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Conformal Actions in Any Dimension

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Abstract

Biconformal gauging of the conformal group gives a scale-invariant volume form, permitting a single form of the action to be invariant in any dimension. We display several $2n$-dim scale-invariant polynomial actions and a dual action. We solve the field equations for the most general action linear in the curvatures for a minimal torsion geometry. In any dimension $n > 2$, the solution is foliated by equivalent n-dim Ricci-flat Riemannian spacetimes, and the full $2n$-dim space is symplectic. Two fields defined entirely on the Riemannian submanifolds completely determine the solution: a metric e^a_μ, and a symmetric tensor $k_{\mu\nu}$.

1 Introduction

One of the problems facing the use of the conformal group as a fundamental spacetime symmetry in n dimensions is the highly restricted set of possible actions. In sharp contrast to general relativity, where the Einstein-Hilbert action is Lorentz and coordinate invariant in every dimension, conformal actions are typically coupled to the dimension. This coupling to dimension occurs because under a rescaling of the metric by a factor $e^{2\phi}$, the volume element of an n-dimensional spacetime scales by $e^{n\phi}$. Therefore, for example, since an action containing k copies of the scale-invariant conformal tensor,

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C^a_{bcd}, requires k inverse metrics (each scaling as $e^{-2\phi}$) to form a Lorentz scalar, we find that an expression such as

$$S = \int \sqrt{-g} C^a_{b\mu\nu} C^b_{c\alpha\beta} \cdots C^c_{a\rho\sigma} \; g^{\mu\alpha} g^{\nu\rho} \cdots g^{\beta\sigma} \; d^n x$$

is scale invariant only if $n - 2k = 0$.

A number of techniques have been developed to overcome this problem, most of them within the context of conformal gauge theory. Generally, these techniques either require additional "compensating" fields or fail to reproduce general relativity in any gauge. Here we show that because of its scale-invariant volume form, the biconformal gauging of the conformal group [1] resolves these problems, allowing us to write an invariant action linear in the curvature without compensating fields. We begin our discussion with a brief overview of some of the previous treatments of conformal gauging.

The gauging of the conformal group in four dimensions has been handled in much the same way as Poincaré gauging, simply treating the dilations and special conformal transformations as generators of additional symmetries. As recounted in [2], it was long believed that special conformal transformations were "ungaugeable" because the conformal matter current is explicitely x-dependent, so that coupling it to the special conformal gauge field would spoil translation invariance. Therefore, prior to 1977, conformal gauging incorporated only Lorentz transformations, dilations and, for the superconformal group SU(2,2—N), the internal U(N) symmetry algebras, i.e., Weyl's theory of gravity was regarded as the unique gauge theory of the conformal group. In order to remain as close as possible to Einstein's theory, Deser [3] coupled a massless Lorentz scalar field $\phi(x)$ (dilaton) of compensating conformal weight -1 to gravitation through the manifestly scale-invariant quantity $\frac{1}{6}\phi^2 R$. Later, Dirac [4], trying to accommodate the Large Numbers Hypothesis, similarly modified Weyl's free Lagrangian by replacing all R^2-type terms by $\phi^2 R$. This method gave rise to various theories involving the "generalized" Einstein equations [5]-[9]. They were shown to reduce to general relativity when expressed in a particular gauge [5]-[6].

In 1977, it was demonstrated by Crispim-Romao, Ferber and Freund [2] and independently by Kaku, Townsend and Van Nieuwenhuizen [10]-[11] that special conformal transformations can indeed be gauged. Using a Weyl-like conformally invariant 4-dimensional action quadratic in the conformal curvatures and the assumption of vanishing torsion, it is found that the gauge fields associated with special conformal transformations are algebraically removable. The action reduces to a scale-invariant, torsion-free Weyl theory of gravity based on the square of the conformal curvature. This auxiliary nature of the special conformal gauge field has been shown to follow for any 4-dimensional action quadratic in the curvatures [12]. Generically, the action reduces to the a linear combination of the square of the conformal curvature and the square of the curl of the Weyl vector.

Alternatively, the special conformal gauge fields may be removed by the curvature constraint [13]

$$R^a_{bac} = 0 \tag{1}$$

This ensures that $\mathbf{R}^a_b$ is just the Weyl conformal curvature tensor, rather than the usual Riemann curvature. Then the constraint of vanishing torsion,

$$\mathbf{T}^a = 0 \tag{2}$$

also renders the spin connection auxiliary. We will refer to conditions (1) and (2) as the conventional constraints. The dilation field (Weyl vector) drops out of the action completely, so instead of a Weyl unified theory one again obtains a 4-dimensional Weyl-trivial theory of gravity, gauge equivalent to a Riemannian geometry.

The quadratic curvature theory was later generalized to dimensions $n > 4$ by unifying it with the compensating field approach [14]. The action involves terms of the form

$$e_{[a}{}^{\mu} e_b{}^{\nu} e_c{}^{\alpha} e_{d]}{}^{\beta} \phi^{2(n-4)/(n-2)} R_{\mu\nu}{}^{ab} R_{\alpha\beta}{}^{cd}$$

While the resulting field equations no longer require the special conformal gauge fields to be removable, the conventional constraints may still be imposed to remove them. These constraints were shown to be conformally invariant if the conformal weight of ϕ is $-(n-2)/2$. Unfortunately, this class of theories is not equivalent to general relativity in any gauge. It is useful, though, in the understanding of superconformal gauge theories in $n = 6$ and 10 dimensions [29].

A different use of a compensating field proves somewhat more successful. In n dimensions, an action of the form $\phi \Box \phi$ is conformally invariant. Because the conformal d'Alembertian contains a term involving the trace of the special conformal gauge field, constraint (1) leads to a $\phi^2 R$ term in the action. Again imposing the conventional constraints, and gauging the Weyl vector to zero and the compensating field to a constant, we recover Einstein gravity in n dimensions.

For $n = 3$, the Chern-Simons form leads to an exactly soluble (super-) conformal gravity theory [15]-[17] characterized by conformal flatness.

In $n = 2$ dimensions, the conformal group is not a Lie group; it is generated by the infinite-dimensional Virasoro algebra [18]. The importance of 2-dimensional conformal field theory is well known as the symmetry of the 2-dimensional world sheet in string theory [19]. In addition, we note the recent surge of interest in conformal field theories due to the celebrated AdS/CFT duality conjecture put forward by Maldacena [20] and made more precise by others [21], which relates type IIB string theory and M theory in certain $(n+1)$-dimensional anti-de-Sitter spacetime backgrounds to conformally invariant field theories in n dimensions.

Recently, a new way of gauging the conformal group [1] has been proposed which resolves the problem of writing scale-invariant actions in arbitrary dimension *without* using compensating fields. In particular, we write the most general *linear* action and find that all minimal torsion solutions are foliated by equivalent n-dimensional Ricci-flat Riemannian spacetimes. Thus, the new gauging establishes a clear connection between conformal gauge theory and general relativity. It does *not* require the conventional constraints.

The new gauging starts with the conformal group that acts on an n-dim spacetime. We will always assume $n > 2$ and can thus identify the conformal group with the $\frac{1}{2}(n+1)(n+2)$-parameter orthogonal group $O(n, 2)$, which acts on an n-dimensional compact spacetime and leaves the null interval $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$, $\eta_{\mu\nu} = diag(1 \ldots 1, -1, -1)$, $\mu, \nu = 1 \ldots n$, invariant. It is generated by Lorentz transformations, dilations (rescalings), translations, and special conformal transformations[1]. The latter are actually translations of the inverse coordinate $y_\mu \equiv -\eta_{\mu\nu} \frac{x^\nu}{x^2}$, or, equivalently, translations of the vertex of the lightcone at infinity. In the new gauging, they are treated on an equal footing with translations and in that context will be referred to as *co-translations*. We retain the term special conformal transformations for conformal gauge theories in which one of the two subsets of n transformations (i.e., translations or special conformal transformations) is treated differently from the other.

By demanding that the translational and co-translational gauge fields together span the base manifold the biconformal technique yields a $2n$-

[1]These transformations have elsewhere been called proper conformal transformations or conformal boosts. See Appendix A for a formal definition of the conformal group and an overview of geometrical gauge theory.

dimensional manifold. A summary of this technique is given in Appendix A and full detail is available in reference [1]. Among the advantages to this procedure is the fact that the resulting volume element is scale-invariant. To see why, notice that the inverse coordinates y_μ scale oppositely to the spacetime coordinates x^μ. The corresponding translational and co-translational gauge fields, ω^a and ω_a scale as e^ϕ and $e^{-\phi}$, respectively. The volume element,

$$\omega^{a_1} \wedge \ldots \wedge \omega^{a_n} \wedge \omega_{b_1} \wedge \ldots \wedge \omega_{b_n},$$

is therefore scale invariant, since there are n translations and n co-translations. The scale invariance of the volume form eliminates the typical coupling of invariance to dimension, opening up a wide range of possible actions.

In the next section, we define our notational conventions.

In Sec.(3) we introduce the zero-weight biconformal Levi-Civita tensor, define the biconformal dual, present a large class of polynomial actions for biconformal geometries valid for all dimensions $n > 2$, and write the most general action linear in the biconformal curvatures and structural invariants. Finally, we note certain topological integrals.

The subsequent three sections examine the consequences of the most general linear action found in Sec.(3). We completely solve the field equations for a minimal torsion biconformal space, and show that the solutions admit two foliations of the $2n$-dim base manifold. The first involution shows that the base space is foliated by conformally flat n-dim submanifolds. The second involution gives a foliation by equivalent n-dim spacetimes constrained by the vanishing of the Ricci tensor. Thus, the solder form satisfies the vacuum Einstein equation despite the overall geometry possessing (minimal) torsion, a non-trivial Weyl vector, and an arbitrary cosmological constant. Each Riemannian geometry is fully determined by the components of the solder form, $e_\mu^{\ a}$, defined entirely on these spacetime submanifolds. The full $2n$-dim solution contains one additional field, a symmetric tensor $h_{\mu\nu}$, also defined entirely on the submanifolds. Except for a single special case, the full $2n$-dim space is necessarily symplectic, hence almost complex and almost Kähler.

