

4-2007

Gauging Newton's Law

James Thomas Wheeler
Utah State University

Follow this and additional works at: https://digitalcommons.usu.edu/physics_facpub

 Part of the [Physics Commons](#)

Recommended Citation

Wheeler, J. T., Gauging Newton's Law, *Canadian Journal of Physics*, vol. 85, issue 4 (2007), pp. 307-344. <http://arxiv.org/pdf/hep-th/0305017>.

This Article is brought to you for free and open access by the Physics at DigitalCommons@USU. It has been accepted for inclusion in All Physics Faculty Publications by an authorized administrator of DigitalCommons@USU. For more information, please contact dylan.burns@usu.edu.



Gauging Newton's Law

James T. Wheeler^{*†}

February 1, 2008

Abstract

We derive both Lagrangian and Hamiltonian mechanics as gauge theories of Newtonian mechanics. Systematic development of the distinct symmetries of dynamics and measurement suggest that gauge theory may be motivated as a reconciliation of dynamics with measurement. Applying this principle to Newton's law with the simplest measurement theory leads to Lagrangian mechanics, while use of conformal measurement theory leads to Hamilton's equations.

1 Introduction

Recent progress in field theory, when applied to classical physics, reveals a previously unknown unity between various treatments of mechanics. Historically, Newtonian mechanics, Lagrangian mechanics and Hamiltonian mechanics evolved as distinct formulations of the content of Newton's second law. Here we show that Lagrangian and Hamiltonian mechanics both arise as local gauge theories of Newton's second law.

While this might be expected of Lagrangian mechanics, which is, after all, just the locally coordinate invariant version of Newton's law, achieving Hamiltonian mechanics as a gauge theory is somewhat surprising. The reason it happens has to do with a new method of gauging scale invariance called *biconformal gauging*. The study of biconformal gauging of Newtonian mechanics serves a dual purpose. First, it sheds light on the meaning in

^{*}Department of Physics, Utah State University, Logan, Utah 84322

[†]Electronic mail: jwheeler@cc.usu.edu

field theory of biconformal gauging, which has already been shown to have symplectic structure and to lead to a satisfactory relativistic gravity theory. Second, we are now able to see a conceptually satisfying unification of Hamiltonian mechanics with its predecessors..

Beyond these reasons for the study, we find a hint of something deeper. The biconformal gauging of Newton’s law actually leads to something midway between configuration space and phase space – a 6-dimensional symplectic space in which motion is described by Hamilton’s equations, but which does not change size with increasing numbers of particles. While many of the mathematical properties of Hamiltonian dynamics continue to hold, we are offered a tantalizing glimpse of a new possibility – perhaps this 6-dimensional space is the proper “arena” for both classical and quantum physics. In our conclusion, we revisit this conjecture.

Although the present article does not dwell on relativistic biconformal spaces *per se*, we give a brief account of their history and properties. The story starts with conformal gauge theories, which are notable for certain pathologies: (1) the requirement for an invariant action in $2n$ dimensions to be of n^{th} order in the curvature and/or the requirement for auxiliary fields to write a linear action, and (2) the presence of unphysical size changes. The existence of a second way to gauge the conformal group was first explored by Ivanov and Niederle [1], who were led to an eight dimensional manifold by gauging the conformal group of a four dimensional spacetime. They restricted the dependence on the extra four dimensions to the minimum needed for consistency with conformal symmetry. Later, Wheeler [2], generalizing to arbitrary dimensions, n , defined the class of biconformal spaces as the result of the $2n$ -dim gauging without imposing constraints, showing it to have symplectic structure and admit torsion free spaces consistent with general relativity and electromagnetism. Wehner and Wheeler [3] went on to write the most general class of actions linear in the biconformal curvatures, eliminating problems (1) and (2) above, and showing that the resulting field equations lead to the Einstein field equations.

For the purposes of the classical gauge theory considered here, it is sufficient at the start to know that the full relativistic picture looks even better: general relativity arises in a natural way from an action principle linear in the biconformal curvatures. Unlike previous conformal gauge theories, the biconformal gauging may be formulated in a uniform way in any dimension, does not lead to unphysical size changes, and does not require auxiliary fields. For further details, see [2] and [3]. In the conclusion to this paper we comment

briefly on the phase-space-like interpretation of biconformal spaces.

In the next two sections, we make some observations regarding dynamical laws, measurement theory and symmetry, then describe the global $ISO(3)$ symmetry of Newton's second law and the global $SO(4, 1)$ symmetry of Newtonian measurement theory. In Sec. 4, we give two ways to make these different dynamical and measurement symmetries agree. After briefly describing our method of gauging in Sec. 5, we turn to the actual gauging of Newtonian mechanics. In Sec. 6 we show that the $ISO(3)$ gauge theory leads to Lagrangian mechanics, while in Sec. 7 we show that biconformal gauging of the $SO(4, 1)$ symmetry leads to Hamiltonian dynamics, including at the end the case of multiple particles. In the penultimate section, we discuss an important question of interpretation, checking that there are no unphysical size changes. Finally, we follow a brief summary with some observations about the possible relevance to biconformal spaces and quantum physics.

2 What constitutes a physical theory?

We focus now on two essential features of any physical theory: dynamics and measurement. Understanding the role played by each of these will lead us to a deeper understanding of symmetry and ultimately to some novel conjectures about the arena for physical theory.

To clarify the difference between dynamics and measurement, we first look at quantum theory where the dynamics and measurement theories are quite distinct from one another. Indeed, the dynamical law of quantum mechanics is the Schrödinger equation,

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This equation gives the time evolution of a state, ψ , but the state has no direct physical meaning. Given a state, we still require a norm or an inner product on states,

$$\langle\psi|\psi\rangle = \int_V \psi^*\psi d^3x$$

to produce anything measurable. In addition, auxiliary rules for interpretation are needed. Thus, the quantum norm above is interpreted as the probability of finding the particle characterized by the state ψ in the volume V . Additional rules govern measurement of the full range of dynamical variables.

Now we return to identify these elements of Newtonian mechanics. Newtonian mechanics is so closely tied to our intuitions about how things move that we don't usually separate dynamics and measurement as conceptually distinct. Still, now that we know what we are looking for it is not difficult. The dynamical law, of course, is Newton's second law:

$$F^i = m \frac{dv^i}{dt}$$

which describes the time evolution of a position vector of a particle. The measurement theory goes back to the Pythagorean theorem – it is based on the line element or vector length in Euclidean space:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \eta_{ij} dx^i dx^j \\ \mathbf{v} \cdot \mathbf{w} &= \eta_{ij} v^i w^j \end{aligned}$$

where

$$\eta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

is the Euclidean metric in Cartesian coordinates. It is metric structure that provides measurable numbers from the position vectors, forces and other elements related by the dynamical equation. As we shall see below, there are also further rules required to associate quantities computed from the dynamical laws with numbers measured in the laboratory.

Once we have both a dynamical law and a measurement theory, we can begin detailed exploration of the physical theory. Generally, this means analyzing the nature of different interactions and making predictions about the outcomes of experiments. For these two purposes – studying interactions and making predictions – the most important tool is symmetry. The use of symmetry for studying interactions follows from the techniques of gauge theory, in which a dynamical law with a global symmetry is modified to be consistent with a local symmetry of the same type. This procedure introduces new fields into the theory, and these new fields generally describe interactions. The use of symmetry for prediction relies on Noether's theorem, which guarantees a conserved quantity corresponding to any continuous symmetry. Once we have such a conserved quantity, we have an immediate

prediction: the conserved quantity will have the same value in the future that it has now.

These three properties – dynamics, measurement, and symmetry – play a role in every meaningful physical theory.

Once again, quantum theory provides a convenient example. Both the dynamical law and the measurement theory make certain multiples of the wave function equivalent. The dynamical law is linear, hence consistent with arbitrary multiples of solutions. However, because of the derivatives involved in the Schrödinger equation, these multiples must be global, $\psi \rightarrow A_0 e^{i\varphi_0} \psi$. In contrast to this, we easily see that the quantum norm is preserved by local multiples, as long as the multiple is a pure phase:

$$\psi \rightarrow e^{i\varphi(x)} \psi$$

Of course, $U(1)$ gauge theory and the usual normalization of the wave function provide one means of reconciling these differences in symmetry. Notice that resolving the differences is accomplished by modifying the symmetry of the dynamical equation to agree with that of the measurement theory, both by restriction (fixing A_0 to normalize ψ) and extension (modifying the dynamical law to be consistent with local $U(1)$ transformations).

Gauging the $U(1)$ phase symmetry plays an extremely important role. By the general procedure of gauging, we replace global symmetries by local ones, and at the same time replace the dynamical law by one consistent with the enlarged symmetry. Well-defined techniques are available for accomplishing the required change in the dynamical laws. When the gauging procedure is applied to the phase invariance of quantum field theory, the result is a theory that includes electromagnetism. Thus, the gauging procedure provides a way to systematically introduce interactions between particles – forces. In relativistic mechanics, gauging provides a successful theory of gravity – general relativity. As we shall see, gauging works well in the Newtonian case too. Although we will not look for new interactions in the Newtonian gauge theory (these include gravity in the relativistic version), we will see that gauging leads directly to Hamilton’s equations.

The symmetry of Newtonian mechanics is often taken to be the set of transformations relating inertial frames. We can arrive at this conclusion by asking what transformations leave the dynamical equation invariant. The answer is that Newton’s second law is invariant under any transformation of

the form

$$\begin{aligned}x^i &\rightarrow J^i_j x^j + v^i t + x_0^i \\F^i &\rightarrow J^i_j F^j\end{aligned}$$

where J^i_j is a constant, nondegenerate matrix and v^i and x_0^i are constant vectors. A shift in the time coordinate and time reversal are also allowed. However, not all of these are consistent with the measurement theory. If we ask which of the transformations above also preserve the Pythagorean norm, we must further restrict J^i_j to be orthogonal. Newtonian mechanics is thus invariant under

$$\begin{aligned}x^i &\rightarrow O^i_j x^j + v^i t + x_0^i \\F^i &\rightarrow O^i_j F^j \\t &\rightarrow t + t_0\end{aligned}$$

While this brief argument leads us to the set of orthogonal inertial frames, it is not systematic. Rather, as we shall see, this is a conservative estimate of the symmetries that are possible.

In the next sections, we treat the symmetries of Newtonian mechanics in a more systematic way. In preparation for this, recall that in the quantum phase example, we both restricted and extended the dynamical law to accommodate a symmetry of the measurement theory, but arriving at the inertial frames for the Newtonian example we only restricted the symmetry of the dynamical law. This raises a general question. When the dynamical law and measurement theory have different symmetries, what do we take as the symmetry of the theory? Clearly, we should demand that the dynamical equations and the measurement theory share a common set of symmetry transformations. If there is a mismatch, we have three choices:

1. Restrict the symmetry to those shared by both the dynamical laws and the measurement theory.
2. Generalize the measurement theory to one with the same symmetry as the dynamical law.
3. Generalize the dynamical equation to one with the same symmetry as the measurement theory.

