère*** if $N_1 N_2 = 0$ and ${N_1}^2 + {N_2}^2 \neq 0$;

[III] I is ***non-Monge-Ampère*** if $N_i \neq 0$ for $i = 1,\, 2$.

The next notion to recall is that of Darboux integrability. Let I be a hyperbolic Pfaffian system satisfying (7.1). We say that I is ***Darboux integrable*** (at level 0) if

[iii] the systems $\left\{\pi^i, \sigma^i\right\}$ are complete, for $i = 1, 2$.

If I is Darboux integrable, then the Pfaffian systems $V_1 = \{\alpha, \alpha^1, \alpha^2,\ \pi^1, \sigma^1\}$ and $V_2 = \{\alpha, \alpha^1, \alpha^2, \pi^2,\ \sigma^2\}$ are called ***singular systems*** of I and the first integrals of V_i are called ***Darboux invariants*** of I. (V_1, V_2) is said to be a ***Vessiot pair*** of I.

Darboux integrable Pfaffian systems of Monge-Ampère type were classified by Goursat [24] over the complex field and lately by Biesecker [5] over the reals.

A geometric analysis of Darboux integrability for hyperbolic PDE in the plane was given by Juráš in his PhD thesis [31] and then in an article coauthored with Anderson [32]. Recently, in [3], Anderson, Fels, and Vassiliou generalize the notion of Darboux integrability for hyperbolic Pfaffian systems. Here we give a specialized version of the main theorems elaborated in this last work. For these, we first need to introduce the following notation.

Let $I_i = \{\theta_i^1,\ \theta_i^2,\ \theta_i^3\}$ be a rank-3 Pfaffian system on a 5-manifold M_i, for each $i = 1,\, 2$. On the cross-product manifold $M_1 \times M_2$ with standard projections $\pi_i : M_1 \times M_2 \to M_i$ we define the rank-6 Pfaffian system $I_1 \oplus I_2$ with basis $\{\tilde{\theta}_i^j = \pi_i^* \theta_i^j\}_{j=1,2,3}^{i=1,2}$. Assume that G is a 3-dimensional symmetry group of both I_1 and I_2, and that G acts freely on M_i with infinitesimal generators $\Gamma_i = \langle E_1^i, E_2^i, E_3^i\rangle$. Then there are uniquely defined local lifts $\tilde{\Gamma}_i = \langle \tilde{E}_1^i, \tilde{E}_2^i, \tilde{E}_3^i\rangle$ on $M_1 \times M_2$ such that $\pi_{i*}\tilde{\Gamma}_i = \Gamma_i$. Then the ***diagonal action*** of G on $M_1 \times M_2$ is defined by the infinitesimal generators $\tilde{E}_j = \tilde{E}_j^1 + \tilde{E}_j^2$, for $j = 1, 2, 3$, and we write $\Gamma = \langle \tilde{E}_1, \tilde{E}_2, \tilde{E}_3\rangle$.

Theorem 7.1.1. *(See [3, Theorem 1.4]) Let I be a Darboux integrable hyperbolic rank-3 Pfaffian system on the 7-manifold M. Then there exist (locally) a 3-dimensional Lie group G and two Pfaffian* *systems I_1 and I_2 respectively defined on the 5-manifolds M_1 and M_2 for which the following are true.*

[i] *I_i has (constant) rank 3 and $I_i^{(\infty)} = \{0\}$, for $i = 1, 2$.*

[ii] *G is a symmetry group of I_i, which acts regularly and freely on M_i and transversely to I_i, for $i = 1, 2$.*

[iii] *The reduction by the diagonal action of G on $M_1 \times M_2$ gives $(M_1 \times M_2)/G = M$ and $(I_1 \oplus I_2)/G = I$.*

The group G of Theorem 7.1.1 is called the ***Vessiot group*** of the Darboux integrable hyperbolic Pfaffian system I. The Pfaffian systems I_i are called ***side systems***.

Theorem 7.1.2. *(See [3, Corollary 3.4]) Let I_1 and I_2 be two rank-3 Pfaffian systems respectively defined on the 5-manifolds M_1 and M_2 such that $I_i^{(\infty)} = \{0\}$, for $i = 1, 2$. Assume the following is true.*

[1] *G is a 3-dimensional symmetry group of I_i, which acts freely on M_i and transversely to I_i, for $i = 1, 2$.*

[2] *The diagonal action of G on $M_1 \times M_2$ is regular.*

Then the following hold.

[i] *The quotient manifold $M = (M_1 \times M_2)/G$ is 7-dimensional.*

[ii] *The rank-6 Pfaffian system $I_1 \oplus I_2$ on $M_1 \times M_2$ is reduced by G to a rank-3 Pfaffian system I on M.*

[iii] *I is a Darboux integrable hyperbolic Pfaffian system, whose Vessiot group is G.*

Notice that Theorem 7.1.1 applies in particular when I is generated by a hyperbolic PDE in plane, while Theorem 7.1.2 does not guarantee that I is generated by a PDE. In the next section we will show that when I_1 and I_2 are both GR_3D_5 Pfaffian systems to which Theorem 7.1.2 can be applied, then I is locally generated by a PDE in plane. Therefore, using the symmetry normal forms of Chapter 5, we can obtain a broad classification of all Darboux integrable non-Monge-Ampère hyperbolic PDE in the plane.

7.2 Non-Monge-Ampère equations

Now we can establish our classification of Darboux integrable non-Monge-Ampère hyperbolic equation. We shall use the same notation as in Theorems 7.1.1 and 7.1.2.

First, we must provide a preliminary result that follows from the recent work of Anderson and Fels [2].

Lemma 7.2.1. *If I is a non-Monge-Ampère Darboux integrable Pfaffian system then each side system is a GR_3D_5 Pfaffian system.*

This results is based on the argument that, in the given hypothesis, the side systems must have derived type $[3, 2, 0]$ in order for the Monge invariants to be both non-vanishing.

We can finally produce our theorem.

Theorem 7.2.2. *Let I be a Darboux integrable hyperbolic rank-3 Pfaffian system on a 7-manifold M. Assume G is the (3-dimensional) Vessiot group of I and that $\mathfrak{g}$ is the Lie Algebra associated to G, of type $[\mathfrak{g}]$ according to [36]. Then I is non-Monge-Ampère if and only if about each point of M there are local coordinates $(A, B, C, u_1, v_1, u_2, v_2)$ such that*

[i] *$\{u_1, v_1\}$ and $\{u_2, v_2\}$ are sets of Darboux invariants of I;*

[ii] *I can be expressed in one of the normal forms of Tables 7.1 and 7.2 (page 211). In these tables $F_1 = F_1(u_1, v_1)$ and $F_2 = F_2(u_1, v_1)$ are assumed to be smooth functions, and the expressions $K_i = K_i(u_i, v_i, F_i)$ are such that $\dfrac{\partial^2 K_i}{\partial {v_i}^2} \neq 0$, for $i = 1, 2$.*

Proof. [$\Leftarrow$] Each Pfaffian system I in Tables 7.1 and 7.2 (page 211) can be checked to be hyperbolic and generated by a PDE in the plane, for instance using the characterizations in [20]. By direct computation one can see that each Monge-Ampère invariant N_i is a multiple of the expression $\frac{\partial^2 K_i}{\partial {v_i}^2}$, and thus non-zero by hypothesis.

This result follows from Theorem 7.1.2 as well, once we show that those in Tables 7.1 and 7.2 are actually reductions of two copies of the systems in Table 1.1. This reduction can be found in the next part of this proof.

[$\Rightarrow$] Applying Theorem 7.1.1 and Lemma 7.2.1, we conclude that the side systems I_1 and I_2 satisfy the hypothesis of Theorem 3. Consequently I is the reduction by G of two copies of one of the systems in Table 1.1. We now describe the algorithm provided in the proof of [3, Theorem 4.5, page

1928], which will lead us to the expressions in Table 7.1 and Table 7.2. After the general description of this algorithm, we shall show the steps at work in one nontrivial example, $A_{3,1}$. Finally, we shall report the main expressions used for each algebraic type.

First, let's recall the notations preluding Theorem 7.1.1 and let's consider local coordinates $(a_i, b_i, c_i, u_i, v_i)$ on the 5-manifolds M_i. A local coordinate system on $M_1 \times M_2$ is $(a_1, b_1, c_1, u_1, v_1, a_2, b_2, c_2, u_2, v_2)$ and, locally, the standard projections can be expressed by

$$\pi_i(a_1, b_1, c_1, u_1, v_1, a_2, b_2, c_2, u_2, v_2) = (a_i, b_i, c_i, u_i, v_i).$$

Now we proceed to the description of the reduction.