Sec.(7) treats one special case which occurs in the general solution. In this case, the Ricci tensor continues to vanish while certain additional fields are allowed. Sec.(8) compares and contrasts the present method with previous conformal and scale-invariant gaugings, while the final section contains a brief summary.

2 Notation

The group $O(n,2)$ preserves the $(n+2)$-dim metric $diag(1\dots 1,-1,-1)$, or equivalently in a null basis

$$\eta_{\tilde{A}\tilde{B}} = \begin{pmatrix} 0 & 0\cdots 0 & 1 \\ 0 & & 0 \\ \vdots & \eta_{ab} & \vdots \\ 0 & & 0 \\ 1 & 0\cdots 0 & 0 \end{pmatrix}$$

where $\tilde{A},\tilde{B},\ldots = 0,1,\ldots,n,n+1$ and $a,b,\ldots = 1,\ldots,n$. The Minkowski metric is written as $\eta_{ab} = diag(1\dots 1,-1)$. The usual antisymmetry of the pseudo-orthonormal connection, $\omega^{\tilde{A}}_{\tilde{B}}$, allows us to express ω^{n+1}_{A}, ω^{A}_{n+1}, and ω^{n+1}_{n+1} (where $A,B,\ldots = 0,1,\ldots,n$) in terms of the remaining set,

$$\omega^A_B = \{\omega^a_b, \omega^a \equiv \omega^a_0, \omega_a \equiv \omega^0_a, \omega^0_0\}$$

These remaining independent connection components (gauge fields) are associated with the Lorentz, translation, co-translation, and dilation generators of the conformal group, respectively. We refer to ω^a_b as the spin-connection, ω^a as the solder-form, ω_a as the co-solder-form, and ω^0_0 as the Weyl vector, where in all cases differential forms are bold and the wedge product is assumed between adjacent forms. The $O(n,2)$ curvature, given by

$$\mathbf{\Omega}^A_B = \mathbf{d}\omega^A_B - \omega^C_A\omega^A_C, \tag{3}$$

divides into corresponding parts, $\{\mathbf{\Omega}^a_b, \mathbf{\Omega}^a \equiv \mathbf{\Omega}^a_0, \mathbf{\Omega}_a \equiv \mathbf{\Omega}^0_a, \mathbf{\Omega}^0_0\}$, called the curvature, torsion, co-torsion, and dilation, respectively. While these parts are not conformally invariant, they are invariant under the fiber symmetry of the biconformal bundle.

When broken into parts based on the homogeneous Weyl transformation properties of the biconformal bundle, i.e. Lorentz and scale transformations, eq.(3) becomes

$$\begin{aligned} \mathbf{\Omega}^a_b &= \mathbf{d}\omega^a_b - \omega^c_b\omega^a_c - \Delta^{ad}_{cb}\omega_d\omega^c && (4) \\ \mathbf{\Omega}^a &= \mathbf{d}\omega^a - \omega^b\omega^a_b - \omega^0_0\omega^a && (5) \\ \mathbf{\Omega}_a &= \mathbf{d}\omega_a - \omega^b_a\omega_b - \omega_a\omega^0_0 && (6) \\ \mathbf{\Omega}^0_0 &= \mathbf{d}\omega^0_0 - \omega^a\omega_a && (7) \end{aligned}$$

where $\Delta^{ab}_{cd} \equiv \delta^a_c \delta^b_d - \eta^{ab}\eta_{cd}$. If we set $\omega_a = 0 = \mathbf{\Omega}_a$ the equations reduce to the structure equations of an n-dim Weyl geometry with torsion. Each curvature may be expanded in the (ω^a, ω_b) basis as

$$\mathbf{\Omega}^A_B = \frac{1}{2}\Omega^A_{Bcd}\omega^{cd} + \Omega^{Ac}_{Bd}\omega_c\omega^d + \frac{1}{2}\Omega^{Acd}_B\omega_{cd}$$

where we introduce the convention of writing $\omega^{bc\ldots d} \equiv \omega^b\omega^c\ldots\omega^d = \omega^b \wedge \omega^c \wedge \ldots \wedge \omega^d$ etc. The three terms will be called the spacetime-, cross-, and momentum-term, respectively, of the corresponding curvature. For each curvature of the set $\{\mathbf{\Omega}^a_b, \mathbf{\Omega}^a, \mathbf{\Omega}_a\}$, each of these three terms is a distinct Weyl-covariant object. Each term of the dilation $\mathbf{\Omega}^0_0$ is Weyl invariant. In addition, the 2-forms $\mathbf{d}\omega^0_0$ and $\omega^a\omega_a$ appearing in eq.(7) are Weyl invariant.

In working with biconformal objects it is simpler to abandon the raising and lowering of indices with the metric, for two reasons. First, it would lead to confusion of fields that are independent, such as the spacetime and cross-terms of the curvature, Ω^a_{bcd} and Ω^{ac}_{bd}, or the necessarily independent 1-forms, ω^a and ω_a. Second, the position of any lower-case Latin index corresponds to the associated scaling weight: each upper index contributes $+1$ to the weight, while each lower index contributes -1. Thus, Ω^a_{bcd} has weight -2, while $\Omega_a^{\ bc}$ has weight $+1$.

3 Biconformal Actions

In order to construct biconformal actions, we must first examine certain special properties of the volume element of a biconformal space. Since the base manifold is spanned by the $2n$ gauge fields $\{\omega^a, \omega_b\}$ we may set

$$\mathbf{\Phi} = \varepsilon_{a_1\cdots a_n}{}^{b_1\cdots b_n}\omega^{a_1}\cdots\omega^{a_n}\omega_{b_1}\cdots\omega_{b_n}$$

where $\varepsilon_{a_1\cdots a_n}{}^{b_1\cdots b_n}$ is the $2n$-dim Levi-Civita symbol. The mixed index positioning indicates the scaling weight of the indices, and <u>not</u> any use of the metric. The positions arise from our notation for $O(n,2)$, in which the generators have a pair of indices, L^A_B.

The Lorentz transformations of the biconformal fields are n- rather than $2n$-dim matrices[2]. Therefore, the two n-dim Levi-Civita symbols are also

[2]While the matrix structure of a Lorentz transformation in biconformal space is n-dim, the functional dependence is $2n$-dim. Thus, we have $\Lambda^a_b = \Lambda^a_b(x^\mu, y_\nu)$, where (x^μ, y_ν) span the full biconformal space.

Lorentz invariant, but have opposite scaling properties,

$$\begin{aligned}\varepsilon^{a_1\cdots a_n} &\rightarrow e^{n\phi}\varepsilon^{a_1\cdots a_n}\\ \varepsilon_{a_1\cdots a_n} &\rightarrow e^{-n\phi}\varepsilon_{a_1\cdots a_n}\end{aligned}$$

Another way to see the intrinsic presence of these two tensors is from the distinguishability of the ω^a from the ω_a by their differing scale weights, giving rise to two n-volumes,

$$\begin{aligned}\mathbf{\Phi}_+ &= \varepsilon_{a_1\cdots a_n}\omega^{a_1}\cdots\omega^{a_n}\\ \mathbf{\Phi}_- &= \varepsilon^{a_1\cdots a_n}\omega_{a_1}\cdots\omega_{a_n}\end{aligned}$$

Therefore,

$$\mathbf{\Phi} = \mathbf{\Phi}_+\mathbf{\Phi}_-$$

and we may set

$$\varepsilon_{a_1\cdots a_n}{}^{b_1\cdots b_n} = \varepsilon_{a_1\cdots a_n}\varepsilon^{b_1\cdots b_n}$$

which is clearly both Lorentz and scale invariant. Whenever the indices on $\varepsilon_{a_1\cdots a_n}{}^{b_1\cdots b_n}$ are not in this standard position, the signs are to be adjusted accordingly. Thus, for example,

$$\varepsilon_{a\ c\cdots d}^{\ \ g}{}^{e\cdots f\ h}_{\qquad b} \equiv -\,\varepsilon_{abc\cdots d}{}^{e\cdots fgh}$$

The Levi-Civita tensor is normalized such that traces are given by

$$\varepsilon_{a_1\cdots a_p c_{p+1}\cdots c_n}\varepsilon^{b_1\cdots b_p c_{p+1}\cdots c_n} = p!(n-p)!\delta^{b_1\cdots b_p}_{a_1\cdots a_p}$$

where the totally antisymmetric δ-symbol is defined as

$$\delta^{b_1\cdots b_p}_{a_1\cdots a_p} \equiv \delta^{[b_1}_{a_1}\cdots\delta^{b_p]}_{a_p}$$

The Levi-Civita tensor allows us to define biconformal duals of forms. Let

$$T \equiv \frac{1}{p!q!}T_{a_1\cdots a_p}{}^{b_1\cdots b_q}\omega^{a_1\cdots a_p}\omega_{b_1\cdots b_q}$$

be an arbitrary $(p+q)$-form, $p, q \in \{0,\ldots,n\}$, with weight $p-q$. Then the dual of T is a $((n-q)+(n-p))$-form, also of weight $(n-q)-(n-p) = p-q$, defined as

$$^*T \equiv \frac{1}{p!q!(n-p)!(n-q)!}T_{a_1\cdots a_p}{}^{b_1\cdots b_q}\varepsilon_{b_1\cdots b_n}{}^{a_1\cdots a_n}\omega^{b_{q+1}\cdots b_n}\omega_{a_{p+1}\cdots a_n}$$

so that

$$^{**}T = (-1)^{(p+q)(n-(p+q))}T$$

Note that the dual map is scale-invariant: both T and $^*T$ have weight $(p-q)$.

We can now write a variety of biconformally invariant and $O(n,2)$ invariant integrals.