We announce and apply the *Goldilocks Principle*: Since we recognize that symmetry sometimes plays an important predictive role in specifying possible interactions, option #1 is *too small*. It is unduly restrictive, and we may miss important physical content. By contrast, the symmetry of measurement is *too large* for option #2 to work – inner products generally admit a larger number of symmetries than dynamical equations. Option #3 is *just right*: there are general techniques for enlarging the symmetry of a dynamical equation to match that of a measurement theory. Indeed, this is precisely what happens in gauge theories. The extraordinary success of gauge theories may be because they extend the dynamical laws to agree with the maximal information permitted within a given measurement theory.

We will take the point of view that the largest possible symmetry is desirable, and will therefore always try to write the dynamical law in a way that respects the symmetry of our measurement theory. This leads to a novel gauging of Newton’s law. In the next section we look in detail at two symmetries of the second law: the usual Euclidean symmetry, $ISO(3)$, and the $SO(4,1)$ conformal symmetry of a modified version of Newton’s law. Each of these symmetries leads to an interesting gauge theory.

3 Two symmetries of classical mechanics

In this section we first find the symmetry of Newton’s second law, then find the symmetry of Newtonian measurement theory.

3.1 Symmetry of the dynamical equation

Newton’s second law

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt} \tag{1}$$

has several well-known symmetries. The point symmetries leaving eq.(1) invariant are derived in Appendix 1. The result is that two allowed coordinate systems must be related by a constant, inhomogeneous, general linear transformation, together with a shift (and possible time reversal) of t :

$$\tilde{x}^m = J_n^m x^n + v_0^m t + x_0^m \tag{2}$$

$$\tilde{t} = t + t_0 \tag{3}$$

$$\tilde{F}^m = J_n^m F^n \tag{4}$$

where J_n^m is any constant, non-degenerate matrix, v_0^m and x_0^m are arbitrary constant vectors, and t_0 is any real constant.

Notice that, setting $e^\lambda = |\det(J_n^m)|$, Newton's second law transforms covariantly with respect to rescaling of units. Both sides of the dynamical law transform by an overall factor of $e^{-2\lambda}$.

Eqs.(2-4) gives a 16-parameter family of transformations: nine for the independent components of the 3×3 matrix J , three for the boosts v_0^i , three more for the arbitrary translation, x_0^m , and a single time translation. The collection of all of these coordinate sets constitutes the maximal set of inertial systems. This gives us the symmetry of the dynamical law.

3.2 Symmetry of Newtonian measurement theory

Newtonian measurement theory begins with the Pythagorean theorem as embodied in the line element and corresponding vector product

$$ds^2 = dx^2 + dy^2 + dz^2 = \eta_{ij} dx^i dx^j \quad (5)$$

$$\mathbf{v} \cdot \mathbf{w} = \eta_{ij} v^i w^j \quad (6)$$

The line element is integrated to find lengths of curves, while the dot product lets us find components of vectors by projecting on a set of basis vectors. But to have a line element or an inner product is not enough to have a theory of measurement. We must be specific about how the numbers found from the inner product relate to numbers measured in the laboratory.

Suppose we wish to characterize the magnitude of a displacement vector, \mathbf{x} , separating two particles by using the Euclidean inner product,

$$\|\mathbf{x}\|^2 = \eta_{ij} x^i x^j \quad (7)$$

The result, to be meaningful, must still be expressed in some set of units, say, meters or centimeters. The fact that either meters or centimeters will do may be expressed by saying that we work with an equivalence class of metrics differing by a positive multiplier. Thus, if we write eq.(7) for the length of \mathbf{x} in meters, then to give the length in centimeters we must write

$$\|\mathbf{x}\|^2 = 10^4 \eta_{ij} x^i x^j$$

We regard these two metrics as equivalent, and indeed, all metrics of the form

$$g_{ij} = e^{2\lambda(x)} \eta_{ij}$$

The factor $e^{2\lambda(x)}$ is called a conformal factor; two metrics which differ by a conformal factor are conformally equivalent.

The symmetry group which preserves conformal equivalence classes of metrics is the global conformal group, locally isomorphic to $O(4, 1)$. The conformal group is comprised of the following transformations:

$$y^i = \begin{cases} O^i_j x^j & \text{Orthogonal transformation} \\ x^i + a^i & \text{Translation} \\ e^\lambda x^i & \text{Dilatation} \\ \frac{x^i + x^2 b^i}{1 + 2b \cdot x + b^2 x^2} & \text{Special conformal transformation} \end{cases}$$

The first three of these are familiar symmetries. We now discuss each of the conformal symmetries, and the relationship between the $SO(4, 1)$ symmetry of classical measurement theory and the $ISO(3)$ symmetry of the dynamical law.

3.3 Relationship between the dynamical and measurement symmetries

As expected, there are some simple relationships between the symmetries of Newton's second law and the symmetries of the Euclidean line element. Indeed, if we restrict to global conformal transformations, the first three – orthogonal transformations, translations, and dilatations – all are allowed transformations to new inertial frames. We only need to restrict the global general linear transformations J_n^m of eqs.(2) and (4) to orthogonal, O_n^m for these to agree, while the $v_0^m t + x_0^m$ part of eq.(2) is a parameterized global translation.

For global dilatations we see the invariance of Newton's second law simply because the units on both sides of the equation match:

$$\begin{aligned} [\mathbf{F}] &= \frac{kg \cdot m}{s^2} \\ [m\mathbf{a}] &= kg \cdot \frac{m}{s^2} \end{aligned}$$

The dilatation corresponds to $e^{-2\lambda} = |\det(J_n^m)|$ in eqs.(2) and (4). Notice that the conformal transformation of units considered here is completely different from the conformal transformations (or renormalization group transformations) often used in quantum field theory. The present transformations

are applied to *all* dimensionful fields, and it is impossible to imagine this simple symmetry broken. By contrast, in quantum field theory only certain parameters are renormalized and there is no necessity for dilatation invariance.

To make the effect of dilatations more transparent, we use fundamental constants to express their units as $(length)^k$ for some real number k called the *conformal weight*. This allows us to quickly compute the correct conformal factor. For example, force has weight $k = -2$ since we may write

$$\left[\frac{1}{\hbar c} \mathbf{F} \right] = \frac{1}{l^2}$$

The norm of this vector then transforms as

$$\left\| \frac{1}{\hbar c} \mathbf{F} \right\| \rightarrow e^{-2\lambda} \left\| \frac{1}{\hbar c} \mathbf{F} \right\|$$

With this understanding, we see that Newton's law, eq.(1), transforms covariantly under global dilatations. With

$$\begin{aligned} \left[\frac{1}{\hbar c} \mathbf{F} \right] &= \frac{1}{l^2} \\ \left[\frac{m\mathbf{v}}{\hbar} \right] &= \frac{1}{l} \\ \left[\frac{1}{c} \frac{d}{dt} \right] &= \frac{1}{l} \end{aligned}$$

the second law,

$$\left(\frac{1}{\hbar c} \right) \mathbf{F} = \frac{1}{c} \frac{d}{dt} \left(\frac{m\mathbf{v}}{\hbar} \right) \quad (8)$$

has units $(length)^{-2}$ throughout:

$$\left[\left(\frac{1}{\hbar c} \right) \mathbf{F} \right] = \left[\left(\frac{1}{c} \frac{d}{dt} \right) \left(\frac{m\mathbf{v}}{\hbar} \right) \right] = \frac{1}{l^2} \quad (9)$$

Under a global dilatation, we therefore have

$$e^{-2\lambda} \left(\frac{1}{\hbar c} \right) \mathbf{F} = e^{-\lambda} \frac{1}{c} \frac{d}{dt} \left(e^{-\lambda} \frac{m\mathbf{v}}{\hbar} \right) = e^{-2\lambda} \frac{1}{c} \frac{d}{dt} \left(\frac{m\mathbf{v}}{\hbar} \right) \quad (10)$$

Newton's law is therefore globally dilatation covariant, of conformal weight -2 .

The story is very different for special conformal transformations. These surprising looking transformations are translations in inverse coordinates. Defining the inverse to any coordinate x^i as

$$y^i = -\frac{x^i}{x^2}$$

we find the general form of a special conformal transformation by inverting, translating by an arbitrary, constant vector, $-b^i$, then inverting once more:

$$x^i \rightarrow -\frac{x^i}{x^2} \rightarrow -\frac{x^i}{x^2} - b^i \rightarrow q^i = \frac{x^i + x^2 b^i}{1 + 2b^i x_i + b^2 x^2}$$

The inverse is given by the same sequence of steps, with the opposite sign of b^i , applied to q^i , i.e.,

$$x^i = \frac{q^i - q^2 b^i}{1 - 2q^i b_i + q^2 b^2}$$

As we show in Appendix 2, the transformation has the required effect of transforming the metric according to

$$\eta_{ab} \rightarrow \left(1 - 2b^i x_i + b^2 x^2\right)^{-2} \eta_{ab} \quad (11)$$

and is therefore conformal. This time, however, the conformal factor is not the same at every point. These transformations are nonetheless global because the parameters b^i are constant – letting b^i be an arbitrary function of position would enormously enlarge the symmetry in a way that no longer returns a multiple of the metric.

In its usual form, Newton’s second law is *not* invariant under global special conformal transformations. The derivatives involved in the acceleration do not commute with the position dependent transformation:

$$e^{-2\lambda(b,x)} \left(\frac{1}{\hbar c}\right) \mathbf{F} \neq e^{-\lambda(x)} \frac{1}{c} \frac{d}{dt} \left(e^{-\lambda(b,x)} \frac{\partial x^i}{\partial q^j} \frac{m v^j}{\hbar} \right) \quad (12)$$

and the dynamical law is not invariant.

4 A consistent global symmetry for Newtonian mechanics

Before we can gauge “the” symmetry of Newtonian mechanics, we face the dilemma described in the second section: our measurement theory and our

dynamical equation have different symmetries. The usual procedure for Newtonian mechanics is to restrict to the intersection of the two symmetries, retaining only global translations and global orthogonal transformations, giving the inhomogeneous orthogonal group, $ISO(3)$. This group can then be gauged to allow local $SO(3)$ transformations. However, in keeping with our (Goldilocks) principal of maximal symmetry, and noting that the conformal symmetry of the measurement theory is larger than the Euclidean symmetry of the second law, we will rewrite the second law with global conformal symmetry, $O(4, 1)$. The global conformal symmetry may then be gauged to allow local $SO(3) \times R^+$ (homothetic) transformations. In subsequent sections we will carry out both of these gaugings.

Our goal in this section is to write a form of Newton's second law which is covariant with respect to global conformal transformations. To begin, we have the set of global transformations

$$\begin{aligned} y^i &= O^i{}_j x^j \\ y^i &= x^i + a^i \\ y^i &= e^\lambda x^i \\ y^i &= \frac{x^i + x^2 b^i}{1 + 2b \cdot x + b^2 x^2} = \beta^{-1} (x^i + x^2 b^i) \end{aligned}$$

As seen above, it is the derivatives that obstruct the full conformal symmetry (see eq.(12)). The first three transformations already commute with ordinary partial differentiation of tensors because they depend only on the constant parameters $O^i{}_j, a^i$ and λ . After a special conformal transformation, however, the velocity becomes a complicated function of position, and when we compute the acceleration,

$$a^i = \frac{dv^i}{dt} = \frac{\partial y^i}{\partial x^j} \frac{d^2 x^j}{dt^2} + v^k \frac{\partial}{\partial x^k} \left(\frac{\partial y^i}{\partial x^j} v^j \right)$$

the result is not only a terrible mess – it is a different terrible mess than what we get from the force (see Appendix 3). The problem is solved if we can find a new derivative operator that commutes with special conformal transformations.