[DI 1] This is the setup. Let the algebraic type of the Vessiot algebra of I be $[\mathfrak{g}]$ as in Theorem 3 (page 5) and Table 1.1 (page 8). We have the corresponding normal form of the side systems I_i, which are GR_3D_5 Pfaffian systems, and the infinitesimal generators Γ_i of the action of G on M_i. To write the local expression of I_i and Γ_i we identify the local coordinates (a, b, c, u, v) in Table 1.1 with $(a_i, b_i, c_i, u_i, v_i)$. In a similar way we define the functions $K_i = {\pi_i}^* K$ on $M_1 \times M_2$, where K is the function defined in Table 1.1 (page 8). Finally, define the Pfaffian system $I_1 \oplus I_2$ on $M_1 \times M_2$ and the diagonal action of G as described before Theorem 7.1.1.

[DI 2] We define the quotient map. As a consequence of our hypothesis, the diagonal action of G on the 10-manifold $M_1 \times M_2$ has 3-dimensional orbits, and thus there are seven invariants. We know that by construction four of them are u_1, v_1, u_2, v_2. We label other three by A, B and C. By Theorem 7.1.1, the diagonal action is regular and on the quotient manifold $(M_1 \times M_2)/G = M$ we can take local coordinates $(A, B, C, u_1, v_1, u_2, v_2)$. Consequently, the quotient map $\mathbf{q} : M_1 \times M_2 \to M$ is locally defined by

$$A = A(a_i, b_i, c_i),\; B = B(a_i, b_i, c_i),\; C = C(a_i, b_i, c_i),\; u_1 = u_1,\; v_1 = v_1,\; u_2 = u_2,\; v_2 = v_2.$$

[DI 3] We obtain the reduced Pfaffian system I. Define the rank-3 Pfaffian subsystems $\tilde{I}_i = \{\tilde{\theta}_i^j\}_{j=1,2,3}$ of $I_1 \oplus I_2$. By the transversality conditions, the matrices $P_i = (\tilde{E}_h \lrcorner\, \tilde{\theta}_i^k)$ are non-singular. We can change the bases of $\tilde{I}_i$ to $\{\bar{\theta}_i^1, \bar{\theta}_i^2, \bar{\theta}_i^3\}$, where we define $(\bar{\theta}_i^1, \bar{\theta}_i^2, \bar{\theta}_i^3)^T = P_i^{-1}(\tilde{\theta}_i^1, \tilde{\theta}_i^2, \tilde{\theta}_i^3)^T$. Now we define the semi-basic forms, that is the rank-3 subsystem $I_{sb} = \{\bar{\theta}_1^j - \bar{\theta}_2^j\}_{j=1,2,3}$ of $I_1 \oplus I_2$. By direct computation, one verifies that $\Gamma \subseteq (I_{sb})^{\perp}$. Consequently the reduced Pfaffian system I is such that $\mathbf{q}^*I = I_{sb}$. Finally, because $I = \mathbf{s}^*I_{sb}$ for any section $\mathbf{s}$, we produce the local expressions in Table 7.1 and Table 7.2 by considering the section $\mathbf{s} : M \to M_1 \times M_2$ given by

$$a_1 = A,\ b_1 = B,\ c_1 = C,\ a_2 = 0,\ b_2 = 0,\ c_2 = 0,\ u_1 = u_1,\ v_1 = v_1,\ u_2 = u_2,\ v_2 = v_2.$$

After step **[DI 3]** we have our local expressions of $I = \{\theta_0^1, \theta_0^2, \theta_0^3\}$. *In general, this is not an adapted basis of* I, which we usually denote by $\{\alpha, \alpha^1, \alpha^2\}$. To obtain an adapted basis one can use a standard procedure, for instance the one outlined in [9]. The first step is to obtain $I' = \{\alpha\}$.

We now proceed to a detailed case, $A_{3,1}$, following the three steps described above. In the other cases we will provide only the data.

[$A_{3,1}$. **DI 1**] We have

$$I_1 \oplus I_2 = \{db_i - (a_i + b_i)\ dc_i + F_i\, du_i + dv_i, \quad da_i - a_i\, dc_i + du_i, \quad dc_i + F_{iv_i}\, du_i\}_{i=1,2}$$

and infinitesimal generators of the diagonal action are

$$\tilde{E}_1 = -e^{c_1}\,\partial_{b_1} - e^{c_2}\,\partial_{b_2}, \quad \tilde{E}_2 = -e^{c_1}\,\partial_{a_1} - c_1 e^{c_1}\,\partial_{b_1} - e^{c_2}\,\partial_{a_2} - c_2 e^{c_2}\,\partial_{b_2}, \quad \tilde{E}_3 = -\partial_{c_1} - \partial_{c_2}.$$

[$A_{3,1}$. **DI 2**] To complete u_1, v_1, u_2, v_2 to a set of invariants of the diagonal action, we take

$$A = a_1 - a_2, \quad B = b_1 - b_2 - (c_1 - c_2)a_2, \quad C = c_1 - c_2.$$

[$A_{3,1}$. **DI 3**] We have

$$P_i = (\tilde{E}_h \lrcorner\, \tilde{\theta}_i^k) = \begin{pmatrix} -1 & -c_i & a_i \\ 0 & -u_i & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Consequently

$$I_{sb} = \begin{cases} c_1\,da_1 - \,db_1 + (F_1v_1c_1 - F_1v_1u_1a_1 - F_1)\,du_1 + a_1\,dv_1 - c_2\,da_2 + \,db_2 \\ \quad + (-F_2v_2c_2 + F_2v_2u_2a_2 + F_2)\,du_2 - a_2\,dv_2, \\ - \,da_1 - F_1v_1\,du_1 + \,da_2 + F_2v_2\,du_2, \\ - \,dc_1 - F_1v_1u_1\,du_1 + \,dv_1 + \,dc_2 + F_2v_2u_2\,du_2 - \,dv_2 \end{cases}$$

For all the other cases, we here report only the invariants A, B and C used.

[$3A_1$] $A = a_1 - a_2, \quad B = b_1 - b_2, \quad C = c_1 - c_2.$

[$A_1 \oplus A_2$] $A = a_1 - a_2e^{c_1-c_2}, \quad B = b_1 - b_2, \quad C = c_1 - c_2.$

[$A_{3,2}$] $A = a_1 - e^{c_1-c_2}a_2, \quad B = b_1 - [(c_1 - c_2)a_2 + b_2]e^{c_1-c_2}, \quad C = c_1 - c_2.$

[$A^{\epsilon}_{3,5}$] $A = a_1 - e^{c_1-c_2}a_2, \quad B = b_1 - e^{\epsilon(c_1-c_2)}b_2, \quad C = c_1 - c_2.$

[$A^{\epsilon}_{3,7}$] $A = a_1 - \cos c_1 - c_2e^{\epsilon(c_1-c_2)}a_2 - \sin c_1 - c_2e^{\epsilon(c_1-c_2)}b_2, \quad B = b_1 + \sin c_1 - c_2e^{\epsilon(c_1-c_2)}a_2 - \cos c_1 - c_2e^{\epsilon(c_1-c_2)}b_2, \quad C = c_1 - c_2.$

[$A_{3,8}$] $A = a_1 - a_2 + ln(1 - b_2c_1 + c_2b_2), \quad B = [b_1(b_2c_1 - b_1c_2b_2 - 1) + b_2](b_2c_1 - c_2b_2 - 1)e^{-2a_2}, \quad C = (c_2 - c_1)\frac{e^{2a_2}}{b_2c_1 - c_2b_2 - 1}$. For this algebraic type we have three possible normal forms.

[$A_{3,8,1}$] Both I_1 and I_2 are GR_3D_5 Pfaffian systems with normal form $A_{3,8,1}$ as in Table 1.1.

[$A_{3,8,2}$] Both I_1 and I_2 are GR_3D_5 Pfaffian systems with normal form $A_{3,8,2}$ as in Table 1.1.

[$A_{3,8,3}$] I_1 and I_2 are GR_3D_5 Pfaffian systems with inequivalent normal forms $A_{3,8}$ as in Table 1.1. In particular different choices of side normal forms will produce equivalent Darboux integrable Pfaffian systems.