To build biconformal invariants, we use the fact that the fiber symmetry (structure group) of the biconformal bundle is the Weyl group, consisting of Lorentz transformations and dilations, while the connection forms corresponding to translations and co-translations span the base space. We can therefore return to the reduced notation, and find a correspondingly increased number of possible actions. First, we note that the bilinear form

$$\omega^a \omega_b$$

is scale invariant. This object allows us to write actions of arbitrary order, $k = 1, \ldots, n$, in the curvatures. In particular, we can write

$$S_{m,k-m} = \int \mathbf{\Omega}^{A_1}_{B_1} \ldots \mathbf{\Omega}^{A_m}_{B_m} \mathbf{\Omega}^0_0 \ldots \mathbf{\Omega}^0_0 \omega^{a_1 \cdots a_{n-k}} \omega_{b_1 \cdots b_{n-k}} Q^{B_1 \cdots B_m b_1 \cdots b_{n-k}}_{A_1 \cdots A_m a_1 \cdots a_{n-k}}$$

where there are m factors of the curvature Ω^A_B and $k-m$ factors of the dilational curvature, Ω^0_0. The invariant tensor $Q^{B \cdots D}_{A \cdots C}$ has $2[n-(k-m)]$ indices and must be built from δ^A_B and the Levi-Civita tensor. Notice that when $m = k$ we can use

$$Q^{B_1 \cdots B_k b_1 \cdots b_{n-k}}_{A_1 \cdots A_k a_1 \cdots a_{n-k}} = \varepsilon^{0 B_1 \cdots B_k b_1 \cdots b_{n-k}} \varepsilon_{0 A_1 \cdots A_k a_1 \cdots a_{n-k}} \longrightarrow \varepsilon^{c_1 \cdots c_n} \varepsilon_{d_1 \cdots d_n}$$

for the invariant tensor. Various combinations of Kronecker deltas are also possible for $Q^{B_1 \cdots B_m b_1 \cdots b_{n-k}}_{A_1 \cdots A_m a_1 \cdots a_{n-k}}$.

A scale-invariant dual action of Yang-Mills type is given by

$$S_{dual} = \int \mathbf{\Omega}^{\tilde{A}}_{\tilde{B}} {}^* \mathbf{\Omega}^{\tilde{B}}_{\tilde{A}}$$

The resulting field equation, however, is more complicated than the usual $\mathbf{D}^* \mathbf{\Omega}^{\tilde{A}}_{\tilde{B}} = 0$, because $\delta \omega^{\tilde{A}}_{\tilde{B}}$ does not commute with the dual operator.

The most general linear Lorentz and scale-invariant (weight zero) action built out of biconformal curvatures and the two invariants $\omega^a \omega_a$ and $\mathbf{d}\omega^0_0$ in a $2n$-dim biconformal space spanned by $\{\omega^{a_i}, \omega_{a_i}; i = 1 \ldots n\}$ is a linear combination of $S_{1,0}, S_{0,1}$ and $S_{0,0}$,

$$S = \int (\alpha \mathbf{\Omega}^{a_1}_{b_1} + \beta \delta^{a_1}_{b_1} \mathbf{\Omega}^0_0 + \gamma \omega^{a_1} \omega_{b_1}) \omega^{a_2 \ldots a_n} \omega_{b_2 \ldots b_n} \varepsilon^{b_1 \ldots b_n} \varepsilon_{a_1 \ldots a_n}$$

where $\alpha, \beta, \gamma \in \mathbf{R}$ are constants. Notice that γ is an arbitrary cosmological constant. An additional term containing $\mathbf{d}\omega_0^0$ would be redundant because of structure equation (7) for Ω_0^0. Moreover, S cannot contain torsion or co-torsion terms, nor is anything further found by using Kronecker deltas in place of the Levi-Civita tensors. This action and the resulting field equations will be considered in detail in the following sections.

Finally, to build $O(n,2)$ invariants, we return to the full $O(n,2)$ notation. We can write

$$S_n = \int \mathbf{\Omega}^{\tilde{A}}_{\tilde{B}} \ldots \mathbf{\Omega}^{\tilde{C}}_{\tilde{D}} Q^{\tilde{B}\cdots\tilde{D}}_{\tilde{A}\cdots\tilde{C}}$$

where $Q^{\tilde{B}\cdots\tilde{D}}_{\tilde{A}\cdots\tilde{C}}$ is an $O(n,2)$-invariant tensor. $Q^{\tilde{B}\cdots\tilde{D}}_{\tilde{A}\cdots\tilde{C}}$ must be built from $\delta^{\tilde{A}}_{\tilde{B}}, \eta_{\tilde{A}\tilde{B}}$ or the $(n+2)$-dimensional Levi-Civita tensor. The only object with the correct index structure is

$$Q^{\tilde{B}\cdots\tilde{D}}_{\tilde{A}\cdots\tilde{C}} = \delta^{\tilde{B}\cdots\tilde{D}}_{\tilde{A}\cdots\tilde{C}} = \frac{1}{2!(n-2)!}\varepsilon^{\tilde{B}\cdots\tilde{D}\tilde{E}\tilde{F}}\varepsilon_{\tilde{A}\cdots\tilde{C}\tilde{E}\tilde{F}}$$

With this specification for $Q^{\tilde{B}\cdots\tilde{D}}_{\tilde{A}\cdots\tilde{C}}$, S_n becomes the n^{th} Pontrijagin class.

4 The Linear Action

As noted above, in a $2n$-dim biconformal space the most general Lorentz and scale-invariant action which is linear in the biconformal curvatures and structural invariants is

$$S = \int (\alpha \mathbf{\Omega}^{a_1}_{b_1} + \beta \delta^{a_1}_{b_1} \mathbf{\Omega}^0_0 + \gamma \omega^{a_1}\omega_{b_1})\omega^{a_2 \ldots a_n}\omega_{b_2 \ldots b_n}\varepsilon^{b_1 \ldots b_n}\varepsilon_{a_1 \ldots a_n} \tag{8}$$

We will always assume non-vanishing α and β. Variation of this action with respect to the connection one-forms yields the following field equations:

$$\begin{aligned}
\delta_{\omega^0_0} S = 0 \Rightarrow 0 &= \beta(\Omega^a{}_{ba} - 2\Omega_{ca}{}^{d}\delta^{ca}_{db}) && (9)\\
0 &= \beta(\Omega_a{}^{ba} - 2\Omega^{cd}{}_{a}\delta^{ab}_{dc}) && (10)\\
\delta_{\omega^a_b} S = 0 \Rightarrow 0 &= \alpha(-\Delta^{af}_{eg}\Omega^b{}_{ab} + 2\Delta^{cf}_{eb}\delta^{ab}_{dg}\Omega_{ac}{}^{d}) && (11)\\
0 &= \alpha(-\Delta^{gf}_{eb}\Omega_a{}^{ab} + 2\Delta^{af}_{ed}\delta^{gc}_{ab}\Omega^{bd}{}_{c}) && (12)\\
\delta_{\omega^a} S = 0 \Rightarrow 0 &= \alpha\Omega^a_{bac} + \beta\Omega^0_{0bc} && (13)\\
0 &= 2(\alpha\Omega^{ac}_{cd} + \beta\Omega^{0a}_{0d})\delta^{ed}_{ab} + (\alpha(n-1) - \beta + \gamma n^2)\delta^e_b && (14)\\
\delta_{\omega_a} S = 0 \Rightarrow 0 &= \alpha\Omega^{bac}_a + \beta\Omega^{0bc}_0 && (15)\\
0 &= 2(\alpha\Omega^{ca}_{dc} + \beta\Omega^{0a}_{0d})\delta^{ed}_{ab} + (\alpha(n-1) - \beta + \gamma n^2)\delta^e_b && (16)
\end{aligned}$$

Combining equations (14) and (16) we see that the latter can be replaced by

$$\Omega_{cd}^{ac} = \Omega_{dc}^{ca} \tag{17}$$

5 Solution for the Curvatures

We now find the most general solution to these equations subject only to a constraint of minimal torsion. Starting with the most general ansatz for the spin connection and Weyl vector, we obtain expressions for the torsion and co-torsion. Then we find the form of the connection required to satisfy eqs.(9)-(12). The result does not permit vanishing torsion without vanishing Weyl vector, so we choose the minimal torsion constraint consistent with a general form for the Weyl vector. The constraint and field equations lead to a foliation by n-dim flat Riemannian manifolds, possibly with torsion. By invoking the gauge freedom on each of these manifolds, we show the existence of a second foliation by n-dim Riemannian spacetimes *without* torsion satisfying the vacuum Einstein equations. Generically, the full biconformal space also has a symplectic structure.

We first write the spin connection ω_b^a as

$$\begin{aligned}
\omega_b^a &= \alpha_b^a + \beta_b^a + \gamma_b^a \\
&= (\alpha_{bc}^a + \beta_{bc}^a + \gamma_{bc}^a)\omega^c + (\alpha_b^{ac} + \beta_b^{ac} + \gamma_b^{ac})\omega_c
\end{aligned}$$

with α_b^a and β_b^a defined by

$$\mathbf{d}\omega^a = \omega^b \alpha_b^a + \frac{1}{2}\Omega^{abc}\omega_{bc} \tag{18}$$

$$\mathbf{d}\omega_a = \beta_a^b \omega_b + \frac{1}{2}\Omega_{abc}\omega^{bc} \tag{19}$$

Using this ansatz as well as the expanded form of the Weyl vector

$$\omega_0^0 = W_a\omega^a + W^a\omega_a$$

in structure equations (5) and (6), Ω^{abc} and Ω_{abc} remain related to derivatives of the solder- and co-solder forms, whereas the other torsion and co-torsion terms are algebraic in α_b^a, β_b^a and γ_b^a:

$$\Omega^a{}_{bc} = \gamma_{cb}^a - \gamma_{bc}^a + \beta_{cb}^a - \beta_{bc}^a + W_c\delta_b^a - W_b\delta_c^a \tag{20}$$

$$\Omega_b^{ac} = \gamma_b^{ac} + \beta_b^{ac} - W^c\delta_b^a \tag{21}$$

$$\Omega_{ac}^{\ b} = \alpha_{ac}^b + \gamma_{ac}^b - W_c\delta_a^b \tag{22}$$

$$\Omega_a^{\ bc} = \alpha_a^{bc} - \alpha_a^{cb} + \gamma_a^{bc} - \gamma_a^{cb} + W^b\delta_a^c - W^c\delta_a^b \tag{23}$$