The mass also poses an interesting problem. If we write the second law as

$$\mathbf{F} = \frac{d}{dt} (m\mathbf{v}) \tag{13}$$

we see that even “constant” scalars such as mass pick up position dependence and contribute unwanted terms when differentiated

$$\begin{aligned} m &\rightarrow e^{-\lambda(x)}m \\ \partial_i m &\rightarrow e^{-\lambda(x)}\partial_i m - e^{-\lambda(x)}m\partial_i\lambda \end{aligned}$$

We can correct this problem as well, with an appropriate covariant derivative.

To find the appropriate derivation, we consider scalars first, then vectors, with differentiation of higher rank tensors following by the Leibnitz rule. For nonzero conformal weight scalars we require

$$D_k s_{(n)} = \partial_k s_{(n)} + n s_{(n)} \Sigma_k$$

while for vectors we require a covariant derivative of the form,

$$D_k v_{(n)}^i = \partial_k v_{(n)}^i + v_{(n)}^j \Lambda_{jk}^i + n v_{(n)}^i \Sigma_k$$

Continuing first with the scalar case, we easily find the required transformation law for Σ_k . Transforming $s_{(n)}$ we demand covariance,

$$D'_k s'_{(n)} = \left(D_k s_{(n)} \right)'$$

where

$$D'_k s'_{(n)} = e^{-\lambda} \partial_k \left(e^{n\lambda} s_{(n)} \right) + n \left(e^{n\lambda} s_{(n)} \right) \Sigma'_k$$

and

$$\left(D_k s_{(n)} \right)' = e^{n'\lambda} \left(D_k s_{(n)} \right)$$

Since derivatives have conformal weight -1 , we expect that¹

$$n' = n - 1$$

Imposing the covariance condition,

$$\begin{aligned} e^{-\lambda} \partial_k \left(e^{n\lambda} s_{(n)} \right) + n \left(e^{n\lambda} s_{(n)} \right) \Sigma'_k &= e^{n'\lambda} \left(D_k s_{(n)} \right) \\ e^{-\lambda} \left(s_{(n)} n \partial_k \lambda + \partial_k s_{(n)} \right) + n s_{(n)} \Sigma'_k &= e^{-\lambda} \left(\partial_k s_{(n)} + n s_{(n)} \Sigma_k \right) \\ s_{(n)} n \partial_k \lambda + n e^{\lambda} s_{(n)} \Sigma'_k &= n s_{(n)} \Sigma_k \end{aligned}$$

¹In field theory, the coordinates and therefore the covariant derivative are usually taken to have zero weight, while dynamical fields and the metric carry the dimensional information. In Newtonian physics, however, the coordinate is also a dynamical variable, and must be assigned a weight.

or

$$\Sigma'_k = e^{-\lambda} (\Sigma_k - \partial_k \lambda)$$

Since we assume the usual form of Newton's law holds in some set of coordinates, Σ_k will be zero for these coordinate systems. Therefore, we can take Σ_k to be zero until we perform a special conformal transformation, when it becomes $-ne^{-\lambda}\partial_k\lambda$. Notice that since λ is constant for a dilatation, Σ_k remains zero if we simply change from furlongs to feet.

Since a special conformal transformation changes the metric from the flat metric η_{ij} to the conformal metric

$$g_{ij} = e^{2\lambda(x)}\eta_{ij} = \beta^{-2}\eta_{ij} \quad (14)$$

where

$$\beta = 1 + 2\mathbf{b} \cdot \mathbf{x} + b^2 x^2 \quad (15)$$

we need a connection consistent with a very limited set of coordinate transformations. This just leads to a highly restricted form of the usual metric compatible Christoffel connection. From eq.(14) we compute immediately,

$$\begin{aligned} \Lambda_{jk}^i &= \frac{1}{2}g^{im} (g_{mj,k} + g_{mk,j} - g_{jk,m}) \\ &= \eta^{im} (\eta_{mj}\lambda_{,k} + \eta_{mk}\lambda_{,j} - \eta_{jk}\lambda_{,m}) \end{aligned} \quad (16)$$

where

$$\lambda_{,k} = -\beta^{-1}\beta_{,k}$$

Notice that Λ_{jk}^i has conformal weight -1 , and vanishes whenever $b_i = 0$.

We can relate Σ_k directly to the special conformal connection Λ_{jk}^i . The trace of Λ_{jk}^i is

$$\Lambda_k \equiv \Lambda_{ik}^i = 3\lambda_{,k}$$

so that

$$\begin{aligned} \Sigma_k &= -\lambda_{,k} \\ &= -\frac{1}{3}\Lambda_k \end{aligned}$$

The full covariant derivative of, for example, a vector of conformal weight n , may therefore be written as

$$D_k v_{(n)}^i = \partial_k v_{(n)}^i + v_{(n)}^j \Lambda_{jk}^i - \frac{n}{3} v_{(n)}^i \Lambda_k \quad (17)$$

where Λ_{jk}^i is given by eq.(16).

4.1 Constant mass and conformal dynamics

Extending the symmetry of classical mechanics to include special conformal transformations introduces an unusual feature: even constants such as mass may appear to be position dependent. But we are now in a position to say what it means for a scalar to be constant. Since mass has conformal weight -1 , we demand

$$D_k m = \partial_k m + \frac{1}{3} \Lambda_k m = 0$$

That is, constant mass now means covariantly constant mass.

This equation is always integrable because Λ_k is curl-free,

$$\Lambda_{k,m} - \Lambda_{m,k} = 3(\lambda_{,km} - \lambda_{,mk}) = 0$$

Integrating,

$$m = m_0 e^{-\lambda}$$

Any set of masses, $\{m_{(1)}, m_{(2)}, \dots, m_{(N)}\}$, in which each element satisfies the same condition,

$$D_k m_{(i)} = 0, \quad i = 1, \dots, N$$

gives rise to an invariant spectrum of measurable mass ratios,

$$M_R = \left\{ \frac{m_1}{m_0}, \frac{m_2}{m_0}, \dots, \frac{m_N}{m_0} \right\}$$

since the conformal factor cancels out. Here we have chosen $\frac{\hbar}{m_0 c}$ as our unit of length.

We can also write Newton's second law in an invariant way. The force is a weight -2 vector. With the velocity transforming as a weight zero vector and the mass as a weight -1 scalar, the time derivative of the momentum now requires a covariant derivative,

$$\frac{D(mv^i)}{Dt} = \frac{d}{dt} (mv^i) + mv^j v^k \Lambda_{jk}^i + \frac{1}{3} mv^i v^k \Lambda_k$$

Then Newton's law is

$$F^i = \frac{D}{Dt} (mv^i)$$

To see how this extended dynamical law transforms, we check conformal weights. The velocity has the dimensionless form

$$\frac{1}{c} \frac{dx^i}{dt}$$

The covariant derivative reduces this by one, so the acceleration has conformal weight -1 . The mass also has weight -1 , while the force, as noted above, has weight -2 . Then we have:

$$\tilde{F}^i = \frac{D}{D\tilde{t}} (\tilde{m}\tilde{v}^i)$$

The first term in the covariant time derivative becomes

$$\begin{aligned} \frac{d}{d\tilde{t}} (\tilde{m}\tilde{v}^i) &= e^{-\lambda} \frac{d}{dt} \left(e^{-\lambda} m \frac{\partial y^i}{\partial x^j} v^j \right) \\ &= e^{-2\lambda} \frac{\partial y^i}{\partial x^j} \frac{d}{dt} (mv^j) + e^{-\lambda} mv^j \frac{dx^k}{dt} \frac{\partial}{\partial x^k} \left(e^{-\lambda} \frac{\partial y^i}{\partial x^j} \right) \end{aligned}$$

The final term on the right exactly cancels the inhomogeneous contributions from Λ_{jk}^i and Λ_k , leaving the same conformal factor and Jacobian that multiply the force:

$$e^{-2\lambda} \frac{\partial y^i}{\partial x^j} F^j = e^{-2\lambda} \frac{\partial y^i}{\partial x^j} \left(\frac{d(mv^j)}{dt} + mv^m v^k \Lambda_{mk}^j + \frac{1}{3} mv^j v^k \Lambda_k \right)$$

The conformal factor and Jacobian cancel, so if the globally conformally covariant Newton's equation holds in one conformal frame, it holds in all conformal frames.

The transformation to the conformally flat metric

$$g_{ij} = e^{2\lambda} \eta_{ij} = \beta^{-2} \eta_{ij}$$

does not leave the curvature tensor invariant. This only makes sense – just as we have an equivalence class of metrics, we require an equivalence class of curved spacetimes. The curvature for g_{ij} is computed in Appendix 4.

We now want to consider what happens when we gauge the symmetries associated with classical mechanics. In the next section, we outline some basics of gauge theory. Then in succeeding sections we consider two gauge theories associated with Newtonian mechanics. First, we gauge the Euclidean $ISO(3)$ invariance of $F^i = ma^i$, then the full $O(4, 1)$ conformal symmetry of $F^i = \frac{D}{Dt} (mv^i)$.

5 Gauge theory

Here we briefly outline the quotient group method of gauging a symmetry group. Suppose we have a Lie group, \mathcal{G} , with corresponding Lie algebra

$$[G_A, G_B] = c_{AB}{}^C G_C$$

Suppose further that \mathcal{G} has a subgroup \mathcal{H} , such that \mathcal{H} itself has no subgroup normal in \mathcal{G} . Then the quotient group \mathcal{G}/\mathcal{H} is a manifold with the symmetry \mathcal{H} acting independently at each point (technically, a fiber bundle). \mathcal{H} is now called the isotropy subgroup. The manifold inherits a connection from the original group, so we know how to take \mathcal{H} -covariant derivatives. We may then generalize both the manifold and the connection, to arrive at a class of manifolds with curvature, still having local \mathcal{H} symmetry. We consider here only the practical application of the method. Full mathematical details may be found, for example, in [4].