$[A_{3,9}]$ $B = -\arcsin L_2, \quad A = \arcsin\left(\frac{L_1}{\cos B}\right), \quad C = \arcsin\left(\frac{L_3}{\cos B}\right)$, where

$$\begin{aligned}
L_1 = &-\cos c_2 \sin a_2 \cos c_1 \cos a_1 - \sin c_2 \sin a_2 \sin c_1 \cos a_1 - \cos c_2 \sin a_2 \sin c_1 \sin b_1 \sin a_1 \\
&+ \sin c_2 \sin a_2 \cos c_1 \sin b_1 \sin a_1 + \cos b_2 \cos a_2 \cos b_1 \sin a_1 - \cos c_2 \sin b_2 \cos a_2 \sin c_1 \cos a_1 \\
&+ \cos c_2 \sin b_2 \cos a_2 \cos c_1 \sin b_1 \sin a_1 + \sin c_2 \sin b_2 \cos a_2 \cos c_1 \cos a_1 \\
&+ \sin c_2 \sin b_2 \cos a_2 \sin c_1 \sin b_1 \sin a_1, \\
L_2 = &-\sin a_2 \sin c_1 \cos b_1 \cos c_2 + \sin a_2 \cos c_1 \cos b_1 \sin c_2 - \cos b_2 \cos a_2 \sin b_1 \\
&+ \cos a_2 \cos c_1 \cos b_1 \cos c_2 \sin b_2 + \cos a_2 \sin c_1 \cos b_1 \sin c_2 \sin b_2, \\
L_3 = &-\cos b_2 \sin a_2 \sin b_1 + \sin a_2 \cos c_1 \cos b_1 \cos c_2 \sin b_2 \\
&+ \sin a_2 \sin c_1 \cos b_1 \sin c_2 \sin b_2 + \sin c_1 \cos b_1 \cos c_2 \cos a_2 - \cos c_1 \cos b_1 \sin c_2 \cos a_2.
\end{aligned}$$

□

7.3 Examples

In the following examples we applied the forementioned algorithm **[DI]** to reduce two copies of the Hilbert-Cartan equation to a Darboux integrable non-Monge-Ampère hyperbolic PDE in the plane. We used the Chevalley basis of $\mathfrak{g}_2$ provided in Table 7.3 (page 213) and the indicated 3-dimensional abelian subalgebra $P3a_i$ of $\mathfrak{g}_2$. A program developed by Biesecker was used to compute the Darboux invariants.

The explicit realizations as hyperbolic PDE in the plane are computationally difficult. One future project is the production of several examples of hyperbolic PDE in the plane obtained from the normal forms in Tables 7.1 and 7.2.

Example 7.3.1. From $P3a_1 = \{Y_1, X_4, X_5\}$ we got the equation

$$rt - s^2 = t^{1/3} s.$$

The Darboux invariants are

$$\begin{aligned}
\mathrm{I}_1 &= W_1 + W_2,\\
\mathrm{I}_2 &= -2\mathrm{I}_1 q W_2 + \frac{1}{3}(p - \frac{qs}{t}),\\
\mathrm{J}_1 &= W_1 - W_2,\\
\mathrm{J}_2 &= 2\mathrm{J}_1 q W_2 + \frac{1}{3}(p - \frac{qs}{t}),
\end{aligned}$$

where $W_1 = \sqrt{\dfrac{3t^{1/3} + 4s}{12t}}$, $W_2 = \dfrac{1}{2t^{1/3}}$. ■

Example 7.3.2. Using $P3a_2 = \{X_4, X_5, X_6\}$ we got

$$rt - s^2 = 3t^4.$$

The Darboux invariants are

$$\begin{aligned}
\mathrm{I}_1 &= \frac{t}{3t^2 - s},\\
\mathrm{I}_2 &= -y\mathrm{I}_1 + x,\\
\mathrm{J}_1 &= -\frac{t}{3t^2 + s},\\
\mathrm{J}_2 &= -y\mathrm{J}_1 + x.
\end{aligned}$$

■

Example 7.3.3. From $P3a_3 = \{X_2, X_5, X_6\}$ we obtained

$$rt - s^2 = \frac{-4t^2 s^2}{s^3 + 4t^2}.$$

The Darboux invariants are

$$\begin{aligned}
\mathrm{I}_1 &= (W_1 - s)\, W_2,\\
\mathrm{I}_2 &= 8xsW_2\mathrm{I}_1 - \frac{1}{27}\left(\frac{q}{4} - 12xtW_2\right),\\
\mathrm{J}_1 &= (W_1 + s)\, W_2,\\
\mathrm{J}_2 &= -8xsW_2\mathrm{J}_1 - \frac{1}{27}\left(\frac{q}{4} - 12xtW_2\right),
\end{aligned}$$

where $W_1 = \sqrt{-\dfrac{s^3 + 16t^2}{3s}}$, $W_2 = \dfrac{st}{12(s^3 + 4t^2)}$. ■

Example 7.3.4. Using $P3a_4 = \{X_3, X_5, X_6\}$ we got

$$r^3\,(yt+2q) = y^3 s^3$$

or equivalently

$$rt - s^2 = -s^2 - \frac{yts}{(yt+2q)^{1/3}}.$$

The Darboux invariants are

$$\begin{aligned}
\mathrm{I}_1 &= \frac{(W_1 - W_2)\,W_3 - 2W_5}{W_4^2},\\
\mathrm{I}_2 &= \frac{\mathrm{I}_1 W_1 x W_4}{W_3} + \frac{2}{3}(p+ys) - x\frac{W_5}{W_3},\\
\mathrm{J}_1 &= (W_1 + W_2)\,,\\
\mathrm{J}_2 &= \frac{\mathrm{J}_1 W_1}{y} + \frac{2}{3}(p+ys),
\end{aligned}$$

where $W_1 = y(yt+2q)^{1/3}$, $W_2 = \sqrt{\frac{4sy^2+3W_1^2}{3}}$, $W_3 = -xy(yx+4W_1)$, $W_4 = (yx+2W_1)$, $W_5 = 2(2qy^3 - W_1^2 W_2 + y^4 t)$. ■

According to the general theory exposed in this dissertation, the PDE obtained in the previous examples are all equivalent, because they are reductions of equivalent GR_3D_5 Pfaffian systems by isomorphic symmetry algebras. By these examples, one realizes how complicated the change of variables involved are.

7.4 Addendum: the exceptional Lie algebra $\mathfrak{g}_2$

This project was motivated by the following question. How many inequivalent symmetry reductions of the Hilbert-Cartan equation $z_x = {y_{xx}}^2$ are there? Or rather, considering Theorem 7.1.2, how many different Darboux integrable generic hyperbolic PDE in the plane can be obtained as reductions of two copies of the HC equation?

Cartan proved that the HC equation is the only Monge equation which has symmetry algebra the (real non-compact form of the) exceptional Lie algebra $\mathfrak{g}_2$. Thus our initial question soon turned out to rise another, rather geometric, one. How many non conjugate 3-dimensional subalgebras does $\mathfrak{g}_2$ have? Searching for an answer we started to study the five variables paper.

In this section we just report a realization of $\mathfrak{g}_2$ as the symmetry algebra of $z_x = {y_{xx}}^2$. A ***Chevalley basis*** for $\mathfrak{g}_2$ is given in Table 7.3 (page 213), then using this and setting $H_3 = 3H_1 + H_2$, $H_4 = 3H_1 + 2H_2$, $H_5 = H_1 + H_2$ and $H_6 = 2H_1 + H_2$, the structure equations of $\mathfrak{g}_2$ are given in Table 7.4 (page 214).

Table 7.1: Non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane (1).