Thus, the separation of the connection allows us to solve the first four field equations algebraically. Imposing field equations (9) and (10) onto (21) and (22) we get

$$\begin{aligned}\beta^a_{ba} &= \alpha^a_{ba} \\ \beta^{ba}_a &= \alpha^{ba}_a\end{aligned}$$

Using this result and imposing field equations (11) and (12) onto (20) and (23) completely determines γ^a_b in terms of α^a_b and β^a_b:

$$\gamma^a_b = -(\alpha^a_{bc}\omega^c + \beta^{ac}_b\omega_c)$$

so the spin connection becomes

$$\omega^a_b = \beta^a_{bc}\omega^c + \alpha^{ac}_b\omega_c$$

Defining the traceless Lorentz tensor

$$\begin{aligned}\sigma^a_b &\equiv \alpha^a_b - \beta^a_b \\ &= \sigma^a_{bc}\omega^c + \sigma^{ac}_b\omega_c\end{aligned}$$

equations (20)-(23) become

$$\Omega^a{}_{bc} = \sigma^a_{bc} - \sigma^a_{cb} + W_c\delta^a_b - W_b\delta^a_c \tag{24}$$
$$\Omega^{ac}_b = -W^c\delta^a_b \tag{25}$$
$$\Omega_{ac}{}^b = -W_c\delta^b_a \tag{26}$$
$$\Omega_a{}^{bc} = \sigma^{bc}_a - \sigma^{cb}_a + W^b\delta^c_a - W^c\delta^b_a \tag{27}$$

While it might seem natural to demand vanishing torsion, $\mathbf{\Omega}^a = 0$, notice that the traces of $\Omega^a{}_{bc}$ and Ω^{ac}_b are given by

$$\begin{aligned}\Omega^b{}_{ba} &= (n-1)W_a \\ \Omega^{ac}_a &= -nW^c\end{aligned}$$

Therefore, constraining either eq.(24) or eq.(25) to vanish unnecessarily constrains the Weyl vector and thus greatly reduces the set of allowed geometries. Instead, we impose the strongest torsion constraint that is consistent with a general Weyl vector, namely,

$$\mathbf{\Omega}^a = \omega^a\omega^0_0 \tag{28}$$

As a consequence,

$$\begin{aligned}\Omega^{abc} &= 0 \\ \sigma^a_{bc} &= \sigma^a_{cb}\end{aligned}$$

Since the antisymmetry of the spin connection,

$$\omega^a_b = -\eta^{ac}\eta_{bd}\omega^d_c$$

is inherited by α^a_b, β^a_b, and therefore σ^a_b, it is now easy to show by permuting the indices of $\eta_{ad}\sigma^d_{bc}$ and taking the usual linear combination, that in fact

$$\sigma^a_{bc} \equiv \alpha^a_{bc} - \beta^a_{bc} = 0$$

so that

$$\omega^a_b = \alpha^a_b$$

We make no assumption concerning the co-torsion, curvature or dilation. In particular, for the co-torsion we see from eqs.(26) and (27) that constraints on $\Omega_a^{\ bc}$ or $\Omega_{ac}^{\ \ b}$ would constrain ω^0_0. Furthermore, we will see below that vanishing spacetime co-torsion, $\Omega_{abc} = 0$, leads to vanishing spacetime curvature, $\Omega^a_{bcd} = 0$, and would therefore be too strong an assumption.

The torsion constraint makes it possible to obtain an algebraic condition on the curvatures from the Bianchi identity associated with eq.(5). Taking the exterior derivative of eq.(5) gives

$$\mathbf{D\Omega}^a \equiv \mathbf{d\Omega}^a + \mathbf{\Omega}^b\omega^a_b - \omega^0_0\mathbf{\Omega}^a = \omega^b\mathbf{\Omega}^a_b - \omega^a\mathbf{\Omega}^0_0$$

Simplifying $\mathbf{D\Omega}^a$ using eqs.(28) and (7)

$$\begin{aligned}\mathbf{D\Omega}^a &= \mathbf{D}(\omega^a\omega^0_0) \\ &= -\omega^a\mathbf{\Omega}^0_0 - \omega^{ab}\omega_b\end{aligned}$$

(recall that $\omega^{ab} \equiv \omega^a\omega^b$) the Bianchi identity reduces to

$$\omega^b\mathbf{\Omega}^a_b = -\omega^{ab}\omega_b$$

which implies

$$\Omega^a_{[bcd]} = 0 \tag{29}$$
$$\Omega^{ac}_{bd} = -\Delta^{ac}_{db} \tag{30}$$
$$\Omega_b^{acd} = 0 \tag{31}$$

If we define

$$\begin{aligned}\mathbf{R}^a_b &\equiv \mathbf{d}\omega^a_b - \omega^c_b\omega^a_c \\ &= \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c\end{aligned}$$

so that $\mathbf{\Omega}^a_b = \mathbf{R}^a_b - \Delta^{ac}_{db}\omega_c\omega^d$, eqs.(30) and (31) show that $\mathbf{R}^a_b$ has vanishing cross- and momentum-terms:

$$\mathbf{R}^a_b = \frac{1}{2}R^a_{bcd}\omega^{cd}$$

This result also follows from the Bianchi identity arising from eq.(18) with $\Omega^{abc} = 0$. Noting from the trace of $\Omega^a_{[bcd]} = 0$ and the antisymmetry condition $\Omega^a_{bcd} = -\eta_{be}\eta^{af}\Omega^e_{fcd}$ that $R^a_{bac} \equiv R_{bc} = R_{cb}$, field equation (13) implies separate vanishing of the α and β terms

$$R_{bc} = 0 \tag{32}$$

$$\Omega^0_{0bc} = 0 \tag{33}$$

while eq.(15) immediately gives

$$\Omega^{0cd}_0 = 0 \tag{34}$$

Notice that, while eq.(32) is certainly similar to the vacuum Einstein equation, $\mathbf{R}^a_b$ has not yet been shown to be the curvature of a Riemannian geometry. In particular, though it has the general form of an n-dim curvature tensor, it might in principle depend on all $2n$ coordinates, on the torsion, and/or on the Weyl vector.

Continuing with the field equations, we see that since $\Delta^{ab}_{ac} = \Delta^{ba}_{ca} = (n-1)\delta^b_c$, eq.(17) is identically satisfied. Finally, eq.(14) implies

$$\Omega^{0a}_{0b} = \lambda\delta^a_b \tag{35}$$

where

$$\lambda \equiv \frac{\alpha n(n-1) - \beta + \gamma n^2}{\beta(n-1)}$$

Thus, the entire 3-parameter class of actions leads to a 1-parameter class of solutions. In particular, the form of the solution is largely independent of the value of the cosmological constant.

We have now satisfied all of the field equations. The curvatures take the form

$$
\begin{aligned}
\mathbf{\Omega}^a_b &= \mathbf{R}^a_b - \Delta^{ac}_{db}\omega_c\omega^d & (36)\\
\mathbf{\Omega}^0_0 &= \lambda\omega_a\omega^a & (37)\\
\mathbf{\Omega}^a &= \omega^a\omega^0_0 & (38)\\
\mathbf{\Omega}_a &= \omega^0_0\omega_a + \sigma_a^{bc}\omega_{bc} + \frac{1}{2}\Omega_{abc}\omega^{bc} & (39)
\end{aligned}
$$

with

$$
\begin{aligned}
R_{ab} &= 0\\
\sigma_a^{ba} &= 0
\end{aligned}
$$

In the next section, we find further constraints on the curvatures arising from the structure equations. We also find an explicit form for the connection that displays clearly the minimal field content of the general solution.

6 Solution for the Connection

While eqs.(36)-(39) for the curvatures satisfy all of the field equations, they do not fully incorporate the form of the biconformal structure equations as embodied in the Bianchi identities. Moreover, it is not yet clear what minimal field content is required to insure a unique solution. Therefore, in this section, we turn to the consequences of the form (36)-(39) of the curvatures on the connection.

Substituting the reduced curvatures into eqs.(4)-(7), the structure equations now take the form

$$
\begin{aligned}
\mathbf{R}^a_b &= \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c & (40)\\
\mathbf{d}\omega^a &= \omega^b\alpha^a_b & (41)\\
\mathbf{d}\omega_a &= \alpha^b_a\omega_b - \sigma^b_a\omega_b + \frac{1}{2}\Omega_{abc}\omega^{bc} & (42)\\
\mathbf{d}\omega^0_0 &= (1-\lambda)\omega^a\omega_a & (43)
\end{aligned}
$$

We begin with eq.(43). For $\lambda \neq 0, 1$, eqs.(43) and (37) show that the three biconformal invariants $\mathbf{d}\omega^0_0$, $\omega_a\omega^a$ and $\mathbf{\Omega}^0_0$ noted in the introduction are all proportional to each other. Moreover, eq.(43) shows in gestalt form that each invariant is a symplectic form: $\mathbf{d}\omega_0^0$ is manifestly closed and $\omega_a\omega^a$ is manifestly nondegenerate. A generic biconformal space subject to the linear action and minimal torsion condition $\mathbf{\Omega}^a = \omega^a\omega_0^0$ is therefore a symplectic manifold. By a well-known theorem [33] it is always possible to construct an almost complex structure and a Kähler metric on a symplectic manifold. Therefore the field equations arising from the linear action constrain the biconformal space to be almost Kähler.

If $\lambda = 0$, the dilation vanishes, while $\omega^a\omega_a = \mathbf{d}\omega_0^0$ remains symplectic. All subsequent calculations hold with $\mathbf{\Omega}_0^0 = 0$. This case has been investigated in [1], where it was argued that for *classical* geometries it is reasonable to assume that no path in phase space encloses a plaquette on which the dilation is nonvanishing. Such spaces were shown to be in $1-1$ correspondence with n-dimensional Einstein-Maxwell spacetimes.

We will consider the special case $\lambda = 1$ further in Sec.(7). For the remainder of this Section let $\lambda \neq 1$.