The generalization of the connection proceeds as follows. Rewriting the Lie algebra in the dual basis of 1-forms defined by

$$\langle G_A, \omega^B \rangle = \delta_A^B$$

we find the Maurer-Cartan equation for \mathcal{G} ,

$$d\omega^C = -\frac{1}{2}c_{AB}{}^C \omega^A \wedge \omega^B$$

This is fully equivalent to the Lie algebra above. We consider the quotient by \mathcal{H} . The result has the same appearance, except that now all of the connection 1-forms ω^A are regarded as linear combinations of a smaller set spanning the quotient. Thus, if the Lie algebra of \mathcal{H} has commutators

$$[H_a, H_b] = c_{ab}{}^c H_c$$

then the Lie algebra for \mathcal{G} may be written as

$$\begin{aligned} [G_\alpha, G_\beta] &= c_{\alpha\beta}{}^\rho G_\rho + c_{\alpha\beta}{}^a H_a \\ [G_\alpha, H_a] &= c_{\alpha a}{}^\rho G_\rho + c_{\alpha a}{}^b H_b \\ [H_a, H_b] &= c_{ab}{}^c H_c \end{aligned}$$

where α and a together span the full range of the indices A . Because \mathcal{H} contains no normal subgroup of \mathcal{G} , the constants $c_{\alpha a}{}^\rho$ are nonvanishing for some

α for all a . The Maurer-Cartan structure equations take the corresponding form

$$\mathbf{d}\omega^\rho = -\frac{1}{2}c_{\alpha\beta}{}^\rho\omega^\alpha \wedge \omega^\beta - \frac{1}{2}c_{\alpha a}{}^\rho\omega^\alpha \wedge \omega^a \quad (18)$$

$$\mathbf{d}\omega^a = -\frac{1}{2}c_{\alpha\beta}{}^a\omega^\alpha \wedge \omega^\beta - \frac{1}{2}c_{ab}{}^a\omega^\alpha \wedge \omega^b - \frac{1}{2}c_{bc}{}^a\omega^b \wedge \omega^c \quad (19)$$

and we regard the forms ω^a as linearly dependent on the ω^α ,

$$\omega^a = \omega_\alpha^a \omega^\alpha$$

The forms ω^α span the base manifold and the ω^a give an \mathcal{H} -symmetric connection.

Of particular interest for our formulation is the fact that eq.(18) gives rise to a covariant derivative. Because \mathcal{H} is a subgroup, $\mathbf{d}\omega^\rho$ contains no term quadratic in ω^a , and may therefore be used to write

$$0 = \mathbf{D}\omega^\rho \equiv \mathbf{d}\omega^\rho + \omega_\alpha{}^\rho \wedge \omega^\alpha$$

with

$$\omega_\alpha{}^\rho \equiv \frac{1}{2}c_{\alpha\beta}{}^\rho\omega^\beta - \frac{1}{2}c_{\alpha a}{}^\rho\omega^a$$

This expresses the covariant constancy of the basis. As we shall see in our $SO(3)$ gauging, this derivative of the orthonormal frames ω^ρ is not only covariant with respect to local \mathcal{H} transformations, but also leads directly to a covariant derivative with respect to general coordinate transformations when expressed in a coordinate basis. This is the reason that general relativity may be expressed as both a local Lorentz gauge theory and a generally coordinate invariant theory, and it is the reason that Lagrangian mechanics with its “generalized coordinates” may also be written as a local $SO(3)$ gauge theory.

Continuing with the general method, we introduce curvature two forms. These are required to be quadratic in the basis forms ω^α only,

$$\mathbf{R}^a = \frac{1}{2}R^a{}_{\alpha\beta}\omega^\alpha \wedge \omega^\beta$$

$$\mathbf{R}^\rho = \frac{1}{2}R^\rho{}_{\alpha\beta}\omega^\alpha \wedge \omega^\beta$$

Then the modified connection is found by solving

$$\begin{aligned} \mathbf{d}\omega^\rho &= -\frac{1}{2}c_{\alpha\beta}{}^\rho\omega^\alpha \wedge \omega^\beta - \frac{1}{2}c_{\alpha a}{}^\rho\omega^\alpha \wedge \omega^a + \mathbf{R}^\rho \\ \mathbf{d}\omega^a &= -\frac{1}{2}c_{\alpha\beta}{}^a\omega^\alpha \wedge \omega^\beta - \frac{1}{2}c_{ab}{}^a\omega^\alpha \wedge \omega^b - \frac{1}{2}c_{bc}{}^a\omega^b \wedge \omega^c + \mathbf{R}^a \end{aligned}$$

and we may use any manifold consistent with this local structure.

This technique is used, for example, as a gauge approach to general relativity by constructing a class of manifolds with local Lorentz structure. One takes the quotient of the Poincaré group by the Lorentz group. Our constructions of Newtonian theory will illustrate the method, although we will not generalize to curved spaces or different manifolds. As a result, the structure equations in the form of eqs.(18, 19) describe the geometry and symmetry of our gauged dynamical law.

6 A Euclidean gauge theory of Newtonian mechanics

We begin by gauging the usual restricted form of the second law, using the Euclidean group as the initial global symmetry. The familiar form (see Appendix 5) of the Lie algebra of the Euclidean group, $iso(3)$, is

$$\begin{aligned} [J_i, J_j] &= \varepsilon_{ij}{}^k J_k \\ [J_i, P_j] &= \varepsilon_{ij}{}^k P_k \\ [P_i, P_j] &= 0 \end{aligned}$$

Using the quotient group method, we choose $so(3)$ as the isotropy subgroup. Then introducing the Lie algebra valued 1-forms ω^i dual to J_i and \mathbf{e}^i dual to P_i we write the Maurer-Cartan structure equations

$$\begin{aligned} \mathbf{d}\omega^m &= -\frac{1}{2}c_{ij}{}^m \omega^i \omega^j = -\frac{1}{2}\varepsilon_{ij}{}^m \omega^i \omega^j \\ \mathbf{d}\mathbf{e}^m &= -c_{ij}{}^m \omega^i \mathbf{e}^j = -\varepsilon_{ij}{}^m \omega^i \mathbf{e}^j \end{aligned}$$

Defining

$$\omega^k = \frac{1}{2}\varepsilon^k{}_{mn} \omega^{mn}$$

these take a form familiar from general relativity,

$$\mathbf{d}\omega^{mn} = \omega^{mk} \omega_k{}^n \tag{20}$$

$$\mathbf{d}\mathbf{e}^m = \mathbf{e}^k \omega_k{}^m \tag{21}$$

with ω^{mn} the spin connection and \mathbf{e}^m the dreibein. These equations are equivalent to the commutation relations of the Lie algebra, with the Jacobi

identity following as the integrability condition $\mathbf{d}^2 = 0$, i.e.,

$$\begin{aligned} \mathbf{d}^2 \omega^{mn} &= \mathbf{d} \left(\omega^{mj} \omega_j{}^n \right) = \mathbf{d} \omega^{mj} \omega_j{}^n - \omega^{mj} \mathbf{d} \omega_j{}^n \\ &\equiv 0 \\ \mathbf{d}^2 \mathbf{e}^m &= \mathbf{d} \left(\omega^m{}_k \mathbf{e}^k \right) = \mathbf{d} \omega^m{}_k \mathbf{e}^k - \omega^m{}_k \mathbf{d} \mathbf{e}^k \\ &\equiv 0 \end{aligned}$$

Eqs.(20) and (21) define a connection on a three dimensional (flat) manifold spanned by the three 1-forms \mathbf{e}^m . We take ω^{mn} to be a linear combination of the \mathbf{e}^m . This completes the basic construction.

The equations admit an immediate solution because the spin connection, ω^{mn} is in involution. The 6-dimensional group manifold therefore admits coordinates y^i such that

$$\omega^{mn} = w^{mn}{}_{\alpha} \mathbf{d}y^{\alpha}$$

and there are submanifolds given by $y^m = \text{const}$. On these 3-dimensional submanifolds, $\omega^{mn} = 0$ and therefore

$$\mathbf{d} \mathbf{e}^m = \omega^m{}_k \mathbf{e}^k = 0$$

with solution

$$\mathbf{e}^m = \delta_{\alpha}^m \mathbf{d}x^{\alpha}$$

for an additional three coordinate functions x^{α} . This solution gives Cartesian coordinates on the $y^{\alpha} = \text{const}$. submanifolds. Identifying these manifolds as a copies of our Euclidean 3-space, we are now free to perform an arbitrary rotation at each point.

Performing such local rotations on orthonormal frames leads us to general coordinate systems. When we do this, the spin connection ω^{mn} takes the more general, pure gauge, form

$$\omega^{mn} = - \left(\mathbf{d} O^m{}_j \right) \bar{O}^{jn}$$

where $O^m{}_j(x)$ is a local orthogonal transformation and $\bar{O}^{jn}(x)$ its inverse. Then \mathbf{e}^i provides a general orthonormal frame field,

$$\mathbf{e}^i = e_{\alpha}{}^i \mathbf{d}x^{\alpha}$$

The coefficients $e_{\alpha}{}^i(x)$ may be determined once we know $O^m{}_j(x)$.

The second Maurer-Cartan equation gives us a covariant derivative as follows. Expand any 1-form in the orthonormal basis,

$$\mathbf{v} = v_i \mathbf{e}^i$$

Then we define the covariant exterior derivative via

$$\begin{aligned} (\mathbf{D}v_i) \mathbf{e}^i &= \mathbf{d}\mathbf{v} \\ &= \mathbf{d}(v_i \mathbf{e}^i) \\ &= \mathbf{d}v_i \mathbf{e}^i + v_i \mathbf{d}\mathbf{e}^i \\ &= (\mathbf{d}v_k - v_i \omega_k^i) \mathbf{e}^k \end{aligned}$$

Similar use of the product rule gives the covariant derivative of higher rank tensors. This local $SO(3)$ -covariant derivative of forms in an orthonormal basis is equivalent to a general coordinate covariant derivative when expressed in terms of a coordinate basis. We see this as follows.

Rewriting eq.(21) in the form

$$\mathbf{d}\mathbf{e}^i + \mathbf{e}^k \omega_k^i = 0$$

we expand in an arbitrary coordinate basis, to find

$$\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta (\partial_\alpha e_\beta^i + e_\alpha^k \omega_{k\beta}^i) = 0$$

The term in parentheses must therefore be symmetric:

$$\partial_\alpha e_\beta^i + e_\alpha^k \omega_{k\beta}^i \equiv \Gamma_{\alpha\beta}^i = \Gamma_{\beta\alpha}^i$$

Writing

$$\Gamma_{\beta\alpha}^i = e_\mu^i \Gamma_{\beta\alpha}^\mu$$

we define the covariant constancy of the basis coefficients,

$$D_\alpha e_\beta^i \equiv \partial_\alpha e_\beta^i + e_\alpha^k \omega_{k\beta}^i - e_\mu^i \Gamma_{\beta\alpha}^\mu = 0 \quad (22)$$

Eq.(22) relates the $SO(3)$ -covariant spin connection for orthonormal frames to the Christoffel connection for general coordinate transformations. Since the covariant derivative of the orthogonal metric $\eta = \text{diag}(1, 1, 1)$ is zero,

$$\begin{aligned} D_\alpha \eta_{ab} &= \partial_\alpha \eta_{ab} - \eta_{cb} \omega_a^c - \eta_{ac} \omega_b^c \\ &= -\eta_{cb} \omega_a^c - \eta_{ac} \omega_b^c \\ &= 0 \end{aligned}$$

where the last step follows by the antisymmetry of the $SO(3)$ connection, we have covariant constancy of the metric:

$$D_\alpha g_{\mu\nu} = D_\alpha (\eta_{ab} e_\mu^a e_\nu^b) = 0$$

This is inverted in the usual way to give the Christoffel connection for $SO(3)$,

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \quad (23)$$

The Christoffel connection may also be found directly from eq.(22) using $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$. There is little practical difference between the ability to perform local rotations on an orthonormal frame field, and the ability to perform arbitrary transformations of coordinates. It is just a matter of putting the emphasis on the coordinates or on the basis vectors (see [5]). It is this equivalence that makes the $SO(3)$ gauge theory equivalent to the use of “generalized coordinates” in Lagrangian mechanics.