$[\mathfrak{g}]$	$I;\ K.$
$3A_1$	$-\,dA - F_1v_1\,du_1 + F_{2v_2}\,du_2, \quad -dB - u_1F_1v_1\,du_1 + u_2F_{2v_2}\,du_2,$ $-\,dC - (v_1F_1v_1 - F_1)\,du_1 + (v_2F_{2v_2} - F_2)\,du_2; \quad K_i = F_i.$
$A_1 \oplus A_2$	$-\,e^{-C}dA - e^{-C}(F_1 + AF_{1v_1})du_1 - e^{-C}dv_1 + F_2du_2 + dv_2,$ $-\,dC - F_{1v_1}du_1 + F_{2v_2}du_2, \quad -dB - u_1F_{1v_1}du_1 + u_2F_{2v_2}du_2; \quad K_i = F_i.$
$A_{3,1}$	$C\,dA - dB + (CF_{1v_1} - Au_1F_{1v_1} - F_1)\,du_1 + A\,dv_1 + F_2\,du_2,$ $-\,dC - u_1F_{1v_1}du_1 + dv_1 + u_2F_{2v_2}du_2 - dv_2,$ $-\,dA - F_{1v_1}du_1 + F_{2v_2}du_2; \quad K_i = F_i.$
$A_{3,2}$	$Ce^{-C}\,dA - e^{-C}\,dB + F_2\,du_2 + \,dv_2$ $\quad + (F_{1v_1}CA - AF_{1v_1} - F_{1v_1}B + C - F_1)e^{-C}\,du_1 - e^{-C}\,dv_1,$ $-\,e^{-C}\,dA - e^{-C}(AF_{1v_1} + 1)\,du_1 + \,du_2,$ $-\,dC - F_{1v_1}\,du_1 + F_{2v_2}\,du_2; \quad K_i = F_i.$
$A^{\epsilon}_{3,5}$ $\epsilon \neq 0$	$-\,e^{-C}\,dA - e^{-C}(\epsilon^{-1}AF_{1v_1} + 1)\,du_1 + \,du_2,$ $-\,e^{-\epsilon C}\,dB - e^{-\epsilon C}(F_{1v_1}B + F_1)\,du_1 - e^{-\epsilon C}\,dv_1 + F_2\,du_2 + \,dv_2,$ $-\,\epsilon\,dC - F_{1v_1}\,du_1 + F_{2v_2}\,du_2; \quad K_i = F_i.$
$A^{\epsilon}_{3,7}$ $\epsilon \geq 0$	$(\epsilon\sin C - \cos C)\,dA + (\epsilon\cos C + \sin C)\,dB + e^{\epsilon C}F_2\,du_2 + e^{\epsilon C}\,dv_2$ $\quad - \Big[\big((A\epsilon^2 + 2\epsilon B - A)(F_1 - \epsilon)F_{1v_1} - (1+\epsilon^2)\big)\,\dfrac{\sin C}{1+\epsilon^2}$ $\quad\quad + \big((B\epsilon^2 - 2A\epsilon - B)(F_1 - \epsilon)F_{1v_1} + (1+\epsilon^2)F_1\big)\,\dfrac{\cos C}{1+\epsilon^2}\Big]\,du_1$ $\quad - \Big[(A\epsilon^2 + 2\epsilon B - A)F_{1v_1}\dfrac{\sin C}{1+\epsilon^2} + \big((B\epsilon^2 - 2A\epsilon - B)F_{1v_1} + (1+\epsilon^2)\big)\,\dfrac{\cos C}{1+\epsilon^2}\Big]\,dv_1,$ $-\,(\epsilon\cos C + \sin C)\,dA + (\epsilon\sin C - \cos C)\,dB + e^{\epsilon C}\,du_2$ $\quad + \Big[\big((A\epsilon^2 + 2\epsilon B - A)(F_1 - \epsilon)F_{1v_1} - (1+\epsilon^2)\big)\,\dfrac{\cos C}{1+\epsilon^2}$ $\quad\quad - \big((B\epsilon^2 - 2A\epsilon - B)(F_1 - \epsilon)F_{1v_1} + (1+\epsilon^2)F_1\big)\,\dfrac{\sin C}{1+\epsilon^2}\Big]\,du_1$ $\quad + \Big[(A\epsilon^2 + 2\epsilon B - A)F_{1v_1}\dfrac{\cos C}{1+\epsilon^2} - ((B\epsilon^2 - 2A\epsilon - B)F_{1v_1} + (1+\epsilon^2))\dfrac{\sin C}{1+\epsilon^2}\Big]\,dv_1,$ $(1+\epsilon^2)\,dC + F_{2v_2}(F_2 - \epsilon)\,du_2 + F_{2v_2}\,dv_2 - F_{1v_1}(F_1 - \epsilon)\,du_1 - F_{1v_1}\,dv_1;$ $K_{iv_i} = (F_i - \epsilon)\,F_{iv_i} - F_{iu_i}.$

Table 7.2: Non-Monge-Ampère Darboux integrable hyperbolic PDE in the plane (2).

$[\mathfrak{g}]$	I; K.
$A_{3,8,1}$	$(-1-2CB)\,dA + C\,dB + F_2\,du_2 + dv_2 + [F_{1v_1}e^{-2A}(Ce^{4A}+CB^2+B) - 1 - 2CB]\,dv_1$ $- \left[e^{-2A}\left(\frac{1}{2} - F_{1v_1}F_1\right)(CB^2 + e^{4A}C + B) + F_1 - e^{2A}C + 2CBF_1\right]\,du_1,$ $-2B\,dA + dB - \left(\frac{1}{2} + F_{2v_2}F_2\right)\,du_2 - F_{2v_2}\,dv_2 + (F_{1v_1}e^{2A} + F_{1v_1}B^2e^{-2A} - 2B)\,dv_1$ $-(e^{-2A}(\frac{1}{2} - F_{1v_1}F_1)(B^2 + e^{4A}) + 2BF_1 - e^{2A})\,du_1,$ $2C(CB+1)\,dA - C^2\,dB + dC - (\frac{1}{2} - F_{2v_2}F_2)\,du_2 + F_{2v_2}\,dv_2$ $+ \{e^{-2A}(\frac{1}{2} - F_{1v_1}F_1)[C^2e^{4A} + (1-CB)^2] - C(1+2CB)F_1 - C^2e^{2A}\}\,du_1$ $+ \{2C(1+CB) - F_{1v_1}e^{-2A}[C^2e^{4A} + (1+CB)^2]\}\,dv_1; \quad K_{iv_i} = F_iF_{iv_i} - v_i - F_{iu_i}.$
$A_{3,8,2}$	$(-1-2CB)\,dA + C\,dB - \frac{1}{2}F_{2v_2}\,du_2 + Ce^{2A}\,dv_1$ $+ [-(1+CB)Be^{-2A} + (BF_{1v_1} + e^{2A}F_1)C + \frac{1}{2}F_{1v_1}]\,du_1,$ $-2B\,dA + dB - F_2\,du_2 - dv_2 + (BF_{1v_1} - B^2e^{-2A} + e^{2A}F_1)\,du_1 + e^{2A}\,dv_1,$ $2C(CB+1)\,dA - C^2\,dB + dC - du_2 - C^2e^{2A}\,dv_1$ $+ [(1+CB)^2e^{-2A} + (-BF_{1v_1} - e^{2A}F_1)C^2 - CF_{1v_1}]\,du_1; \quad K_i = F_i + {v_i}^2.$
$A_{3,8,3}$	$(-1-2CB)\,dA + C\,dB - \frac{F_{2v_2}}{2}\,du_2 + [e^{-2A}((1+CB)B + Ce^{4A})F_{1v_1} - 1 - 2CB]\,dv_1$ $+ e^{-2A}[(-\frac{1}{2} + F_{1v_1}F_1)(CB^2 + Ce^{4A} + B) - 2CBe^{2A}F_1 + e^{4A}C - e^{2A}F_1]\,du_1,$ $-2B\,dA + dB - F_2\,du_2 - dv_2 + [(e^{-2A}B^2 + e^{2A})F_{1v_1} - 2B]\,dv_1$ $+ [(-\frac{1}{2} + F_{1v_1}F_1)(e^{-2A}B^2 + e^{2A}) - 2BF_1 + e^{2A}]\,du_1,$ $2C(CB+1)\,dA - C^2\,dB + dC - du_2 + [-((1+CB)^2e^{-2A} + C^2e^{2A})F_{1v_1}$ $+ 2(1+CB)C]\,dv_1 + [(\frac{1}{2} - F_{1v_1}F_1)(e^{-2A}(1+CB)^2 + e^{2A}C^2)$ $+ 2(1+CB)F_1 - e^{2A}]\,du_1; \quad K_{1v_1} = F_1F_{1v_1} - v_1 - F_{1u_1}, \quad K_2 = F_2 + {v_2}^2.$
$A_{3,9}$	$-\cos C\cos B\,dA + \sin C\,dB + du_2$ $-[F_1(F_{1v_1}\sin A - \cos A)\sin C + (F_1(F_{1v_1}\cos A + \sin A)\sin B + \cos B)\cos C]\,du_1$ $-[(F_{1v_1}\sin A - \cos A)\sin C + (F_{1v_1}\cos A + \sin A)\sin B\cos C]\,dv_1,$ $-\sin C\cos B\,dA - \cos C\,dB + F_2\,du_2 + dv_2$ $-[(F_1(F_{1v_1}\cos A + \sin A)\sin B + \cos B)\sin C - F_1(F_{1v_1}\sin A - \cos A)\cos C]\,du_1$ $-[(F_{1v_1}\cos A + \sin A)\sin B\sin C - (F_{1v_1}\sin A - \cos A)\cos C]\,dv_1,$ $\sin B\,dA - dC + F_2F_{2v_2}\,du_2 + F_{2v_2}\,dv_2 - [(F_{1v_1}\cos A + \sin A)F_1\cos B - \sin B]\,du_1$ $-(\cos AF_{1v_1} + \sin A)\cos B\,dv_1; \quad K_{iv_i} = F_iF_{iv_i} + v_i - F_{iu_i}.$