The Bianchi identity obtained by taking the exterior derivative of eq.(43) is

$$\begin{aligned}
0 &= \mathbf{d}\omega_a\omega^a - \omega_a\mathbf{d}\omega^a \\
&= (\alpha_a^b\omega_b - \sigma_a^b\omega_b + \frac{1}{2}\Omega_{abc}\omega^{bc})\omega^a - \omega_a\omega^b\alpha_b^a \\
&= \sigma_a^b\omega^a\omega_b + \frac{1}{2}\Omega_{abc}\omega^{abc}
\end{aligned}$$

so that

$$\begin{aligned}
\Omega_{[abc]} &= 0 \\
\sigma_a^b &= 0
\end{aligned}$$

Here the vanishing of σ_a^b follows from the simultaneous vertical antisymmetry and horizontal symmetry of σ_a^{bc}. This vanishing of σ_a^b amounts to the vanishing of the momentum term of the co-torsion.

Next we examine consequences of eq.(41), which is in involution. By the Frobenius theorem, we can consistently set ω^a to zero and obtain a foliation by submanifolds where the spin connection and Weyl vector reduce to

$$\begin{aligned}
\hat{\alpha}_b^a &\equiv \alpha^a_{b|\omega^a=0} = \alpha_b^{ac}\omega_c \\
\hat{\omega}_0^0 &\equiv \omega^0_{0|\omega^a=0} = W^a\omega_a
\end{aligned}$$

If we define

$$\mathbf{f}_a \equiv \omega_{a|\omega^a=0}$$

then each submanifold is described by the reduced structure equations

$$\begin{aligned}
\mathbf{d}\hat{\alpha}^a_b &= \hat{\alpha}^c_b\hat{\alpha}^a_c \\
\mathbf{d}\mathbf{f}_a &= \hat{\alpha}^b_a\mathbf{f}_b + \mathbf{f}_a\hat{\omega}^0_0 \\
\mathbf{d}\hat{\omega}^0_0 &= 0
\end{aligned}$$

Since $\hat{\omega}^0_0$ is closed, we can scale-gauge the Weyl vector to zero on each subspace, i.e. $W^a = 0$. The remaining two equations then describe a flat n-dimensional Riemannian spacetime. Since the spin-connection is involute, there also exists a Lorentz gauge transformation such that $\hat{\alpha}^a_b = 0$ on each submanifold, i.e. $\alpha^{ac}_b = 0$. With these gauge choices the system reduces to simply $\mathbf{d}\mathbf{f}_a = 0$, with solution $\mathbf{f}_a = \mathbf{d}\theta_a$ for some 0-forms θ_a.

Returning to the full biconformal space, we now have a gauge such that the spin connection and Weyl vector are

$$\alpha^a_b = \alpha^a_{bc}\omega^c \tag{44}$$

$$\omega^0_0 = W_a\omega^a \tag{45}$$

while the co-solder form may be written in terms of $\mathbf{f}_a$ and an additional term linear in the solder form,

$$\omega_a = \mathbf{f}_a + h_{ab}\omega^b$$

Notice that $\mathbf{f}_a$ is essentially unchanged by this extension, except that the 0-forms θ_a must be regarded as dependent on all $2n$ coordinates. This means that $\mathbf{d}\mathbf{f}_a$ remains at least linear in $\mathbf{f}_a$, and is consequently involute (see Appendix B). We can therefore turn the problem around, setting $\mathbf{f}_a = 0$ to obtain a second foliation of the biconformal space. We can define $\mathbf{h}_a$ in terms of this involution, setting

$$\mathbf{h}_a \equiv \omega_{a|\mathbf{f}_a=0} = h_{ab}\ \omega^b$$

with h_{ab} arbitrary. Now, with $\mathbf{f}_a = 0$, the new submanifolds are described by

$$\begin{aligned}
\mathbf{R}^a_b &= \mathbf{d}\alpha^a_b - \alpha^c_b\alpha^a_c \\
\mathbf{d}\omega^a &= \omega^b\alpha^a_b \\
\mathbf{d}\mathbf{h}_a &= \alpha^b_a\mathbf{h}_b + \frac{1}{2}\Omega_{abc}\omega^{bc} \\
\mathbf{d}\omega^0_0 &= (1-\lambda)\omega^a\mathbf{h}_a
\end{aligned}$$

The first two equations are unchanged from their full biconformal form, showing that the curvature $\mathbf{R}^a_b$ and connection α^a_b (and of course ω^a, by the first involution) are fully determined on the $\mathbf{f}_a = 0$ submanifold. Thus, α^a_b is the usual spin connection compatible with ω^a, while $\mathbf{R}^a_b$ is its curvature. Therefore, the vanishing of the Ricci tensor, $R_{ab} = 0$, now shows that these n-dim submanifolds satisfy the vacuum Einstein equations. Even though the torsion and dilation have nonvanishing spacetime projections, Ω_{abc} and $\Omega^{0a}_{0b|\mathbf{f}_a=0} = -\lambda\omega^a\mathbf{h}_a$, respectively, the curvature is the one computed from the solder form ω^a alone; even though our action included an arbitrary cosmological constant, γ, the Ricci tensor vanishes. This is our most important result, since it establishes a direct connection between the usual Ricci-flat Riemannian structure of general relativity and the more general structure of conformal gauge theory.

Finally, we seek a minimum set of fields required to uniquely specify a complete solution. We can easily find such a minimum set by choosing coordinates. Based on the involution for ω^a there exist n coordinates x^μ such that

$$\omega^a = e_\mu^{\ a}\mathbf{d}x^\mu$$

with the component matrices necessarily invertible. From eq.(41), we immediately find that $e_\mu^{\ a} = e_\mu^{\ a}(x)$. Similarly, we show in Appendix B that there exist coordinates y_ν such that $\mathbf{f}_a$ takes the form

$$\mathbf{f}_a = e_a^{\ \mu}\mathbf{d}y_\mu + \psi_{a\mu}\mathbf{d}x^\mu$$

where $e_a^{\ \mu}$ is the inverse to $e_\mu^{\ a}$ and $\psi_{a\mu} = \psi_{a\mu}(x, y)$.

Using this coordinate choice and writing the co-solder form as

$$\begin{aligned} \omega_a &= e_a^{\ \mu}(\mathbf{d}y_\mu + h_{\mu\nu}\mathbf{d}x^\nu + \psi_{\mu\nu}\mathbf{d}x^\nu) \\ &\equiv e_a^{\ \mu}(\mathbf{d}y_\mu + \hat{h}_{\mu\nu}\mathbf{d}x^\nu) \end{aligned} \qquad (46)$$

eq.(43) yields

$$\begin{aligned} \partial_{[\nu}W_{\mu]} &= (\lambda - 1)\hat{h}_{[\mu\nu]} \\ \partial^\nu W_\mu &= (\lambda - 1)\delta^\nu_\mu \end{aligned}$$

where $e_a^{\ \mu}$ and $e_\mu^{\ a}$ are used to interchange coordinate and orthonormal indices in the usual way and $(\partial_\nu, \partial^\nu)$ denote derivatives with respect to (x^μ, y_ν). The second equation can immediately be integrated,

$$W_\mu = (\lambda - 1)(y_\mu - A_\mu(x)) \qquad (47)$$

where the integration "constant" $A_\mu(x)$ determines the antisymmetric part of $\hat{h}_{[\mu\nu]}$:

$$\hat{h}_{[\mu\nu]} = \partial_{[\nu} A_{\mu]} \tag{48}$$

Clearly, $\hat{h}_{[\mu\nu]}$ is a function of x only.

The fields A_μ and $\hat{h}_{[\mu\nu]}$ in eqs.(47) and (48) are purely coordinate dependent. To see this, note that under coordinate transformations of the form $\bar{y}_\alpha = y_\alpha + \gamma_\alpha(x)$, eq.(46) changes to

$$\omega_a = e_a^{\ \mu}(\mathbf{d}\bar{y}_\mu + (\hat{h}_{\mu\nu} + \gamma_{\mu,\nu})\mathbf{d}x^\nu)$$

Then eqs.(47) and (48) become

$$\begin{aligned} \bar{W}_\mu &= (\lambda - 1)(\bar{y}_\mu - \gamma_\mu - A_\mu(x)) \\ \bar{h}_{[\mu\nu]} &= \partial_{[\nu} A_{\mu]} + \gamma_{[\mu,\nu]} \end{aligned}$$

Therefore, if we choose $\gamma_\mu = -A_\mu$, we have simply

$$\begin{aligned} \bar{W}_\mu &= (\lambda - 1)\hat{y}_\mu \\ \bar{h}_{[\mu\nu]} &= 0 \end{aligned}$$

We make this coordinate choice below, dropping the overbars.

Finally, with $\sigma_a^b = 0$, eq.(42) for the co-solder form reduces to

$$\mathbf{d}\omega_a = \alpha_a^b \omega_b + \frac{1}{2}\Omega_{abc}\omega^{bc}$$

or in coordinate form

$$\begin{aligned} (\partial_\mu h_{a\nu} - \alpha_{a\mu}^b h_{b\nu}) - (\partial_\nu h_{a\mu} - \alpha_{a\nu}^b h_{b\mu}) &= \Omega_{a\mu\nu} && (49) \\ \partial^\nu h_{a\mu} &= \partial_\mu e_a^{\ \nu} - \alpha_{ac}^b e_\mu^c e_b^{\ \nu} && (50) \end{aligned}$$

Using the well-known relation between the orthonormal connection $\alpha_b^a(x)$ and the Christoffel connection $\Gamma_{\alpha\mu}^\nu(x)$ of a metric compatible geometry,

$$D_\mu e_a^{\ \nu} \equiv \partial_\mu e_a^{\ \nu} - \alpha_{a\mu}^b e_b^{\ \nu} + e_a^{\ \alpha}\Gamma_{\alpha\mu}^\nu = 0$$

we find from eq.(50) that

$$\partial^\nu h_{\alpha\mu} = e_\alpha^{\ a}\partial^\nu h_{a\mu} = -\Gamma_{\alpha\mu}^\nu$$

which integrates to

$$h_{\alpha\mu} = -y_\nu \Gamma^\nu_{\alpha\mu} + k_{\alpha\mu}(x)$$

where the symmetric tensor $k_{\alpha\mu}(x)$ is a second integration "constant". The contraction with y_μ permits h_{ab} to behave as a Lorentz tensor. Notice that the covariant curl of $h_{\alpha\mu}$ has a term proportional to the curvature tensor because the contraction $-y_\nu \Gamma^\nu_{\alpha\mu}$ eliminates the extra connection terms that normally prevent the covariant curl of $\Gamma^\nu_{\alpha\mu}$ from being simply related to curvature. When this result for $h_{\alpha\mu}$ is substituted into eq.(49), the spacetime co-torsion is expressed in terms of $k_{\alpha\mu}(x)$ and the curvature. After restoring the basis forms,

$$\frac{1}{2}\Omega_{abc}\omega^{bc} = -y_b \mathbf{R}^b_a + \ \mathbf{D}\mathbf{k}_a$$

where $\mathbf{D}$ is the metric compatible covariant exterior derivative and $\mathbf{k}_a = k_{ab}\omega^b$. As claimed in Sec.(5), vanishing Ω_{abc} implies vanishing curvature since $\mathbf{k}_a$ depends on x^μ only.