Since Newtonian 3-space is Euclidean and we have not generalized to curved spaces, the metric is always just a diffeomorphism away from orthonormal, that is,

$$\begin{aligned} e_\alpha^a &= J_\alpha^a = \frac{\partial y^a}{\partial x^\alpha} \\ g_{\alpha\beta} &= \eta_{ab} e_\alpha^a e_\beta^b = \eta_{ab} \frac{\partial y^a}{\partial x^\alpha} \frac{\partial y^b}{\partial x^\beta} \end{aligned} \quad (24)$$

and the connection takes the simple form

$$\Gamma_{\mu\nu}^\alpha = - \frac{\partial x^\alpha}{\partial y^a} \frac{\partial^2 y^a}{\partial x^\mu \partial x^\nu} \quad (25)$$

which has, of course, vanishing curvature. Notice that $(\mathbf{e}^a, \omega^a_c)$ or equivalently, $(g_{\alpha\beta}, \Gamma_{\mu\nu}^\alpha)$, here describe a much larger class of coordinate transformations than the global conformal connection Λ_{jk}^i of Sec. 4. The connection of eq.(25) gives a derivative which is covariant for *any* coordinate transformation.

This completes our description of Euclidean 3-space in general $SO(3)$ frames or general coordinates. We now generalize Newton’s second law to be consistent with the enhanced symmetry.

6.1 Generally covariant form of Newton's law

The generalization of Newton's second law to a locally $SO(3)$ covariant form of mechanics is now immediate. We need only replace the time derivative by a directional covariant derivative,

$$F^i = v^k D_k (m v^i)$$

where

$$D_k v^i \equiv \partial_k v^i + v^j \Gamma_{jk}^i \quad (26)$$

and Γ_{jk}^i is given by eq.(25). This is the principal result of the $SO(3)$ gauging.

If F^i is curl free, then it may be written as the contravariant form of the gradient of a potential

$$F^i = -g^{ij} \frac{\partial \phi}{\partial x^j}$$

The covariant form of Newton's law then follows as the extremum of the action

$$S = \int dt \left(\frac{m}{2} g_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt} - \phi \right)$$

The integrand is identical to the usual Lagrangian for classical mechanics, so the $ISO(3)$ gauge theory has led us to Lagrangian mechanics. Indeed, it is straightforward to check that the terms of the Euler-Lagrange equation involving the kinetic energy

$$T = \frac{m}{2} g_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt}$$

where g_{mn} is given by eq.(24), combine to give precisely the covariant acceleration:

$$\frac{d}{dt} \frac{\partial T}{\partial v^i} - \frac{\partial T}{\partial x^i} = m v^m D_m v^i \quad (27)$$

with $D_m v^i$ as in eq.(26) and Γ_{jk}^i as in eq.(23). This equivalence is the central result of this section. This result is expected, since Lagrangian mechanics was formulated in order to allow "generalized coordinates", i.e., general coordinate transformations, but the equivalence is rarely pointed out.

We treat multiple particles in the usual way. Suppose we have Newton's second law for each of N variables. Then we have the system

$$F_A^i = m_A a_A^i$$

for $A = 1, 2, \dots, N$. The symmetry, however, applies in the same way to each particle's coordinates, so the symmetry of the full system is still described by $ISO(3)$. When we gauge $ISO(3)$ we therefore still have a coordinate-invariant formulation of Euclidean 3-space with connection and metric as given in eqs.(24) and (25). The only change comes in the action, which we now write as a sum over all particles:

$$S = \sum_A \int dt \left(\frac{m_A}{2} g_{mn}(x_A) \frac{dx_A^m}{dt} \frac{dx_A^n}{dt} - \phi(x_A) \right)$$

Notice how the metric suffices for all N particles because it is a function of position. Each term in the sum causes the metric to be evaluated at a different point.

We have therefore shown that the locally $SO(3)$ -covariant gauge theory of Newton's second law is Lagrangian mechanics.

We now repeat the procedure for the full conformal symmetry associated with Newtonian measurement theory.

7 A conformal gauge theory of Newtonian mechanics

Now we gauge the full $O(4, 1)$ symmetry of our globally conformal form of Newton's law. The Lie algebra of the conformal group (see Appendix 5) is:

$$\begin{aligned} [M^a{}_b, M^c{}_d] &= \delta_b^c M^a{}_d - \eta^{ca} \eta_{be} M^e{}_d - \eta_{bd} M^{ac} + \delta_d^a M_b{}^c \\ [M^a{}_b, P_c] &= \eta_{bc} \eta^{ae} P_e - \delta_c^a P_b \\ [M^a{}_b, K^c] &= \delta_b^c K^a - \eta^{ca} \eta_{be} K^e \\ [P_b, K_d] &= -\eta_{be} M^e{}_d - \eta_{bd} D \\ [D, P_a] &= -P_a \\ [D, K^a] &= K^a \end{aligned} \tag{28}$$

where $M^a{}_b, P_a, K_a$ and D generate rotations, translations, special conformal transformations and dilatations, respectively.

As before, we write the Lie algebra in terms of the dual basis of 1-forms, setting

$$\langle M^a{}_b, \omega^c{}_d \rangle = \delta_b^c \delta_d^a - \eta^{ca} \eta_{be}$$

$$\begin{aligned}
\langle P_b, \mathbf{e}^a \rangle &= \delta_b^a \\
\langle K^a, \mathbf{f}_b \rangle &= \delta_b^a \\
\langle D, \mathbf{W} \rangle &= 1
\end{aligned}$$

The Maurer-Cartan structure equations are therefore

$$d\omega^a{}_b = \omega^c{}_b \omega^a{}_c + \mathbf{f}_b \mathbf{e}^a - \eta^{ac} \eta_{bd} \mathbf{f}_c \mathbf{e}^d \quad (29)$$

$$d\mathbf{e}^a = \mathbf{e}^c \omega^a{}_c + \mathbf{W} \mathbf{e}^a \quad (30)$$

$$d\mathbf{f}_a = \omega^c{}_a \mathbf{f}_c + \mathbf{f}_a \mathbf{W} \quad (31)$$

$$d\mathbf{W} = \mathbf{e}^a \mathbf{f}_a \quad (32)$$

So far, these structure equations look the same regardless of how the group is gauged. However, there are different ways to proceed from here because there is more than one sensible subgroup. In principle, we may take the quotient of the conformal group by any subgroup, as long as that subgroup contains no normal subgroup of the conformal group. However, we certainly want the final result to permit local rotations and local dilatations. Looking at the Lie algebra, we see only three subgroups² satisfying this condition, namely, those generated by one of the following three sets of generators

$$\begin{aligned}
&\{M^a{}_b, P_a, D\} \\
&\{M^a{}_b, K_a, D\} \\
&\{M^a{}_b, D\}
\end{aligned}$$

The first two generate isomorphic subgroups, so there are really only two independent choices, $\{M^a{}_b, K_a, D\}$ and $\{M^a{}_b, D\}$. The most natural choice is the first because it results once again in a gauge theory of a 3-dim Euclidean space. However, it leads only to a conformally flat 3-geometry with no new features. The final possibility, $\{M^a{}_b, D\}$, is called biconformal gauging. It turns out to be interesting.

Therefore, we perform the biconformal gauging, choosing the homogeneous Weyl group generated by $\{M^a{}_b, D\}$ for the local symmetry. This means that the forms \mathbf{e}^a and \mathbf{f}_a are independent, spanning a 6-dimensional sub-manifold of the conformal group manifold.

²We consider only rotationally and dilatationally covariant subgroups, which restricts consideration to subsets of $\{M^a{}_b, P_a, K_a, D\}$ and not, for example, collections such as $\{P_2, K_2, D\}$.

The solution of the structure equations (see [2]), eqs.(29-32) may be put in the form:

$$\begin{aligned}
\omega^a{}_b &= (\delta_a^c \delta_b^d - \eta^{ac} \eta_{db}) y_c \mathbf{d}x^d \\
\mathbf{W} &= -y_a \mathbf{d}x^a \\
\mathbf{e}^a &= \mathbf{d}x^a \\
\mathbf{f}_a &= \mathbf{d}y_a - \left(y_a y_b - \frac{1}{2} y^2 \eta_{ab} \right) \mathbf{d}x^b
\end{aligned}$$

Notice that if we hold y_a constant, these forms are defined on a 3-dim space with orthonormal basis $\mathbf{e}^a = \mathbf{d}x^a$. Since this is also a coordinate basis, this subspace is Euclidean, and we identify it with the original configuration space.

We can see that $\mathbf{e}^a \mathbf{f}_a$ is a symplectic form because \mathbf{e}^a and \mathbf{f}_a are independent, making this 2-form non-degenerate, while the structure equation, eq.(32),

$$\mathbf{d}\mathbf{W} = \mathbf{e}^a \mathbf{f}_a \tag{33}$$

shows that $\mathbf{e}^a \mathbf{f}_a$ is closed, $\mathbf{d}(\mathbf{e}^a \mathbf{f}_a) = \mathbf{d}^2 \mathbf{W} = 0$. This is also evident from the solution, where

$$\begin{aligned}
\mathbf{e}^a \mathbf{f}_a &= \mathbf{d}x^a \left(\mathbf{d}y_a - \left(y_a y_b - \frac{1}{2} y^2 \eta_{ab} \right) \mathbf{d}x^b \right) \\
&= \mathbf{d}x^a \mathbf{d}y_a
\end{aligned}$$

is in canonical form. Because of this symplectic form we are justified in identifying the solution as a relative of phase space.

Since we are in some 6-dimensional space, we cannot simply write Newton's law as before. Moreover, with the interpretation as a relative of phase space, we do not expect physical paths to be geodesics. We start then with an action. Noting that the geometry contains a new one-form, the Weyl vector, it is reasonable to examine what paths are determined by its extremals. Therefore, we consider the action

$$S_0 = \int \mathbf{W}$$

We add a function to make it interesting (note that the relativistic case adds this function automatically). Then

$$S = \int (\mathbf{W} + f \mathbf{d}t)$$

$$\begin{aligned}
&= \int (-y_m dx^m + f dt) \\
&= - \int \left(y_m \frac{dx^m}{dt} - f \right) dt
\end{aligned}$$

Since the function now depends on six variables, x^m, y_m the variation gives

$$\begin{aligned}
0 &= \delta S \\
&= - \int \left(\delta y_m \frac{dx^m}{dt} + y_m \frac{d\delta x^m}{dt} - \frac{\partial f}{\partial x^m} \delta x^m - \frac{\partial f}{\partial y_m} \delta y_m \right) dt \\
&= - \int \left(\left(\frac{dx^m}{dt} - \frac{\partial f}{\partial y_m} \right) \delta y_m + \left(-\frac{dy_m}{dt} - \frac{\partial f}{\partial x^m} \right) \delta x^m \right) dt
\end{aligned}$$

so that

$$\begin{aligned}
\frac{dx^m}{dt} &= \frac{\partial f}{\partial y_m} \\
\frac{dy_m}{dt} &= -\frac{\partial f}{\partial x^m}
\end{aligned}$$

We recognize these immediately as Hamilton's equations, if we identify the function f with the Hamiltonian.