Table 7.3: A Chevalley basis for $\mathfrak{g}_2$.

$$
\begin{aligned}
H_1 &= y\partial_y + y_x\partial_{y_x} + y_{xx}\partial_{y_{xx}} + 2z\partial_z,\\
H_2 &= -x\partial_x - 3y\partial_y - 2y_x\partial_{y_x} - y_{xx}\partial_{y_{xx}} - 3z\partial_z,\\
X_1 &= \left(-4y_x^2 + 6y_{xx}y\right)\partial_x + \left(-3zy + 6yy_xy_{xx} - \frac{8}{3}y_x^3\right)\partial_y + \left(3yy_{xx}^2 - 3y_xz\right)\partial_{y_x}\\
&\quad + \left(-3y_{xx}z + 2y_{xx}^2y_x\right)\partial_{y_{xx}} + \left(2y_{xx}^3y - 3z^2\right)\partial_z,\\
Y_1 &= \frac{1}{3}\partial_z,\\
X_2 &= x\partial_y + \partial_{y_x},\\
Y_2 &= \left(-4y_xx + 3y + \frac{3}{2}y_{xx}x^2\right)\partial_x + \left(-2y_x^2x - \frac{3}{4}zx^2 + \frac{3}{2}y_xy_{xx}x^2\right)\partial_y\\
&\quad + \left(\frac{3}{4}y_{xx}^2x^2 - \frac{3}{2}zx - y_x^2\right)\partial_{y_x} + \left(y_{xx}^2x - y_{xx}y_x - \frac{3}{2}z\right)\partial_{y_{xx}} + \left(\frac{1}{2}y_{xx}^3x^2 - 3y_xz\right)\partial_z,\\
X_3 &= \left(8y_x - 6y_{xx}x\right)\partial_x + \left(3zx - 6y_xy_{xx}x + 4y_x^2\right)\partial_y + 3\left(z - y_{xx}^2x\right)\partial_{y_x}\\
&\quad - 2y_{xx}^2\partial_{y_{xx}} - 2y_{xx}^3x\partial_z,\\
Y_3 &= \frac{1}{4}x^2\partial_y + \frac{1}{2}x\partial_{y_x} + \frac{1}{2}\partial_{y_{xx}} + y_x\partial_z,\\
X_4 &= -4\partial_x,\\
Y_4 &= \frac{1}{4}x^2\partial_x + \frac{3}{4}yx\partial_y + \left(\frac{1}{4}y_xx + \frac{3}{4}y\right)\partial_{y_x} + \left(y_x - \frac{1}{4}y_{xx}x\right)\partial_{y_{xx}} + y_x^2\partial_z,\\
X_5 &= \frac{4}{3}\partial_y,\\
Y_5 &= \left(\frac{1}{2}y_xx^2 - \frac{3}{4}yx - \frac{1}{8}y_{xx}x^3\right)\partial_x + \left(\frac{1}{4}y_x^2x^2 - \frac{1}{8}y_xy_{xx}x^3 - \frac{3}{4}y^2 + \frac{1}{16}zx^3\right)\partial_y\\
&\quad + \left(-\frac{1}{16}y_{xx}^2x^3 + \frac{3}{16}zx^2 - \frac{3}{4}y_xy + \frac{1}{4}y_x^2x\right)\partial_{y_x}\\
&\quad + \left(-\frac{1}{8}y_{xx}^2x^2 + \frac{3}{8}zx - \frac{1}{2}y_x^2 + \frac{1}{4}y_xy_{xx}x\right)\partial_{y_{xx}}\\
&\quad + \left(-\frac{1}{2}4y_{xx}^3x^3 + \frac{3}{4}zy_xx - \frac{3}{4}zy - \frac{1}{3}y_x^3\right)\partial_z,\\
X_6 &= -8y_{xx}\partial_x + \left(-8y_{xx}y_x + 4z\right)\partial_y - 4y_{xx}^2\partial_{y_x} - \frac{8}{3}y_{xx}^3\partial_z,\\
Y_6 &= -\frac{1}{48}x^3\partial_y - \frac{1}{16}x^2\partial_{y_x} - \frac{1}{8}x\partial_{y_{xx}} + \left(-\frac{1}{4}y_xx + \frac{1}{4}y\right)\partial_z
\end{aligned}
$$

Table 7.4: Lie brackets in $\mathfrak{g}_2$.

	H_1	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	0	$2X_1$	$-2Y_1$	$-X_2$	Y_2	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
H_2	0	0	$-3X_1$	$3Y_1$	$2X_2$	$-2Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
X_1	$-2X_1$	$3X_1$	0	H_1	X_3	0	0	$-Y_2$	0	0	X_6	0	0	$-Y_5$
Y_1	$2Y_1$	$-3Y_1$	$-H_1$	0	0	$-Y_3$	X_2	0	0	0	0	$-Y_6$	X_5	0
X_2	X_2	$-2X_2$	$-X_3$	0	0	H_2	$-2X_4$	$3Y_1$	$3X_5$	$2Y_3$	0	$-Y_4$	0	0
Y_2	$-Y_2$	$2Y_2$	0	Y_3	$-H_2$	0	$-3X_1$	$2Y_4$	$-2X_3$	$-3Y_5$	X_4	0	0	0
X_3	$-X_3$	X_3	0	$-X_2$	$2X_4$	$3X_1$	0	H_3	$3X_6$	$-2Y_2$	0	0	0	$-Y_4$
Y_3	Y_3	$-Y_3$	Y_2	0	$-3Y_1$	$-2Y_4$	$-H_3$	0	$2X_2$	$-3Y_6$	0	0	X_4	0
X_4	0	$-X_4$	0	0	$-3X_5$	$2X_3$	$-3X_6$	$-2X_2$	0	H_4	0	Y_2	0	Y_3
Y_4	0	Y_4	0	0	$-2Y_3$	$3Y_5$	$2Y_2$	$3Y_6$	$-H_4$	0	$-X_2$	0	$-X_3$	0
X_5	X_5	$-3X_5$	$-X_6$	0	0	$-X_4$	0	0	0	X_2	0	H_5	0	Y_1
Y_5	$-Y_5$	$3Y_5$	0	Y_6	Y_4	0	0	0	$-Y_2$	0	$-H_5$	0	$-X_1$	0
X_6	$-X_6$	0	0	$-X_5$	0	0	0	$-X_4$	0	X_3	0	X_1	0	H_6
Y_6	Y_6	0	Y_5	0	0	0	Y_4	0	$-Y_3$	0	$-Y_1$	0	$-H_6$	0

CHAPTER 8

FIVEVARIABLES PACKAGE

In this Chapter we give a brief account of the equivalence method applied to a general rank-3 Pfaffian system I, as shown by Cartan [10] and Hsiao [29]. A recent description of this method was provided by Stormark [38, Chapter 16].

From Theorem 3.3.5 and Hsiao [29], we know that there exist a local coframe $\omega^1, \ldots, \omega^5$ and 1-forms c^1, c^2, d^1, d^2, e^1, e^2 and e^3 on M such that:

[i] $I = \{\omega^1, \omega^2, \omega^3\}$;

[ii] the following structure equations are satisfied

$$\begin{aligned} d\omega^1 &= c^1 \wedge \omega^1 + c^2 \wedge \omega^2 + \omega^3 \wedge \omega^4, \\ d\omega^2 &= d^1 \wedge \omega^1 + d^2 \wedge \omega^2 + \omega^3 \wedge \omega^5, \\ d\omega^3 &= e^1 \wedge \omega^1 + e^2 \wedge \omega^2 + e^3 \wedge \omega^3 + \omega^4 \wedge \omega^5. \end{aligned} \tag{8.1}$$

[iii] the ***Hsiao requirement*** is satisfied

$$\omega^3 \wedge \left(e^3 - \frac{1}{3}\left(c^1 + d^2\right) \right) = 0. \tag{8.2}$$

A change of variables on M, say $\phi^{-1} : M \to M$, is equivalent to a linear change of the given coframe on M, say $\overline{\Omega} = \phi^* \Omega$. Using the vector notation $\vec{\Omega} = \left(\omega^1 \ \ldots \ \omega^5\right)^T$, this linear transformation has to be represented by a 5×5 matrix T such that $\vec{\overline{\Omega}} = \mathrm{T}\vec{\Omega}$. Because ϕ preserves the structure equations of Ω, $\overline{\Omega}$ satisfies properties **[i]** to **[iii]**.