Collecting the results for the connection, we immediately see the essential field content:

$$\begin{aligned}
\alpha^a_b &= \alpha^a_b(e_\mu^{\ a}(x)) \\
\omega^a &= e_\mu^{\ a}(x)\mathbf{d}x^\mu \\
\omega_a &= e_a^{\ \mu}(x)(\mathbf{d}y_\mu - y_\nu \Gamma^\nu_{\mu\alpha}\mathbf{d}x^\alpha + k_{\mu\alpha}(x)\ \mathbf{d}x^\alpha) \\
&= \mathbf{D}y_a + \mathbf{k}_a(x) \\
\omega^0_0 &= (\lambda - 1)\ y_\mu \mathbf{d}x^\mu
\end{aligned} \tag{51}$$

The entire solution depends on two fields, $e_\mu^{\ a}(x)$ and $k_{\mu\alpha} = k_{(\mu\alpha)}(x)$. Both of these fields are defined entirely on the vacuum Einstein spacetime submanifolds (with coordinates x^μ). Otherwise $k_{\mu\alpha}(x)$ is an arbitrary integration constant, while $e_\mu^{\ a}(x)$ is the usual solder form.

Finally, we write the final form of the curvatures. Decomposing the Riemann curvature into its traceless and Ricci parts

$$\mathbf{R}^a_b = \mathbf{C}^a_b + \Delta^{ac}_{db}\mathcal{R}_c \mathbf{e}^d \tag{52}$$

where

$$\mathcal{R}_a \equiv (R_{ab} - \frac{1}{2(n-1)}\eta_{ab}R)\mathbf{e}^b \tag{53}$$

we use the Ricci-flat condition together with the results of this section for σ_a^b and the co-torsion to write the curvatures as

$$\begin{aligned}
\mathbf{\Omega}_b^a &= \mathbf{C}_b^a - \Delta_{db}^{ac}\omega_c\omega^d && (54)\\
\mathbf{\Omega}_0^0 &= \lambda\omega_a\omega^a && (55)\\
\mathbf{\Omega}^a &= \omega^a\omega_0^0 && (56)\\
\mathbf{\Omega}_a &= \omega_0^0\omega_a - y_b\mathbf{C}_a^b + \mathbf{D}\mathbf{k}_a && (57)
\end{aligned}$$

7 Special Case

For the special case $\lambda = 1$, eq.(43) implies a closed and hence locally removable Weyl vector, i.e. $W_a = 0$. In that case, eqs.(38) and (39) for the torsion and co-torsion become

$$\begin{aligned}
\mathbf{\Omega}^a &= 0\\
\mathbf{\Omega}_a &= \sigma_a^{bc}\omega_{bc} + \frac{1}{2}\Omega_{abc}\omega^{bc}
\end{aligned}$$

with

$$\sigma_a^{ba} = 0$$

As before, the involution in eq.(41) allows us to gauge the spin connection α_b^a so that $\alpha_b^{ac} = 0$. Then structure equation (6) becomes

$$\mathbf{d}\omega_a = \alpha_{ac}^b\omega^c\omega_b - \sigma_a^{bc}\omega_{bc} + \frac{1}{2}\Omega_{abc}\omega^{bc} \tag{58}$$

Writing the co-solder form again as

$$\omega_a = \mathbf{f}_a + \mathbf{h}_a$$

such that $\mathbf{f}_a \equiv \omega_{a|\omega^a=0}$, we have on the $\omega^a = 0$ subspace

$$\mathbf{d}\mathbf{f}_a = \sigma_a^{bc}\mathbf{f}_{bc}$$

This can be solved in the usual way giving σ_a^{bc} in terms of the projected part of $\mathbf{f}_a$ and its y derivatives. Since this solution has the same form at each point x, the expression for σ_a^{bc} remains valid when the x-dependence of $\mathbf{f}_a$ is restored.

For the $\mathbf{f}_a = 0$ subspace,

$$\mathbf{dh}_a = \alpha_{ac}^{b}\omega^c\mathbf{h}_b - \sigma_a^{bc}\mathbf{h}_{bc} + \frac{1}{2}\Omega_{abc}\omega^{bc}$$

In general, this equation determines the spacetime co-torsion, Ω_{abc}, once $\mathbf{h}_a$ is given.

Extending back to the full space, and introducing coordinates as before, we write

$$\begin{aligned}
\mathbf{df}_a &= \partial_\alpha f_a{}^\beta \mathbf{d}x^\alpha \mathbf{d}y_\beta + \partial^\alpha f_a{}^\beta \mathbf{d}y_\alpha \mathbf{d}y_\beta \\
\mathbf{dh}_a &= \partial_\alpha h_{a\beta}\mathbf{d}x^\alpha \mathbf{d}x^\beta + \partial^\alpha h_{a\beta}\mathbf{d}y_\alpha \mathbf{d}x^\beta
\end{aligned}$$

where we can no longer restrict the functional dependence of $f_a{}^\beta$. Now eq.(58) implies

$$\partial_\alpha f_a{}^\beta - \partial^\beta h_{a\alpha} = \alpha_{a\alpha}^{b} f_b{}^\beta - (\sigma_a^{bc} - \sigma_a^{cb})h_{b\alpha}f_c{}^\beta$$

Any solution of this equation for $f_a{}^\beta$ and $h_{a\alpha}$ gives a complete solution. It is clear that solutions do exist, since the $\lambda \neq 1$ condition $\sigma_a^{bc} = 0$ permits the generic solution to hold.

8 Comparisons with previous theories

As mentioned in the introduction, there have been a number of studies of conformal and superconformal gauge theories for $n > 2$. In this section, we compare our results with these other approaches. The gravitational sectors of standard conformal actions fall into three principal types:

1. Chern-Simons action ($n = 3$)
2. Curvature-linear action with compensating fields ($n = 4$)
3. Curvature-quadratic action with compensating fields ($n \geq 4$)

We will treat each of these cases in turn, comparing the results to ours.

8.1 Chern-Simons action

The topological Chern-Simons action, which is intrinsically odd-dim, is of particular interest in 3-dimensional conformal gravity [15], where it becomes

$$S = \int \omega^a_b (\mathbf{d}\omega^b_a - \frac{2}{3}\omega^c_a \omega^b_c) \tag{59}$$

Here ω^a_b is the spin connection of a torsion-free Riemannian geometry. When this action is varied with respect to the solder form $\mathbf{e}^a$, the resulting field equation is

$$\mathbf{D}\mathcal{R}_a = 0 \tag{60}$$

with $\mathcal{R}_a$ defined as in eq.(53). This is precisely the condition for spacetime to be conformally flat in 3-dim [26], so the model is exactly soluble with $\mathbf{e}^a = e^\phi \mathbf{d}x^a$ for any function $\phi(x)$. It has been observed that this action can be derived from the Chern-Simons action for the whole (super)conformal group $O(3,2)$ by imposing the constraints of vanishing curvature and torsion [16]. The same result follows *without* constraints if one replaces the Riemannian spin connection in eq.(59) by the conformal connection,

$$\omega^a_b \to \omega^A_B$$

and performs a Palatini variation [17]. Then all conformal curvatures vanish and gauging the Weyl vector to zero renders the first- and second-order formalisms equivalent again giving condition (60) for conformal flatness. As in the quadratic 4-dim theory (see below), the special conformal gauge field is found to be equal to $\mathcal{R}_a$.

In biconformal space, the 3-dim example does not lead to many simplifications over the general method of solution, though an explicit check in that dimension did confirm our previous results. A generalization of condition (60) obviously arises in this case (and in fact for $n > 3$ as well), since our solution shows the existence of 3-dimensional embedded Ricci-flat spacetimes. While our proof demonstrates the existence of the appropriate gauge choice directly, it is clear that any other x-dependent gauge transformation must lead to a slicing satisfying eq.(60). In addition, the biconformal model permits y-dependent gauge choices. Thus, the field equations of the linear biconformal field theory generalize eq.(60) to an embedding biconformal or phase space. Note further that constraint (1) follows from the field equations as in [17] rather than being imposed as in [16].