As expected, the symmetry of these equations includes local rotations and local dilatations, but in fact is larger since, as we know, local symplectic transformations preserve Hamilton's equations.

Hamilton's principal function follows by evaluating the action along the classical paths of motion. This function is well-defined because the Lagrangian one-form is curl free on these paths:

$$\begin{aligned}
\mathcal{L} &= y_m \mathbf{d}x^m - f \mathbf{d}t \\
\mathbf{d}\mathcal{L} &= \mathbf{d}y_m \wedge \mathbf{d}x^m - \mathbf{d}f \wedge \mathbf{d}t \\
&= \mathbf{d}y_m \wedge \mathbf{d}x^m - \frac{\partial f}{\partial y_m} \mathbf{d}y_m \wedge \mathbf{d}t - \frac{\partial f}{\partial x^m} \mathbf{d}x^m \wedge \mathbf{d}t \\
&= \left(\mathbf{d}y_m + \frac{\partial f}{\partial x^m} \mathbf{d}t \right) \wedge \left(\mathbf{d}x^m - \frac{\partial f}{\partial y_m} \mathbf{d}t \right) \\
&= 0
\end{aligned}$$

where the last step follows by imposing the equations of motion. Thus, the integral of

$$\mathcal{L} = y_m \mathbf{d}x^m - f \mathbf{d}t$$

along solutions to Hamilton's equations is independent of path, and

$$\mathcal{S}(x) = \int_{x_0}^x \mathcal{L}$$

unambiguously gives Hamilton's principal function.

7.1 Multiple particles

Multiple particles in the conformal gauge theory are handled in a way similar to the Newtonian case. This is an important difference from the usual treatment of Hamiltonian dynamics, where the size of the phase space depends on the number of particles. Here, there is a single conformal connection on the full conformal group manifold, and gauging still gives a 6-dimensional space with a symplectic form. The variational principal becomes a sum over all particles,

$$\begin{aligned} S &= \sum_{A=1}^N \int (\mathbf{W}(x_A^m, y_m^A) + f(x_A^m, y_m^A) dt) \\ &= - \sum_{A=1}^N \int \left(y_m^A \frac{dx_A^m}{dt} - f(x_A^m, y_m^A) dt \right) \end{aligned}$$

Variation with respect to all $6N$ particle coordinates then yields the usual set of Hamilton's equations. If we regard the Lagrangian 1-form as depending independently on all N particle positions,

$$\mathbf{L} = - \sum_{A=1}^N \left(y_m^A \frac{dx_A^m}{dt} - f(x_A^m, y_m^A) dt \right)$$

then $d\mathbf{L} = 0$ as before and Hamilton's principal function is again well-defined.

The situation here is to be contrasted with the usual N -particle phase space, which is of course $6N$ -dimensional. Instead we have a 6-dimensional symplectic space in which all N particles move. The idea is to regard bi-conformal spaces as fundamental in the same sense as configuration spaces, rather than derived from dynamics the way that phase spaces are. This means that in principle, dynamical systems could depend on position and momentum variables independently. Classical solutions, however, have been shown to separate neatly into a pair of 3-dimensional submanifolds with the usual properties of configuration space and momentum space. Similarly, relativistic solutions with curvature separate into a pair of 4-dim manifolds with

the properties of spacetime and energy-momentum space, with the Einstein equation holding on the spacetime submanifold. These observations suggest that the symplectic structure encountered in dynamical systems might actually have kinematic (i.e., symmetry based) origins.

8 Is size change measurable?

While we won't systematically introduce curvature, there is one important consequence of dilatational curvature that we must examine. A full examination of the field equations for curved biconformal space ([2],[3]) shows that the dilatational curvature is proportional (but not equal) to the curl of the Weyl vector. When this curvature is nonzero, the relative sizes of physical objects may change. Specifically, suppose two initially identical objects move along paths forming the boundary to a surface. If the integral of the dilatational curvature over that surface does not vanish the two objects will no longer have identical sizes. This result is inconsistent with macroscopic physics. However, we now show that the result never occurs classically. A similar result has been shown for Weyl geometries [6].

If we fix a gauge, the change in any length dimension, l , along any path, C , is given by the integral of the Weyl vector along that path:

$$\begin{aligned} dl &= lW_i dx^i \\ l &= l_0 \exp\left(\int_C W_i dx^i\right) \end{aligned}$$

It is this integral that we want to evaluate for the special case of classical paths. Notice that this factor is gauge dependent, but if we compare two lengths which follow different paths with common endpoints, the ratio of their lengths changes in a gauge independent way:

$$\frac{l_1}{l_2} = \frac{l_{10}}{l_{20}} \exp\left(\oint_{C_1-C_2} W_i dx^i\right)$$

This dilatation invariant result represents measurable relative size change.

We now show that such measurable size changes never occur classically. From the expression for the action we have

$$\int W_i dx^i = \int W_i \frac{dx^i}{dt} dt = S(x) - \int^x f dt$$

where we write the action as a function of x because we are evaluating only along classical paths, where S becomes Hamilton's principle function. This is precisely what we need. The right side of this expression is a function; its value is independent of the path of integration. Therefore, the integral of the Weyl vector along every classical path may be removed (all of them at once) by the gauge transformation

$$e^{-S(x)+\int f dt}$$

Then in the new gauge,

$$\begin{aligned} \int W'_i dx^i &= \int \left(W_i - \partial_i S(x) + \partial_i \int f dt \right) dx^i \\ &= \int W_i dx^i - S(x) + \int f dt \\ &= 0 \end{aligned}$$

regardless of the (classical) path of integration. Therefore, no classical objects ever display measurable length change.

9 Conclusions

We have shown the following

1. The $SO(3)$ gauge theory of Newton's second law is Lagrangian mechanics
2. The $SO(4,1)$ gauge theory of Newton's second law gives Hamilton's equations of motion on a fundamental 6-dim symplectic space.

These results provide a new unification of classical mechanics using the tools of gauge theory.

We note several further insights.

First, by identifying the symmetries of a theory's dynamical law from the symmetry of its measurement theory, we gain new insight into the meaning of gauge theory. Generally speaking, dynamical laws will have global symmetries while the inner products required for measurement will have local symmetries. Gauging may be viewed as enlarging the symmetry of the dynamical law to match the symmetry of measurement, thereby maintaining closer contact with what is, in fact, measurable.

Second, we strengthen our confidence and understanding of the interpretation of relativistic biconformal spaces as relatives of phase space. The fact that the same gauging applied to classical physics yields the well-known and powerful formalism of Hamiltonian dynamics suggests that the higher symmetry of biconformal gravity theories may in time lead to new insights or more powerful solution techniques.

Finally, it is possible that the difference between the 6-dimensional symplectic space of $SO(4,1)$ gauge theory and $2N$ -dimensional phase space of Hamiltonian dynamics represents a deep insight. Like Hamiltonian dynamics, quantum mechanics requires both position and momentum variables for its formulation – without both, the theory makes no sense. If we take this seriously, perhaps we should look closely at a higher dimensional symplectic space as the fundamental arena for physics. Rather than regarding $2N$ -dim phase space as a convenience for calculation, perhaps there is a 6-dim (or, relativistically, 8-dim) space upon which we move and make our measurements. If this conjecture is correct, it will be interesting to see the form taken by quantum mechanics or quantum field theory when formulated on a biconformal manifold.

The proof of Sec. 8 is encouraging in this regard, for not only do classical paths show no dilatation, but a converse statement holds as well: non-classical paths generically do show dilatation. Since quantum systems may be regarded as sampling all paths (as in a path integral), it may be possible to regard quantum non-integrability of phases as related to non-integrable size change. There is a good reason to think that this correspondence occurs: biconformal spaces have a metric structure (consistent with classical collisions [2]) which projects separately onto its position and momentum subspaces. However, while the configuration space metric and momentum space metric are necessarily identical, the biconformal projections have opposite signs. This difference is reconciled if we identify the biconformal coordinate y_k with i times the momentum. This identification does not alter the classical results, but it changes the dilatations to phase transformations when y_k is replaced by ip_k . If this is the case, then the evolution of sizes in biconformal spaces, when expressed in the usual classical variables, gives unitary evolution just as in quantum physics. The picture here is much like the familiar treatment of quantum systems as thermodynamic systems by replacing time by a complex temperature parameter, except it is now the energy-momentum vector that is replaced by a complex coordinate in a higher dimensional space. A full examination of these questions takes us too far afield to pursue here, but

they are under current investigation.

Acknowledgments

The author wishes to thank Stefan Hollands for a question prompting parts of this investigation, and Lara B. Anderson, Clayton Call, Charles Torre, and David Peak for interesting and useful discussions.

References

- [1] Ivanov, E. A. and Niederle, J., Gauge formulation of gravitation theories II. The special conformal case, *Phys. Rev.* **D25** (1982) 988.
- [2] Wheeler, James T., New conformal gauging and the electromagnetic theory of Weyl, *J. Math. Phys.* **39** (1998) 299.
- [3] Wehner, A. and James T. Wheeler, Conformal actions in any dimension, *Nucl. Phys B* 557 (1-2) (1999) 380.
- [4] Kobayashii, S. and K. Nomizu, *Foundations of Differential Geometry*, Wiley, New York, (1963).
- [5] Isham, C. J., *Modern Differential Geometry For Physicists*, World Scientific (1989), p.162.
- [6] Wheeler, James T., Quantum measurement and geometry, *Phys. Rev* **D41** (1990) 431.
- [7] Weinberg, S., *Gravitation and cosmology*, Wiley, 1972, p.28.

Appendices

Appendix 1: Point transformations of Newton's second law

Here we derive the point transformations leaving the second law invariant, assuming the force to transform as a vector.