The preservation of the derived system implies that

$$\mathrm{T} = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 \\ \gamma_1 & \delta_1 & 0 & 0 & 0 \\ x & y & z & 0 & 0 \\ p_1 & p_2 & p_3 & \alpha & \beta \\ p_1 & p_2 & p_3 & \gamma & \delta \end{pmatrix} \in G_{19},$$

that is, T has only nineteen arbitrary entries.

A complete consideration of equations (8.1) reveals that

$$\mathrm{T} = \begin{pmatrix} L\alpha & L\beta & 0 & 0 & 0 \\ L\gamma & L\delta & 0 & 0 & 0 \\ x & y & L & 0 & 0 \\ p_1 & p_2 & p_3 & \alpha & \beta \\ q_1 & q_2 & q_3 & \gamma & \delta \end{pmatrix} \in G_{12}, \quad L = \alpha\delta - \beta\gamma \neq 0. \tag{8.3}$$

Consequently, two GR_3D_5 Pfaffian systems $\{\omega^1, \omega^2, \omega^3\}$ and $\{\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3\}$ are equivalent if and only if the local coframes Ω and $\bar{\Omega}$ (satisfying **[i]** and **[ii]**) are related by a transformation $T \in G_{12}$, defined in (8.3).

Cartan proved that one can actually reduce G_{12} to a 7-dimensional group G_7. Hsiao provided explicit computations for the reduction from G_{12} to G_7, showing first of all that Hsiao's requirement (8.2) (implicitly assumed by Cartan) reduces G_{12} to a 10-dimensional group G_{10} defined by

$$\mathrm{T} = \begin{pmatrix} L\alpha & L\beta & 0 & 0 & 0 \\ L\gamma & L\delta & 0 & 0 & 0 \\ x & y & L & 0 & 0 \\ p_1 & p_2 & \frac{4F}{3L} & \alpha & \beta \\ q_1 & q_2 & \frac{4G}{3L} & \gamma & \delta \end{pmatrix} \in G_{10}, \quad F = \alpha y - \beta x, \quad G = y\gamma - x\delta, \tag{8.4}$$

where L is defined in (8.3).

Finally, G_{10} is reduced to a 7-dimensional group G_7, namely

$$\mathrm{T} = \begin{pmatrix} L\alpha & L\beta & 0 & 0 & 0 \\ L\gamma & L\delta & 0 & 0 & 0 \\ x & y & L & 0 & 0 \\ \dfrac{3\sigma\alpha L + 2xF}{3L^2} & \dfrac{3\sigma\beta L + 2yF}{3L^2} & \dfrac{4F}{3L} & \alpha & \beta \\ \dfrac{3\sigma\gamma L + 2xG}{3L^2} & \dfrac{3\sigma\delta L + 2yG}{3L^2} & \dfrac{4G}{3L} & \gamma & \delta \end{pmatrix} \in G_7, \quad \sigma = \frac{1}{2}(p_1\delta - p_2\gamma - q_1\beta + q_2\alpha), \tag{8.5}$$

where L, F and G are defined in (8.3) and (8.4). The Maurer-Cartan matrix associated to G_7 is

$$\begin{pmatrix} 2\,\pi^1 + \pi^4 & \pi^2 & 0 & 0 & 0 \\ \pi^3 & \pi^1 + 2\,\pi^4 & 0 & 0 & 0 \\ \pi^5 & \pi^6 & \pi^1 + \pi^4 & 0 & 0 \\ \pi^7 & 0 & \frac{4}{3}\,\pi^6 & \pi^1 & \pi^2 \\ 0 & \pi^7 & -\frac{4}{3}\,\pi^5 & \pi^3 & \pi^4 \end{pmatrix}.$$

Consider the principal bundle $P_{12} = M \times G_7$ with local coframe $\omega^1, \ldots, \omega^5$, $\pi^1, \ldots, \pi^7$. Then the structure equations [10, page 149, equation (5)]

$$\begin{aligned} d\omega^1 &= \omega^1 \wedge \left(2\,\pi^1 + \pi^4\right) + \omega^2 \wedge \pi^2 + \omega^3 \wedge \omega^4, \\ d\omega^2 &= \omega^1 \wedge \pi^3 + \omega^2 \wedge \left(\pi^1 + 2\,\pi^4\right) + \omega^3 \wedge \omega^5, \\ d\omega^3 &= \omega^1 \wedge \pi^5 + \omega^2 \wedge \pi^6 + \omega^3 \wedge \left(\pi^1 + \pi^4\right) + \omega^4 \wedge \omega^5, \end{aligned} \tag{8.6}$$

called *Cartan's first formulas* are satisfied. These equations are invariant under the equivalences G_7.

Introducing two new auxiliary variables υ_1, υ_2 one defines the group G_9 of block matrices

$$\begin{pmatrix} \mathrm{T}_7 & 0 \\ \mathrm{T}_2 & \mathrm{I}_7 \end{pmatrix} \in G_9$$

such that $\mathrm{T}_7 \in G_7$, I_7 is the identity matrix of order seven and

$$\mathrm{T}_2 = \begin{pmatrix} \upsilon_1 & 0 & 0 & 0 & 0 \\ \upsilon_2 & 0 & 0 & 0 & 0 \\ 0 & \upsilon_1 & 0 & 0 & 0 \\ 0 & \upsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \upsilon_1 & 0 & 0 \\ 0 & 0 & \upsilon_2 & 0 & 0 \\ 0 & 0 & 0 & \upsilon_1 & \upsilon_2 \end{pmatrix}.$$

Consequently we can consider the principal bundle $P_{14} = M \times G_9$ with coframe $\omega^1, \ldots, \omega^5, \pi^1, \ldots, \pi^7, \chi^1, \chi^2$. Cartan's first formulas (8.6) will still hold and be invariant under the equivalences G_9. Moreover the following set of invariant structure equations, called *Cartan's second formulas*, is revealed [10, page 151, equation (8)]