8.2 Curvature-linear actions

In standard 4-dimensional Weyl gauge theory [5]-[9], one obtains a Lorentz- and scale invariant linear action through the introduction of a Brans-Dicke-like [22] compensating field $\phi(x)$ in the manner suggested by Deser [3] and Dirac [4]. In close analogy to the geometrical gauge approach of identifying the gauge fields with connections on spacetime developed by Utiyama and Kibble [24] for Poincaré gravity, a Weyl-covariant derivative D_a is built out of the spin connection (usually assumed to be metric compatible and torsion-free) and the Weyl vector. The free (vacuum) action comprises a kinetic term $\phi\Box\phi$, a Yang-Mills-type term $F_{ab}F^{ab}$ for the curl of the Weyl vector $F_{ab}{=}D_{[a}W_{b]}$, a gravitational term $R\phi^2$, and possibly a cosmological term $\Lambda\phi^4$:

$$S = \int \sqrt{-g}(6\phi\Box\phi + \frac{1}{4}F_{ab}F^{ab} - R\phi^2 + \Lambda\phi^4)d^4x \tag{61}$$

Dropping the cosmological term, the corresponding gravitational field equations change the vacuum Einstein equations [8] to the "generalized" Einstein equations:

$$2\phi^2(R^a_b - \frac{1}{2}\delta^a_b R) + 4(D^a\phi D_b\phi - \frac{1}{2}\delta^a_b D^c\phi D_c\phi) = T^a_b$$

where T^a_b is the generalized Maxwell stress tensor:

$$T^a_b = F_{ac}F^{bc} - \frac{1}{4}\delta^b_a F_{cd}F^{cd}$$

In the Einstein gauge one sets $\phi = 1$, so the vacuum Einstein equations, coupled to a spin-1 field, are recovered. However, note that the geometric meaning of $F_{\mu\nu}$ as producing changes in the lengths of transported vectors precludes interpreting $F_{\mu\nu}$ as the Maxwell field [23].

It is also possible to couple n-dimensional *conformal* gravity to compensating fields of conformal weight $-(n-2)/2$ [14]. This approach does not require an explicit gravity term in the action, since the d'Alembertian is built out of derivatives that are also covariant with respect to special conformal transformations and hence contain the special conformal gauge field f^a_b. Then the Lagrangian $\phi\Box\phi$, when broken up based on its conformal invariance properties, contains a term of the form $f^a_a\phi^2$, which under the conventional constraint (1) reduces to $R\phi^2$ when the Weyl vector is gauged to zero. As in the Weyl case, this theory is equivalent to Einstein gravity when ϕ is expressed in a particular gauge using the special conformal gauge freedom.

Biconformal space improves on these results in two important ways: (1) biconformal space does not require compensating fields, and (2) the first conventional constraint, eq.(1) follows from the field equations and is not required as a constraint. A third point developed elsewhere [1] is that it is possible to include electromagnetism without the usual interpretational difficulties.

There are interesting differences between these treatments and our results regarding the effect of constraint (2). In standard conformal gaugings vanishing torsion leads to vanishing Weyl vector as a possible gauge choice and identification of the special conformal gauge field with $\mathcal{R}_a$:

$$f_a = -\left(\frac{1}{n-2}\right)\mathcal{R}_a \tag{62}$$

We find that the same results occur if the biconformal torsion is set to zero and attention is restricted to the $y = 0$ subspace. However, on the full biconformal space, where y is allowed to vary, this solution proves to be inconsistent. Instead, the torsion may be fixed intrisically by the minimal torsion constraint, eq.(28), resulting in a non-trivial y-dependance for the Weyl vector and independence of the projected co-solder form, $k_{a\alpha}$.

To further compare these standard results to the biconformal solution, consider the final form of the biconformal curvatures

$$\begin{aligned}
\mathbf{\Omega}^a_b &= \mathbf{C}^a_b - \Delta^{ac}_{db}\omega_c\omega^d && (63)\\
\mathbf{\Omega}^0_0 &= \lambda\omega_a\omega^a && (64)\\
\mathbf{\Omega}^a &= \omega^a\omega^0_0 && (65)\\
\mathbf{\Omega}_a &= \omega^0_0\omega_a - y_b\mathbf{C}^b_a + \mathbf{D}\mathbf{k}_a && (66)
\end{aligned}$$

The first constraint, eq.(1), already holds for the *spacetime* components of $\mathbf{\Omega}^a_b$, namely, $\mathbf{\Omega}^a_{bcd} = C^a_{bcd}$. In the standard conformal gauging, this constraint also includes the term $-\Delta^{ac}_{db}\omega_c\omega^d$, so that the constraint fixes ω_c. However, in biconformal space, ω_c is independent of ω^d, and even its $y = const.$ projection, $\mathbf{k}_a$, remains arbitrary. If we restrict attention to the $y = 0$ submanifold, the curvatures take the form

$$\begin{aligned}
\mathbf{\Omega}^a_b &= \mathbf{C}^a_b - \Delta^{ac}_{db}\mathbf{k}_c\mathbf{e}^d\\
\mathbf{\Omega}^0_0 &= 0\\
\mathbf{\Omega}^a &= 0\\
\mathbf{\Omega}_a &= \mathbf{D}\mathbf{k}_a
\end{aligned}$$

It is amusing to notice that if we demand that $\mathbf{\Omega}_b^a$ be the Riemannian curvature of the submanifold, then not only do we immediately have

$$\mathbf{k}_a = -\left(\frac{1}{n-2}\right)\mathcal{R}_a \tag{67}$$

but also the second Bianchi identity $\mathbf{D\Omega}_b^a = 0$ implies $\mathbf{D}\mathcal{R}_a = 0$, i.e., the spacetime is conformally Ricci flat. However, $\mathbf{\Omega}_b^a$ is *not* a Riemannian curvature, and satisfies a different Bianchi identity that leaves $\mathbf{k}_a$ arbitrary. These same comments apply when $n = 3$ by simply setting $\mathbf{C}_b^a = 0$.

8.3 Curvature-quadratic actions

In 4 dimensions, all invariant Lagrangians of a Weyl geometry with curvature tensor R^a_{bcd} and Weyl vector W_a have been classified [25]:

$$c_1 F_{ab}F^{ab} + c_2 R^2 + c_3 R_{ab}R^{ab} + c_4 R_{abcd}R^{abcd}$$

The last term may be written as a linear combination of R^2, $R_{ab}R^{ab}$, and $C_{abcd}C^{abcd}$, where C^a_{bcd} is Weyl's conformal tensor defined by eq.(52), or it may be eliminated altogether using the Gauss-Bonnet invariant. All of these actions lead to higher order field equations. For example, Weyl's original free action,

$$S = \frac{1}{4}\int \sqrt{-g}(F_{ab}F^{ab} + R^2)d^4x$$

yields the fourth-order field equation [27]

$$R(R^a_b - \frac{1}{4}\delta^a_b R) + T^a_b = 0$$

As a result of this field equation, the metric is underdetermined. For example, when $T^a_b = 0$, the single condition $R = 0$ already provides a solution. Since almost every metric is scale equivalent to one with $R = 0$, almost every metric is gauge equivalent to a solution.

Following the approach of MacDowell and Mansouri [28] for obtaining Einstein (super-)gravity through squaring the curvatures of the de-Sitter group, Crispim-Romao, Ferber, and Freund [2] and independently Kaku, Townsend, and van Nieuwenhuizen [10] derived Weyl (super-)gravity as a gauge theory of the full conformal group. Gauging $O(4,2)$ under the conventional constraints (1) and (2), vanishing torsion and tracefree curvature, their $R_{abcd}R^{abcd}$-type Lagrangian reduces to

$$C_{abcd}C^{abcd} = R_{ab}R^{ab} - \frac{1}{3}R^2$$

All Weyl vector-dependent terms drop out of the action, whereas eq.(1) renders the special conformal gauge field auxiliary. As in the Chern-Simons case, it is given by $\mathcal{R}_a(\mathbf{e}^a)$. All other possible actions built out of the $O(4,2)$ curvatures under the conventional constraints were shown to reduce to a Weyl geometry [12]. It was concluded that Weyl's theory of gravity is the unique conformally invariant gravity theory in 4 dimensions.

These results were generalized to any dimension $n \geq 4$ by including a compensating field [14], so that the Lagrangian assumes the form

$$e_{[a}^{\ \mu} e_b^{\ \nu} e_c^{\ \alpha} e_{d]}^{\ \beta} \phi^{\frac{2(n-4)}{(n-2)}} R_{\mu\nu}^{\ \ \ ab} R_{\alpha\beta}^{\ \ \ cd}$$

Under the conventional constraints (1) and (2) this reduces to

$$(R_{ab}R^{ab} - \frac{n}{4(n-1)}R^2)\phi^{\frac{2(n-4)}{n-2}}$$

As in the 4-dim case, none of the quadratic action theories provide obvious contact to Einstein gravity, but instead lead to higher derivative field theories. Nonetheless, supersymmetrization of an R^2-action in $n = 10$ [14] and $n = 6$ [29] is an important issue which arises in the study of the low-energy limit of superstrings.

The comments of the preceeding two subsections regarding eq.(60) and the relationship between the special conformal gauge field and $\mathcal{R}_a$ hold here as well (thought it should be noted that eq.(62) follows from the quadratic field equations rather than only as a constraint). Thus, in contrast to these quadratic-curvature theories, the linear biconformal theory:

- provides direct contact with Einstein gravity,
- does not require compensating fields,
- does not require the conventional constraints (1) or (2)

Of course, biconformal space also permits curvature-quadratic actions for any $n > 2$ *without* the use of compensating fields, although these theories are not explored further here.

9 Conclusion

By finding the most general class of biconformal scale-invariant curvature-poly-nomial actions for any dimension $n > 2$, we have overcome the well-known restrictions to the set of possible scale-invariant actions in standard conformal gauge theory imposed by the coupling of the action to the dimension, without the use of compensating fields. All of the displayed polynomial actions rely on the existence of certain biconformally invariant tensors as well as the scaling properties of the connection forms. Since the solder and the co-solder forms that span the $2n$-dimensional biconformal space scale with opposite weights, they provide a manifestly scale-invariant volume element consisting of n solder and n co-solder forms. We also displayed a Yang-Mills type scale invariant dual action, which hinges on the existence of a scale-invariant biconformal dual operator.

For the most general linear action we computed and solved the field equations by imposing them onto the minimal torsion biconformal structure equations. With one exceptional case, all solutions have the following properties:

1. The full $2n$-dim space has a symplectic form, and is therefore almost complex and almost Kähler.

2. There are two n-dim involutions. The first leads to a foliation by conformally flat manifolds spanned by weight -1 co-solder forms. The second leads to a foliation by equivalent Ricci-flat Riemannian spacetimes spanned by the weight $+1$ solder forms. The Riemann curvature is computed from the solder form alone, despite the inclusion of minimal torsion, general co-torsion and a general Weyl vector, and the spacetime is Ricci-flat despite an arbitrary cosmological constant.