Consider a general coordinate transformation in which we replace the Cartesian coordinates, x^i , as well as the time parameter, by

$$\begin{aligned} q^i &= q^i(\mathbf{x}, t) \\ \tau &= \tau(\mathbf{x}, t) \end{aligned}$$

We have four functions, each of four variables. These functions must be invertible, so we may also write

$$\begin{aligned} x^i &= x^i(\mathbf{q}, \tau) \\ t &= t(\mathbf{q}, \tau) \end{aligned}$$

The limitation on covariance comes from the acceleration. First, the velocity is given by

$$\begin{aligned} v^i &= \frac{dx^i(\mathbf{q}, \tau)}{dt} \\ &= \frac{d\tau}{dt} \left(\frac{\partial x^i}{\partial q^j} \frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau} \right) \end{aligned}$$

where we use the usual summation convention on repeated indices, e.g.,

$$\sum_{j=1}^3 \frac{\partial x^i}{\partial q^j} \frac{dq^j}{d\tau} = \frac{\partial x^i}{\partial q^j} \frac{dq^j}{d\tau}$$

The acceleration is

$$\begin{aligned} a^i &= \frac{dv^i(\mathbf{q}, \tau)}{dt} \\ &= \frac{d\tau}{dt} \frac{d}{d\tau} \left(\frac{d\tau}{dt} \left(\frac{\partial x^i}{\partial q^j} \frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau} \right) \right) \\ &= \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial x^i}{\partial q^j} \frac{d^2 q^j}{d\tau^2} + \frac{\partial x^i}{\partial \tau} \right) + \frac{d\tau}{dt} \frac{d^2 \tau}{dt^2} \left(\frac{\partial x^i}{\partial q^j} \frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau} \right) \\ &\quad + \left(\frac{d\tau}{dt} \right)^2 \frac{dq^k}{d\tau} \left(\frac{\partial^2 x^i}{\partial q^k \partial q^j} \frac{dq^j}{d\tau} + \frac{\partial^2 x^i}{\partial q^k \partial \tau} \right) \\ &\quad + \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial^2 x^i}{\partial \tau \partial q^j} \frac{dq^j}{d\tau} + \frac{\partial^2 x^i}{\partial \tau^2} \right) \end{aligned}$$

The first term is proportional to the acceleration of q^i , but the remaining terms are not. Since we assume that force is a vector, it changes according to:

$$F^i(\mathbf{x}, t) = \frac{\partial x^i}{\partial q^j} F^j(\mathbf{q}, \tau) \quad (34)$$

where $\frac{\partial x^i}{\partial q^j}$ is the Jacobian matrix of the coordinate transformation. Substituting into the equation of motion, we have

$$\begin{aligned} \frac{1}{m} \frac{\partial x^i}{\partial q^j} F^j(\mathbf{q}, \tau) &= \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial x^i}{\partial q^j} \frac{d^2 q^j}{d\tau^2} + \frac{\partial x^i}{\partial \tau} \right) \\ &+ \frac{d\tau}{dt} \frac{d^2 \tau}{dt^2} \left(\frac{\partial x^i}{\partial q^j} \frac{dq^j}{d\tau} + \frac{\partial x^i}{\partial \tau} \right) \\ &+ \left(\frac{d\tau}{dt} \right)^2 \frac{dq^k}{d\tau} \left(\frac{\partial^2 x^i}{\partial q^k \partial q^j} \frac{dq^j}{d\tau} + \frac{\partial^2 x^i}{\partial q^k \partial \tau} \right) \\ &+ \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial^2 x^i}{\partial \tau \partial q^j} \frac{dq^j}{d\tau} + \frac{\partial^2 x^i}{\partial \tau^2} \right) \end{aligned} \quad (35)$$

Multiplying by the inverse to the Jacobian matrix, $\left(\frac{\partial q^m}{\partial x^i} \right)$, eq.(35) becomes

$$\begin{aligned} \frac{1}{m} F^m(\mathbf{q}, \tau) &= \left(\frac{d\tau}{dt} \right)^2 \left(\frac{d^2 q^m}{d\tau^2} + \frac{\partial q^m}{\partial x^i} \frac{\partial x^i}{\partial \tau} \right) \\ &+ \frac{d\tau}{dt} \frac{d^2 \tau}{dt^2} \left(\frac{dq^m}{d\tau} + \frac{\partial q^m}{\partial x^i} \frac{\partial x^i}{\partial \tau} \right) \\ &+ \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial q^j} \frac{dq^k}{d\tau} \frac{dq^j}{d\tau} + \frac{dq^k}{d\tau} \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial \tau} \right) \\ &+ \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial \tau \partial q^j} \frac{dq^j}{d\tau} + \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial \tau^2} \right) \end{aligned} \quad (36)$$

Therefore, Newton's second law holds in the new coordinate system

$$F^m(\mathbf{q}, \tau) = m \frac{d^2 q^m}{d\tau^2}$$

if and only if:

$$1 = \left(\frac{d\tau}{dt} \right)^2 \quad (37)$$

$$0 = \frac{\partial q^m}{\partial x^i} \frac{\partial x^i}{\partial \tau} \quad (38)$$

$$\begin{aligned} 0 &= \frac{d\tau}{dt} \frac{d^2\tau}{dt^2} \left(\frac{dq^m}{d\tau} + \frac{\partial q^m}{\partial x^i} \frac{\partial x^i}{\partial \tau} \right) \\ &+ \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial q^j} \frac{dq^k}{d\tau} \frac{dq^j}{d\tau} + \frac{dq^k}{d\tau} \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial \tau} \right) \\ &+ \left(\frac{d\tau}{dt} \right)^2 \left(\frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial \tau \partial q^j} \frac{dq^j}{d\tau} + \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial \tau^2} \right) \end{aligned} \quad (39)$$

Therefore, we have

$$\tau = t + t_0$$

together with the possibility of time reversal,

$$\tau = -t + t_0$$

for the time parameter. Using the first two equations to simplify the third (including $\frac{d^2\tau}{dt^2} = 0$),

$$\begin{aligned} 0 &= \left(\frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial q^j} \right) \frac{dq^k}{d\tau} \frac{dq^j}{d\tau} \\ &+ 2 \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial \tau} \frac{dq^k}{d\tau} \\ &+ \frac{\partial^2 x^i}{\partial \tau^2} \frac{\partial q^m}{\partial x^i} \end{aligned} \quad (40)$$

Now, since the components of the velocity, $\frac{dq^k}{d\tau}$, are independent we get three equations,

$$0 = \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial q^j} \quad (41)$$

$$0 = \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial q^k \partial \tau} \quad (42)$$

$$0 = \frac{\partial q^m}{\partial x^i} \frac{\partial^2 x^i}{\partial \tau^2} \quad (43)$$

Since the Jacobian matrix is invertible, these reduce to

$$0 = \frac{\partial^2 x^i}{\partial q^k \partial q^j} \quad (44)$$

$$0 = \frac{\partial^2 x^i}{\partial q^k \partial \tau} \quad (45)$$

$$0 = \frac{\partial^2 x^i}{\partial \tau^2} \quad (46)$$

and integrating,

$$0 = \frac{\partial^2 x^i}{\partial \tau^2} \Rightarrow x^i = x_0^i(q^m) + v_0^i(q^m) \tau \quad (47)$$

$$0 = \frac{\partial^2 x^i}{\partial q^k \partial \tau} \Rightarrow 0 = \frac{\partial v_0^i}{\partial q^k} \Rightarrow v_0^i = \text{const.} \quad (48)$$

The remaining equation implies that the Jacobian matrix is constant,

$$\frac{\partial x^m}{\partial q^j} = \frac{\partial x_0^m}{\partial q^j} = J_j^m = \text{const.} \quad (49)$$

Integrating, the coordinates must be related by a constant, inhomogeneous, general linear transformation,

$$x^m = J_j^m q^j + v_0^i \tau + x_0^m \quad (50)$$

$$t = \tau + \tau_0 \quad (51)$$

together with a shift and possible time reversal of t .

We get a 16-parameter family of coordinate systems: nine for the independent components of the nondegenerate 3×3 matrix J , three for the boosts v_0^i , three more for the arbitrary translation, x_0^m , and a single time translation.

Notice that the transformation includes the possibility of an arbitrary scale factor, $e^{-2\lambda} = |\det(J^m_n)|$.

Appendix 2: Special conformal transformations

In three dimensions, there are ten independent transformations preserving the inner product (or the line element) up to an overall factor: three rotations, three translations, one dilatation and three special conformal transformations. The first six of these are well-known for leaving ds^2 invariant – they form the Euclidean group for 3-dimensional space (or, equivalently the inhomogeneous orthogonal group, $ISO(3)$). The single dilatation is a simple rescaling. In Cartesian coordinates it is just

$$x^i = e^\lambda y^i$$

where λ is any constant. The special conformal transformations are actually a second kind of translation, performed in inverse coordinates, given by:

$$q^i = \frac{x^i + x^2 b^i}{1 + 2b^i x_i + b^2 x^2}$$

The inverse is given by:

$$x^i = \frac{q^i - q^2 b^i}{1 - 2q^i b_i + q^2 b^2}$$

Here we prove directly that this transformation has the required effect of transforming the metric conformally, according to

$$\eta_{ab} \rightarrow \left(1 - 2b^i x_i + b^2 x^2\right)^{-2} \eta_{ab} \quad (52)$$

Under *any* coordinate transformation the metric changes according to

$$g_{ij}(q) = \eta_{mn} \frac{\partial x^m}{\partial q^i} \frac{\partial x^n}{\partial q^j}$$

For the particular case of the special conformal coordinate transformation we have

$$x^i = \frac{q^i - q^2 b^i}{1 - 2q \cdot b + q^2 b^2} = \frac{1}{\beta} (q^i - q^2 b^i)$$

and therefore

$$\frac{\partial x^m}{\partial q^i} = \frac{1}{\beta^2} \left(\beta \delta_i^m - 2\beta b^m q_i + 2b^2 q^2 b^m q_i + 2q^m b_i - 2b^2 q^m q_i - 2q^2 b^m b_i \right)$$

Substituting into the metric transformation we find

$$\begin{aligned}
g_{ij} &= \eta_{mn} \frac{\partial x^m}{\partial q^i} \frac{\partial x^n}{\partial q^j} \\
\beta^4 g_{ij} &= \beta^2 \eta_{ij} - 2\beta^2 b_i q_j + 2b^2 q^2 \beta b_i q_j + \beta 2q_j b_i - 4\beta (b \cdot q) b_i q_j \\
&\quad + 4b^2 q^2 (b \cdot q) b_i q_j - 4b^2 q^2 b_i q_j + 4\beta b^2 q^2 b_i q_j - 4b^2 q^2 q^2 b^2 b_i q_j \\
&\quad + 4b^2 q^2 (b \cdot q) b_i q_j + 2\beta q_i b_j - 2\beta^2 b_j q_i - 4\beta (b \cdot q) q_i b_j + 4q^2 \beta b^2 q_i b_j \\
&\quad + 2\beta b^2 q^2 b_j q_i + 4b^2 q^2 (b \cdot q) q_i b_j - 4q^2 b^2 q^2 b^2 q_i b_j - 4b^2 q^2 q_i b_j \\
&\quad + 4q^2 b^2 (b \cdot q) q_i b_j - 2b^2 \beta q_i q_j + 4\beta^2 b^2 q_i q_j - 4b^2 q^2 \beta b^2 q_i q_j \\
&\quad + 4b^2 \beta (b \cdot q) q_i q_j - 4\beta b^2 q^2 b^2 q_i q_j + 4b^2 q^2 b^2 q^2 b^2 q_i q_j \\
&\quad - 4b^2 b^2 q^2 (b \cdot q) q_i q_j - 2\beta b^2 q_j q_i + 4\beta b^2 (b \cdot q) q_i q_j \\
&\quad - 4b^2 q^2 b^2 (b \cdot q) q_i q_j + 4b^2 b^2 q^2 q_i q_j - 2q^2 \beta b_i b_j + 4q^2 b_i b_j \\
&\quad - 4q^2 (b \cdot q) b_i b_j - 2\beta q^2 b_j b_i - 4q^2 (b \cdot q) b_i b_j + 4q^2 q^2 b^2 b_i b_j
\end{aligned}$$

The terms proportional to $q_i q_j$ cancel identically, while the remaining terms coalesce into β factors which cancel, leaving simply

$$g_{ij} = \beta^{-2} \eta_{ij}$$

This is a direct proof that the given transformation is conformal.