$$\begin{aligned}
d\pi^1 &= \omega^1 \wedge \left(\chi^1 + 2B_2\,\omega^3 + 2A_2\,\omega^4 + 2A_3\,\omega^5\right) + \omega^2 \wedge \left(B_3\,\omega^3 + A_3\,\omega^4 + A_4\,\omega^5\right) \\
&\quad + \frac{1}{3}\,\omega^3 \wedge \pi^7 - \frac{2}{3}\,\omega^4 \wedge \pi^5 + \frac{1}{3}\,\omega^5 \wedge \pi^6 - \pi^2 \wedge \pi^3, \\
d\pi^2 &= \omega^1 \wedge \chi^2 + \omega^2 \wedge \left(B_4\,\omega^3 + A_4\,\omega^4 + A_5\,\omega^5\right) - \omega^4 \wedge \pi^6 + \pi^2 \wedge \left(\pi^1 - \pi^4\right), \\
d\pi^3 &= -\omega^1 \wedge \left(B_1\,\omega^3 + A_1\,\omega^4 + A_2\,\omega^5\right) + \omega^2 \wedge \chi^1 - \omega^5 \wedge \pi^5 - \pi^3 \wedge \left(\pi^1 - \pi^4\right), \\
d\pi^4 &= -\omega^1 \wedge \left(B_2\,\omega^3 + A_2\,\omega^4 + A_3\,\omega^5\right) + \omega^2 \wedge \left(\chi^2 - 2B_3\,\omega^3 - 2A_3\,\omega^4 - 2A_4\,\omega^5\right) \\
&\quad + \frac{1}{3}\,\omega^3 \wedge \pi^7 + \frac{1}{3}\,\omega^4 \wedge \pi^5 - \frac{2}{3}\,\omega^5 \wedge \pi^6 + \pi^2 \wedge \pi^3, \\
d\pi^5 &= \omega^1 \wedge \left(\frac{9}{32}D_1\,\omega^2 + \frac{9}{8}C_1\,\omega^3 + \frac{3}{4}B_1\,\omega^4 + \frac{3}{4}B_2\,\omega^5\right) + \omega^3 \wedge \left(\chi^1 + A_2\,\omega^4 + A_3\,\omega^5\right) \\
&\quad + \omega^2 \wedge \left(\frac{9}{8}C_2\,\omega^3 + \frac{3}{4}B_2\,\omega^4 + \frac{3}{4}B_3\,\omega^5\right) - \omega^5 \wedge \pi^7 + \pi^1 \wedge \pi^5 + \pi^3 \wedge \pi^6, \\
d\pi^6 &= \omega^1 \wedge \left(\frac{9}{32}D_2\,\omega^2 + \frac{9}{8}C_2\,\omega^3 + \frac{3}{4}B_2\,\omega^4 + \frac{3}{4}B_3\,\omega^5\right) + \omega^3 \wedge \left(\chi^2 - A_3\,\omega^4 - A_4\,\omega^5\right) \\
&\quad + \omega^2 \wedge \left(\frac{9}{8}C_3\,\omega^3 + \frac{3}{4}B_3\,\omega^4 + \frac{3}{4}B_2\,\omega^5\right) + \omega^4 \wedge \pi^7 + \pi^2 \wedge \pi^5 + \pi^4 \wedge \pi^6, \\
d\pi^7 &= \omega^1 \wedge \left(\frac{9}{64}E\,\omega^2 - \frac{3}{8}D_1\,\omega^3\right) - \frac{3}{8}D_2\,\omega^2 \wedge \omega^3 - \omega^3 \wedge \left(B_2\,\omega^4 - B_3\,\omega^5\right) \\
&\quad + \omega^4 \wedge \left(\chi^1 + 2A_3\,\omega^5\right) + \omega^5 \wedge \chi^2 + \left(\pi^1 + \pi^4\right) \wedge \pi^7 + \frac{4}{3}\,\pi^5 \wedge \pi^6.
\end{aligned} \tag{8.7}$$

where $A_1, \ldots, A_5,\ B_1, \ldots, B_4,\ C_1, C_2, C_3,\ D_1, D_2$ and E are functions on P_{14}. Using the notation $\alpha^{[i+1]} = \alpha^{[1]} \odot \alpha^{[i]} = \alpha \odot \alpha^{[i]}$ for symmetric tensor product of forms, the *symmetric* $(0,4)$-*tensors*

$$\mathcal{F}_I\left(\omega^4, \omega^5\right) = A_1 {\omega^4}^{[4]} + 4A_2 {\omega^4}^{[3]} \odot \omega^5 + 6A_3 {\omega^4}^{[2]} \odot {\omega^5}^{[2]} + 4A_4 \omega^4 \odot {\omega^5}^{[3]} + A_5 {\omega^5}^{[4]}$$

and

$$\begin{aligned}
\mathcal{G}_I\left(\omega^3, \omega^4, \omega^5\right) = \mathcal{F}_I\left(\omega^4, \omega^5\right) & \\
& + 4\left(B_1 {\omega^4}^{[3]} + 3B_2 {\omega^4}^{[2]} \odot \omega^5 + 3B_3 \omega^4 \odot {\omega^5}^{[2]} + B_4 {\omega^5}^{[3]}\right) \odot \omega^3 \\
& + 6\left(C_1 {\omega^4}^{[2]} + 2C_2 \omega^4 \odot \omega^5 + C_3 {\omega^5}^{[2]}\right) \odot {\omega^3}^{[2]} \\
& + 4\left(D_1 \omega^4 + D_2 \omega^5\right) \odot {\omega^3}^{[3]} \\
& + E {\omega^3}^{[4]}
\end{aligned}$$

are respectively called Cartan 2-tensor and Cartan 3-tensor associated to the general rank-3 Pfaffian system in five variables I. They are relative invariant with respect to G_9, in the following sense. Consider the two homogeneous polynomials, which we call ***Cartan tensors***,

$$\mathcal{F}_I = A_1 {x_1}^4 + 4A_2 {x_1}^3 x_2 + 6A_3 {x_1}^2 {x_2}^2 + 4A_4 x_1 {x_2}^3 + A_5 {x_2}^4,$$

and

$$\begin{aligned}
\mathcal{G}_I = \mathcal{F}_I &+ 4\left(B_1 {x_1}^3 + 3B_2 {x_1}^2 x_2 + 3B_3 x_1 {x_2}^2 + B_4 {x_2}^3\right) x_3 \\
&+ 6\left(C_1 {x_1}^2 + 2C_2 x_1 x_2 + C_3 {x_2}^2\right) {x_3}^2 + 4\left(D_1 x_1 + D_2 x_2\right) {x_3}^3 + E {x_3}^4.
\end{aligned}$$

The linear factors of $\mathcal{F}_I$ can be called roots and we can call ***root type*** of these polynomials their factorization by means of roots. For instance, if K_i is a linear expression in x_1 and x_2, then $\mathcal{F}_I$ must be of one of the following types:

[∞] infinitely many roots, $\mathcal{F}_I = 0$;

[**4**] one root of multiplicity four, $\mathcal{F}_I = {K_1}^4$;

[**3,1**] one triple root and one simple root, $\mathcal{F}_I = {K_1}^3 K_2$;

[**2,2**] two double roots, $\mathcal{F}_I = {K_1}^2 {K_2}^2$;

[**2,1,1**] one double root and two simple roots, $\mathcal{F}_I = {K_1}^2 K_2 K_3$;

[**1,1,1,1**] four simple roots, $\mathcal{F}_I = K_1 K_2 K_3 K_4$.

If two GR_3D_5 Pfaffian systems I and J are equivalent then their Cartan tensors $\mathcal{F}_I$ and $\mathcal{F}_J$ have the same type of roots. However, we must notice that if $\mathcal{F}_I$ and $\mathcal{F}_J$ have the same root type, I and J need not to be equivalent (unless $\mathcal{F}_I \equiv 0$). The root type of a GR_3D_5 Pfaffian system I is defined as the root type of its Cartan 2-tensor $\mathcal{F}_I$.

The root type [**3,1**] and [**2,1,1**] were treated simultaneously by Cartan [10, §VII], while Stormak [38] does not point out the type [**2,1,1**]. For the first time in the literature we provided, in Chapter 6, examples of GR_3D_5 Pfaffian systems for all the root types.

We implemented the above procedure in Maple language with commands defined in the *DifferentialGeometry* package. In this way we built a program, called *FiveVariables*. Using FiveVariables we can compute the Cartan tensors and the root type of any GR_3D_5 Pfaffian system. Moreover, our program can handle even nonlinear involutive systems or general Goursat equations, when realized as in Section 4.3. Using FiveVariables we can determine if two given nonlinear involutive systems (or two general Goursat equations) are not equivalent. The examples in Chapter 6 were determined using our software.

A detailed guide for FiveVariables is the subject of a future work. We note that this software handles GR_3D_5 Pfaffian systems and their lifts to 6-manifold, that is, the Pfaffian systems generated by nonlinear involutive systems or the Pfaffian systems associated to general Goursat equations (see Section 4.3). Here we report the main routines defined in FiveVariables and their purpose.

When executing the Maple command

```
> with(FiveVariables);
```

the user will see the following routines

[*CartanTensor, ClassifyCartanTensor, CreateCartanTensor, FiveVariablesChecks, GeneralForm, HsiaoRequirement, ModuleApply, PointCartan3Tensor, PointCartanTensor, PolyCartanTensor, Prolongation, Reduction1, Reduction2*].

The main routine "ModuleApply" is executed by the command "FiveVariables". The arguments of this command are of various nature, depending on the goal of the user. For instance, one can start a computation from a previous one, using stored data, or one can start a new computation providing an adapted basis (see Definition 2.6.2) for the GR_3D_5 Pfaffian system at hand. Several options are available.

The command "GeneralForm" produces an adapted coframe satisfying properties **[i]** and **[ii]**.

Starting from an adapted coframe produced by "GeneralForm", the command "HsiaoRequirement" computes an adapted coframe and the auxiliary 1-forms c^i, d^i and e^i satisfying (8.2).

One can apply the equivalence method described above to this adapted coframe. The routine "Reduction1" reduces the 19-dimensional structure G_{19} to G_{12}.

With the routine "Reduction2", one obtains G_7.