3. The full $2n$-dim minimal torsion solutions are fully determined by two fields, each defined entirely on the n-dimensional Riemannian spacetimes: the solder form $e_\mu{}^a(x)\mathbf{d}x^\mu$ and a symmetric tensor field, $k_{\alpha\beta}(x)$.

For the single special case, $\lambda = 1$, there still exists a foliation by Ricci-flat Riemannian spacetimes, but the minimal field content includes the one field beyond the solder form and $h_{a\alpha}$: the co-solder coefficient $f_a{}^\beta$, which is coupled to $h_{a\alpha}$ by a differential equation.

Certain important subclasses of biconformal spaces described in [1] turned out to be special cases of the general solution. Because of the symmetry between solder and co-solder form, analogous results to the ones obtained hold for co-torsion-free biconformal spaces, e.g. spacetime sector flatness. Spaces of vanishing torsion and co-torsion are conformally flat.

A Biconformal Gauging

The conformal (Möbius) group $C(n)$ is the group of transformations preserving angles or ratios of infinitesimal lengths when acting on an n-dim space or, equivalently, leaving the null interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$$

with $\eta_{\mu\nu} = diag(-1, 1 \ldots 1)$, $\mu, \nu = 1 \ldots n$, invariant. While $C(2)$ is the infinite-dimensional diffeomorphism group of the plane, the conformal group for $n > 2$ is a Lie group of dimension $\frac{1}{2}(n+1)(n+2)$ and locally isomorphic to the pseudo-orthogonal group $O(n, 2)$. It can be shown [30] that $C(n)$ is the projective group $O(n, 2)/\{1, -1\}$ of rays through the origin in $O(n, 2)$. It possesses a real linear representation in $\mathbf{R}^{n+2}$, a complex linear representation in $\mathbf{R}^n$, and a real nonlinear representation in an n-dim compact spacetime (Möbius space).

Biconformal space was first introduced in [1] using methods similar to the geometric construction of general relativity as an ECSK theory. In the standard Poincaré gauge theory of gravitation one postulates the invariance of some action integral under local Poincaré transformations [24]. The field equations are derived by "soldering" the Lorentz fibers to the base manifold, i.e. by identifying those gauge fields (connection forms) that correspond to the translation generators of the Poincaré group with an orthonormal basis $\{\mathbf{e}^a\}$. This gauging was later recognized as being equivalent to Cartan's orthonormal frame bundle formalism [31]. In this formalism, a homogeneous space is first constructed as the quotient space of a group G and a subgroup with trivial core, i.e. a subgroup G_0 that itself contains no subgroup which is normal in G other than the identity. This subgroup will act as the isotropy subgroup of any point in the orbit space G/G_0. The group action on this space is effective (only the identity of the group acts as the identity transformation) and transitive (only one orbit). The orbit space is a manifold with a Lie-algebra-valued connection if the group is a Lie group. The affine connection of this frame bundle $\{\pi : G \rightarrow G/G_0\}$ is then generalized to a Cartan connection by including curvatures in the structure equations of the group. Holonomy considerations require these curvatures to be horizontal, i.e. bilinear in the base connections. The formalism provides *a priori* locally group symmetric geometries without requiring an action integral.

In this way, Minkowski space is built as the quotient space of the Poincaré group acting on $\mathbf{R}^4$and the Lorentz group $O(3,1)$. The curvature 2-forms associated with Lorentz transformations and translations, Riemann curvature $\mathbf{R}^a_b$ and torsion $\mathbf{T}^a$, respectively, are defined through the Poincaré structure equations:

$$\begin{aligned}\mathbf{R}^a_b &= \mathbf{d}\omega^a_b - \omega^c_b \wedge \omega^a_c \\ \mathbf{T}^a &= \mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \omega^a_b\end{aligned}$$

The resulting spacetime is a curved four-dimensional manifold with torsion. The result can be generalized to n dimensions as well as applied to manifolds with topology other than the usual $\mathbf{R}^n$ topology. *Any* action constructed within this locally Poincaré invariant geometry, such as the Einstein-Hilbert action

$$S = \int \eta^{a_1 b} \mathbf{R}^{a_2}_b \wedge \mathbf{e}^{a_3} \wedge \cdots \wedge \mathbf{e}^{a_n} \varepsilon_{a_1 \ldots a_n} = \int \sqrt{-g} R \; d^n x$$

provides a field theory.

In the frame bundle formalism language the standard conformal gauge theories [2]-[14] correspond to a quotienting of the conformal group by the inhomogeneous Weyl group generated by Poincaré transformations and dilations [32]. While this construction retains the largest possible continuous symmetry on the fibers, it does not take the discrete symmetry of the conformal algebra into account, according to which translation and co-translation generators are essentially interchangeable.

Biconformal space is the $2n$-dimensional homogeneous space obtained by quotienting the conformal group $C(n)$ acting on Möbius space by the homogeneous Weyl group C_0, consisting of Lorentz transformations and dilations. Thus, symmetry of the fibers is exchanged for increased coordinate freedom for the base manifold. In this gauging, the conformal translation and co-translation generators are treated on an equal footing: Their associated connection forms span the base space together. When broken up into components based on their biconformal covariance properties, the $O(n,2)$ curvatures defined through the conformal structure equations (4)-(7) are bilinear in the these connection forms. There are no *a priori* conditions on the torsion and the curvature.

Definition 1 *Let $A, B, ... = 0, ..., n$ and $a, b, ... = 1, ..., n$. A biconformal space is a principal fiber bundle $\pi : C \to \mathcal{B}$ with conformal connection $\omega^A_B = \{\omega^a_b,\ \omega^a, \omega_a, \omega^0_0\}$, where π is the canonical projection of the $(n+1)(n+2)/2$-dimensional conformal bundle onto the $2n$-dimensional base manifold $\mathcal{B}$ induced by C/C_0, where the structure (or symmetry or gauge) group C_0 is the Weyl group of an n-dimensional Minkowski space.*

Biconformal space possesses a preferred orthonormal basis $\{\omega^a, \omega_a\}$ defined through the conformal Killing metric g , so that $g(\omega^a, \omega^b) = 0$, $g(\ \omega_a =$ 0, and $\omega_b) = 0$ and $g(\omega^a,\ \omega_b) = \delta^a_b$. It provides a natural nondegenerate, invariant 2-form $\omega^a \wedge \omega_a$. In the generic case of a biconformal space subject to the linear action and the minimal torsion constraint discussed in this paper, the 2-form is closed and hence symplectic. By a well-known theorem [33] it is always possible to construct on a symplectic manifold an almost complex structure and a Kähler metric. Therefore, the field equations arising from the linear action constrain the biconformal space to be almost Kähler.

As a result of the increased dimension of biconformal space, there are many new fields that could be identified with the electromagnetic potential or other internal symmetries.

B The projected co-solder form, $\mathbf{f}_a$

We first define the projected co-solder form

$$\mathbf{f}_a \equiv \omega_{a|\omega^a=0}$$

Since the involution (41) allowed us to single out a set of n biconformal coordinates $\{x^\mu\}$ for the weight $+1$ sector such that $\omega^a = e_\mu{}^a \mathbf{d}x^\mu$, we can find a complimentary set of n coordinates $\{z_\mu\}$ for the weight -1 sector so that $\mathbf{f}_a$ is of the form

$$\mathbf{f}_a = f^\mu_a(x,z)\mathbf{d}z_\mu \tag{68}$$

This form shows that $\mathbf{f}_a$ is necessarily involute, since $\mathbf{df}_a$ is at least linear in $\mathbf{d}z_\mu$ and $\mathbf{d}z_\mu = f_\mu{}^a \mathbf{f}_a$. On each $x_0 = const.$ submanifold we can gauge ω^a_b and ω^0_0 to zero which implies $\mathbf{df}_a = 0$ or, by the converse of the Poincaré Lemma,

$$\mathbf{f}_a = \mathbf{d}\theta_a$$

for some 0-form $\theta_a(x_0, z)$. When the last equation is extended to the full biconformal space, θ_a becomes a function of x and z, and therefore

$$\begin{aligned} \mathbf{f}_a &= \mathbf{d}\theta_a + \chi_{a\mu}\mathbf{d}x^\mu \\ &= (\partial_\mu\theta_a + \chi_{a\mu})\mathbf{d}x^\mu + (\partial^\mu\theta_a)\mathbf{d}z_\mu \end{aligned} \tag{69}$$

where ∂_μ and ∂^μ denote derivatives with respect to x^μ and z_μ, respectively. Since by eq.(68) $\mathbf{f}_a$ can have no part proportional to $\mathbf{d}x^\mu$, this implies

$$\chi_{a\mu} = -\partial_\mu\theta_a$$

so that

$$\mathbf{f}_a = (\partial^\mu\theta_a)\mathbf{d}z_\mu$$

Defining a new set of coordinates by

$$y_\mu \equiv e_\mu^{\ a}\theta_a$$

and regarding θ_a as a function of x and y, we have

$$\begin{aligned} \mathbf{f}_a &= (\partial^\mu\theta_a)\mathbf{d}z_\mu \\ &= (e_a^{\ \alpha}\frac{\partial y_\alpha}{\partial z_\mu})\mathbf{d}z_\mu \\ &= e_a^{\ \beta}(\mathbf{d}y_\beta - \frac{\partial y_\beta}{\partial x^\alpha}\mathbf{d}x^\alpha) \end{aligned}$$

The partial derivative $\frac{\partial y_\beta}{\partial x^\alpha}$ is computed holding z_μ constant. Writing this partial as a function $\psi_{\alpha\beta}(x^\mu, y_\nu)$, $\mathbf{f}_a$ takes the desired form,

$$\mathbf{f}_a = e_a^{\ \beta}\mathbf{d}y_\beta - \psi_{a\beta}\mathbf{d}x^\alpha$$

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