It is possible to show that the ten transformations described here are the only conformal transformations in 3 dimensions.

Appendix 3: What is the velocity after a special conformal transformation?

Suppose a particle follows the path $\mathbf{x}(t)$ with velocity

$$\mathbf{v} = \frac{d\mathbf{x}(t)}{dt}$$

If we introduce new coordinates

$$\mathbf{y} = \frac{\mathbf{x} + x^2\mathbf{b}}{1 + 2\mathbf{x} \cdot \mathbf{b} + b^2x^2} = \beta^{-1}(\mathbf{x} + x^2\mathbf{b})$$

where $\beta = (1 + 2(\mathbf{x} \cdot \mathbf{b}) + b^2x^2)$. Then

$$\begin{aligned} \mathbf{y}(t) &= \frac{\mathbf{x}(t) + x^2(t)\mathbf{b}}{1 + 2\mathbf{x}(t) \cdot \mathbf{b} + b^2x(t)^2} \\ \mathbf{x}(t) &= \frac{\mathbf{y} - y^2\mathbf{b}}{1 - 2\mathbf{y} \cdot \mathbf{b} + b^2y^2} \end{aligned}$$

Then

$$\frac{\partial y^i}{\partial x^j} = \beta^{-1}(\delta_j^i + 2x_j b^i) - \beta^{-2}(x^i + x^2 b^i)(2b_j + 2b^2 x_j) \quad (53)$$

This is just as complicated as it seems. The velocity in the new coordinates is

$$\begin{aligned} \frac{dy^i}{dt} &= \beta^{-1}(\mathbf{v} + 2(\mathbf{x} \cdot \mathbf{v})\mathbf{b}) - \beta^{-2}(\mathbf{x} + x^2\mathbf{b})(2\mathbf{v} \cdot \mathbf{b} + 2b^2(\mathbf{x} \cdot \mathbf{v})) \\ &= v^j \left(\beta^{-1}(\delta_j^i + 2x_j b^i) - \beta^{-2}(x^i + x^2 b^i)(2b_j + 2b^2 x_j) \right) \end{aligned} \quad (54)$$

The explicit form is probably the basis for Weinberg's claim [7], that under conformal transformations "...the statement that a free particle moves at constant velocity [is] not an invariant statement..." This is clearly the case – if $\frac{dx^i}{dt} = v^i$ is constant, $\frac{dy^i}{dt}$ depends on position in a complicated way. However, we note that using eq.(53) we may rewrite eq.(54) in the usual form for the transformation of a vector.

$$\frac{\partial y^i}{\partial x^j} = \frac{\partial y^i}{\partial x^j} v^j$$

This is the reason we must introduce a derivative operator covariant with respect to special conformal transformations. The statement $v^k D_k v^i = 0$ is then a manifestly conformally covariant expression of constant velocity.

Appendix 4: The geometry of special conformal transformations

We have shown that

$$g_{ij} = \beta^{-2} \eta_{ij}$$

But notice that, if we perform such a transformation, the connection and curvature no longer vanish, but are instead given by

$$\begin{aligned} \mathbf{e}^a &= \beta^{-1} \mathbf{d}x^a \\ \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \omega_b^a \\ \mathbf{R}_b^a &= \mathbf{d}\omega_b^a - \omega_b^c \omega_c^a \end{aligned}$$

This system is simple to solve. From the second equation, we have

$$\omega_b^a = - \left(\beta_{,b} \mathbf{e}^a - \eta^{ac} \eta_{bd} \beta_{,c} \mathbf{e}^d \right)$$

Then substituting into the curvature,

$$\begin{aligned} \mathbf{R}_b^a &= \mathbf{d}\omega_b^a - \omega_b^c \omega_c^a \\ &= -\delta_d^a \beta_{,bc} \mathbf{e}^c \mathbf{e}^d + \eta^{ae} \eta_{bd} \beta_{,ec} \mathbf{e}^c \mathbf{e}^d + \delta_d^a \beta_{,c} \beta_{,b} \mathbf{e}^c \mathbf{e}^d - \eta^{ae} \eta_{bd} \beta_{,c} \beta_{,e} \mathbf{e}^c \mathbf{e}^d \\ &\quad - \delta_d^a \beta_{,b} \beta_{,c} \mathbf{e}^c \mathbf{e}^d + \delta_d^a \eta_{bc} \eta^{fe} \beta_{,f} \beta_{,e} \mathbf{e}^c \mathbf{e}^d - \eta_{bc} \eta^{af} \beta_{,d} \beta_{,f} \mathbf{e}^c \mathbf{e}^d \\ R_{bcd}^a &= \delta_c^a \beta_{,bd} - \delta_d^a \beta_{,bc} + \eta^{ae} \eta_{bd} \beta_{,ec} - \eta^{ae} \eta_{bc} \beta_{,ed} \\ &\quad + (\delta_d^a \eta_{bc} - \delta_c^a \eta_{bd}) \eta^{fe} \beta_{,f} \beta_{,e} \end{aligned}$$

which is pure Ricci. We knew this in advance because the Weyl curvature tensor vanishes for conformally flat metrics. The Ricci tensor and Ricci scalar are

$$\begin{aligned} R_{bd} &= (n-2) \beta_{,bd} + \eta_{bd} \eta^{ce} \beta_{,ec} - (n-1) \eta_{bd} \eta^{fe} \beta_{,f} \beta_{,e} \\ R &= 2(n-1) \eta^{bd} \beta_{,bd} - n(n-1) \eta^{fe} \beta_{,f} \beta_{,e} \end{aligned}$$

where

$$\begin{aligned} \partial_a \beta &= \partial_a (1 - 2x \cdot b + x^2 b^2) \\ &= -2b_a + 2b^2 x_a \\ \partial_{ab} \beta &= \partial_b (-2b_a + 2b^2 x_a) \\ &= 2b^2 \eta_{ab} \end{aligned}$$

so finally,

$$\begin{aligned} R_{bd} &= 4(n-1) b^2 (1 - \beta) \eta_{bd} \\ R &= 4n(n-1) b^2 (1 - \beta) \end{aligned}$$

The full curvature therefore is determined fully by the Ricci scalar:

$$\begin{aligned} R_{bcd}^a &= (\delta_c^a \eta_{bd} - \delta_d^a \eta_{bc}) 4b^2 (1 - \beta) \\ &= \frac{R}{n(n-1)} (\delta_c^a \eta_{bd} - \delta_d^a \eta_{bc}) \end{aligned}$$

where

$$R = 4n(n-1)b^2 (2x \cdot b - x^2 b^2)$$

Appendix 5: The $ISO(3)$ and conformal $SO(4,1)$ Lie algebras

For our gauging, we require the form of the Lie algebras $iso(3)$ and $so(4,1)$. We can find both from the general form of any pseudo-orthogonal Lie algebra. Let $\eta_{AB} = \text{diag}(1, \dots, 1, -1, \dots, -1)$ with p +1s and q -1s. Then the Lie algebra $o(p, q)$ is, up to normalization,

$$[M_{AB}, M_{CD}] = \eta_{BC}M_{AD} - \eta_{BD}M_{AC} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC}$$

The Lie algebra $iso(3)$ may be found as a contraction of $o(4)$ or $o(3,1)$. Let $\eta_{AB} = \text{diag}(1, 1, 1, 1)$, let $\eta_{ij} = \text{diag}(1, 1, 1)$, and replace M_{i4} and M_{4i} by

$$\lambda P_i = M_{i4} = -M_{4i}$$

Separating the $i = 1, 2, 3$ parts from the $i = 4$ parts,

$$\begin{aligned} [M_{ij}, M_{kl}] &= \eta_{jk}M_{il} - \eta_{jl}M_{ik} - \eta_{ik}M_{jl} + \eta_{il}M_{jk} \\ [M_{ij}, \lambda P_k] &= \eta_{jk}\lambda P_i - \eta_{ik}\lambda P_j \\ [\lambda P_i, \lambda P_k] &= -\eta_{44}M_{ik} \end{aligned}$$

so in the limit as $\lambda \rightarrow \infty$,

$$\begin{aligned} [M_{ij}, M_{kl}] &= \eta_{jk}M_{il} - \eta_{jl}M_{ik} - \eta_{ik}M_{jl} + \eta_{il}M_{jk} \\ [M_{ij}, P_k] &= \eta_{jk}P_i - \eta_{ik}P_j \\ [P_i, P_j] &= -\frac{1}{\lambda^2}M_{ik} \rightarrow 0 \end{aligned}$$

While these relations hold in any dimension, in 3-dim we can simplify the algebra using the Levi-Civita tensor to write

$$\begin{aligned} J_i &= -\frac{1}{2}\varepsilon_i{}^{jk}M_{jk} \\ M_{ij} &= -\varepsilon_{ij}{}^k J_k \end{aligned}$$

Then

$$\begin{aligned} [J_m, J_n] &= \varepsilon_{mn}{}^i J_i \\ [J_i, P_j] &= \varepsilon_{ij}{}^n P_n \\ [P_i, P_j] &= 0 \end{aligned}$$

The conformal Lie algebra is just $o(4,1)$, given by

$$[M_{AB}, M_{CD}] = \eta_{BC}M_{AD} - \eta_{BD}M_{AC} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC}$$

with $A, B, \dots = 1, 2, \dots, 5$. To see this algebra in terms of the usual definite-weight generators we write the metric in the form

$$\eta_{AB} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} = \eta^{AB}$$

The Lie algebra of generators $M^a{}_b, P_a, K^a$ and D is then found by setting

$$\begin{aligned} P_a &= M^4{}_a \\ K^a &= \eta^{ab} M^5{}_b \\ D &= M^4{}_4 = -M^5{}_5 \end{aligned}$$

With $a, b, \dots = 1, 2, 3$ we find

$$\begin{aligned} [M^a{}_b, M^c{}_d] &= \delta_b^c M^a{}_d - \eta^{ca} \eta_{be} M^e{}_d - \eta_{bd} M^{ac} + \delta_d^a M_b{}^c \\ [M^a{}_b, P_c] &= \eta_{bc} \eta^{ae} P_e - \delta_c^a P_b \\ [M^a{}_b, K^c] &= \delta_b^c K^a - \eta^{ca} \eta_{be} K^e \\ [P_b, K_d] &= -\eta_{be} M^e{}_d - \eta_{bd} D \\ [D, P_a] &= -P_a \\ [D, K^a] &= K^a \end{aligned} \tag{55}$$

This is the usual form of the conformal algebra. The matrices $M^a{}_b$ generate $SO(3)$, the three generators P_a lead to translations, K^a give translations of the point at infinity (special conformal transformations), and D generates dilatations.