Finally, calling "PointCartanTensor" one realizes G_9. Using the Cartan's formulas, the Cartan 2-tensor (default) or the Cartan 3-tensor are computed.

Implementing an algorithm from [35, Exercise 3.53, page 103], the command "ClassifyCartanTensor" returns the root type of the Cartan 2-tensor handed.

REFERENCES

[1] I. M. Anderson and M. Fels, "Exterior differential systems with symmetry." *Acta Appl. Math.*, vol. 87, no. 1-3, pp. 3–31.

[2] ——, "Symmetry reduction of Pfaffian systems and Bäcklund transformations." *preprint*.

[3] I. M. Anderson, M. E. Fels, and P. J. Vassiliou, "Superposition formulas for exterior differential systems." *Advances in Mathematics*, no. 221, pp. 1910–1963.

[4] I. M. Anderson and N. Kamran, "The cohomology of the variational bicomplex for parabolic second order equations in the plane (in French)." *C. R. Acad. Sci. Paris*, vol. 321, pp. 1213–1217.

[5] M. Biesecker, *Geometric Studies in Hyperbolic Systems in the Plane.* Ph.D. dissertation, Utah State University, 2004.

[6] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, revised second ed. Orlando: Academic Press, 2003.

[7] R. Bryant, S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior Differential Systems.* New York-Berlin: Springer-Verlag, 1991.

[8] R. Bryant and P. Griffiths, "Characteristic cohomology of differential systems (II). Conservation laws for a class of parabolic equations." *Duke Math. J.*, vol. 78, no. 3, pp. 531–676.

[9] R. Bryant, P. Griffiths, and L. Hsu, "Hyperbolic exterior differential systems and their conservation laws (I)." *Selecta Math. (N.S.)*, vol. 1, no. 1, pp. 21–112.

[10] É. Cartan, "Les systèmes de Pfaff à cinq variables et les équations aux dèrivèes partielles du second ordre." *Ann. Sci. École Norm. Sup. (3)*, pp. 109–192.

[11] ——, "Sur certaines expressions différentielles et le problème de Pfaff." *Ann. Sci. École Norm. Sup. (3)*, vol. 16, pp. 239–332.

[12] ——, "Sur la structure des groupes infinis de transformation." *Ann. Sci. École Norm. Sup. (3)*, vol. 21, pp. 153–206.

[13] ——, "Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes." *Bull. Soc. Math. France*, vol. 42, pp. 12–48.

[14] É. Cartan, J. Glazebrook, and R. Hermann, *Geometry of Reimannian Spaces*, ser. Lie Groups: history, frontiers and applications. Brookline, Massachusetts: Math Sci Press, 1951-1983, vol. 13.

[15] M. Fels, "Integrating scalar ordinary differential equations with symmetry revisited." *Found. Comput. Math.*, vol. 7, no. 4, pp. 417–454, 2007.

[16] R. B. Gardner, "Invariants of Pfaffian systems." *Trans. Amer. Math. Soc.*, vol. 126, no. 3, pp. 514–533, 1967.

[17] ——, "A differential geometric generalization of characteristics." *Comm. Pure Appl. Math.*, vol. 22, pp. 597–626, 1969.

[18] ——, *Differential Geometric Methods interfacing Control Theory*, ser. Progress in Mathematics. Birkhäuser, 1983, vol. 27.

[19] ——, *The Method of Equivalence and its Applications*, ser. CBMS-NSF Regional Conference in Applied Mathematics. Society for Industrial and Applied Mathematics, 1989, vol. 58.

[20] R. B. Gardner and N. Kamran, "Characteristics and the geometry of hyperbolic equations in the plane." *J. Differential Equations*, no. 104, pp. 60–116, 1993.

[21] R. B. Gardner and W. Shadwick, "The GS algorithm for exact linearization to Brunovsky normal form." *IEEE Trans. Automat. Control*, vol. 37, no. 2, pp. 224–230, 1992.

[22] A. Giaro, A. Kumpera, and C. Ruiz, "Sur la lecture correcte d'un rèsultat d'Élie Cartan." *C. R. Acad. Sci. Paris Sér. A-B*, vol. 287, no. 4, pp. A241–A244, 1978.

[23] E. Goursat, "Sur une class d'èquations aux dèrivèes partielles du secon ordre et sur la thèorie des intègrales intermèdiares." *Acta Math.*, vol. 19, no. 1, pp. 285–340, 1895.

[24] ——, *Lecons sur l'intègration des èquations aux dèrivèes partielles du secon ordre.* Paris: Hermann, 1896, vol. I.

[25] ——, *Lecons sur l'intègration des èquations aux dèrivèes partielles du secon ordre.* Paris: Hermann, 1898, vol. II.

[26] ——, *Lecons sur le problem de Pfaff.* Paris: Hermann, 1922.

[27] P. Griffiths and G. Jensen, *Differential Systems and Isometric Embeddings.* Princeton, New Jersey: Princeton University Press, 1987.

[28] N. J. Hicks, *Notes on Differential Geometry.* Princeton, New Jersey: Van Nostrand Company, 1965.

[29] S. H. C. Hsiao, "On Cartan-Sternberg's example of the reduction of a G-structure II-III." *Tamkang J. Math.*, vol. 11, no. 2, pp. 277–312, 1980.

[30] T. Ivey and J. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, ser. GSM. Providence: American Mathematical Society, 2003, vol. 61.

[31] M. Juráš, *Geometric Aspects of Second-Order Scalar Hyperbolic Partial Differential Equations in the Plane.* Ph.D. dissertation, Utah State University, 1997.

[32] M. Juráš and I. M. Anderson, "Generalized Laplace invariants and the method of Darboux." *Duke Math. J.*, vol. 89, no. 2, pp. 350–375, 1997.

[33] R. M. Murray, "Applications and extensions of Goursat normal form to control of nonlinear systems." *IEEE CNF on Decision and Control*, vol. 32, pp. 3425–3430, 1993.

[34] P. Olver, *Application of Lie Groups to Differential Equations*, 2nd ed., ser. GTM. New York - Berlin: Springer-Verlag, 1975, vol. 107.

[35] ——, *Equivalence, Invariants, and Symmetry.* Cambridge, United Kinkdom: Cambridge University Press, 1995.

[36] J. Patera and P. Winternitz, "Subalgebras of real 3- and 4-dimensional Lie algebras." *J. Math. Phys.*, vol. 18, no. 7, pp. 1449–1455, 1977.

[37] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, 2nd ed. Houston, Texas: Public or Perish Inc., 1979.

[38] O. Stormark, *Lie's Structural Approach to PDE Systems.* Cambridge, United Kingdom: Cambridge University Press, 2000.

[39] D. Tilbury and S. S. Sastry, "On Goursat normal forms, prolongations, and control systems." *IEEE CNF on Decision and Control*, vol. 33, pp. 1797–1802, 1994.

[40] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, corrected reprint of the 1971 edition ed., ser. GTM. New York-Berlin: Springer-Verlag, 1983, vol. 94.

[41] K. Yamaguchi, "Geometrization of Jet bundles." *Hokkaido Math. J.*, vol. 12, no. 1, pp. 27–40, 1983.

VITA

Francesco Strazzullo

(August 2009)

PERSONAL INFORMATION

Born in Napoli, Italy, on 05/15/1974.

DEGREES:

M.S. in Mathematics at University of Studies in Napoli "Federico II" (IT, 2000).

ADDITIONAL EDUCATION:

"*Research Program on Symmetries of PDE and computer algebra systems*" at Silesian University in Opava (CZ, 2002), Grant by the University of Salerno (IT).

AREAS OF INTEREST:

Geometry of Partial Differential Equations, Exterior Differential Systems, Differential Geometry, Lie Groups and Topology. MSC numbers: 53, 35, and 58.

TEACHING EXPERIENCES AT COLLEGE LEVEL:

Intermediate Algebra, Calculus Techniques, Calculus 1, Calculus 2, Linear Algebra and Differential Equations (at USU, USA, 2002 - 2009). *Group Theory, Linear Algebra, Real Analysis* (at CEPU, IT, 1998 - 2001).

PUBLICATIONS:

Darboux integrable hyperbolic PDE in the plane of generic Type: a classification by means of Cartan Tensor and Maple. International Conference SPT, World Scientific, 2007.

Jets: a software package for symbolic computations. Research Program Resume, University of Studies in Salerno, 2002.

Hopf's problem and its development. M.S. Thesis, University of Studies in Napoli "Federico II", 2001.