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SCALE-INVARIANT PHASE SPACE AND THE CONFORMAL GROUP

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1. Introduction

Certain difficulties associated with any theory of quantum gravity stem from the different mathematical framing of our gravity and quantum theories. While general relativity is based on a real, deterministic, differentiable manifold structure, the elements of quantum theory are complex-valued and probabilistic. Consider two ways to reconcile these differences by placing one of the theories into the context of the other:

A. Find an abstract quantum system (such as quantum string) with general coordinate invariance and a massless, spin-2 state. Gravitation may emerge in terms of excited states of the system.
B. Find a curved geometry (necessarily non-Riemannian) which naturally places the same restrictions on measurement as does quantum theory. Quantum theory then emerges from the geometry.

In this work and elsewhere [1, 2] we display some progress toward approach B. The geometric arena for our approach is the biconformal fiber bundle, defined as the gauge bundle of the 4-dim conformal group over an 8-dim base space. Here, we show how biconformal space can be used to characterize the dynamics of Hamiltonian systems. Specifically, we show that the classical Hamiltonian dynamics of a point particle is equivalent to the specification of a 7-dim hypersurface in flat biconformal space and the consequent necessary existence of a set of preferred curves. The central importance of this result is in definitively establishing the physical interpretation of conformal gauging. Given the result of section 3, that a classical Hamiltonian system generates a class of biconformal spacetimes, there can remain little doubt as to the meaning of the geometric variables.

In [1, 2], we go on to derive the general solution for flat biconformal space and show how that solution predicts the electromagnetic potential including its form, $\alpha(\mathbf{x})$ as a vector field on 4-dim spacetime, its gauge dependence $\alpha'(\mathbf{x}) = \alpha(\mathbf{x}) + \mathbf{d}\phi$, its minimal coupling $p_a \longrightarrow p_a - \lambda\alpha_a$, and the correct equation of motion for a charged particle moving under its influence. This second work therefore provides part of the consistent realization of Weyl's goal of expressing the electrodynamics of a charged particle in terms of a Weyl geometry [3]. Full realization of the goal requires in addition the specification of Maxwell's field equations in terms of the biconformal variables. Reference [2] contains an examination of measurement when the biconformal space is no longer dilationally flat. We show there that the rules of quantum mechanics provide a consistent way to make measurements of physical size change in dilationally curved biconformal space.

To provide the background setting for the present discussion of Hamiltonian dynamics in this new geometric context, we describe below the conformal group and its gauging to give the structure equations of biconformal space, and display the flat solution for a local frame field satisfying those structure equations. Then, we establish our central claim in three steps. First, we show that the classical Hamiltonian description of a point particle defines a class of 7-dim differential geometries with structure equations of manifestly biconformal type. Next, we show that the introduction of an independent eighth coordinate in place of the Hamiltonian allows, up to locally symplectic changes of basis, definition of a unique flat biconformal geometry. Finally, specializing to the hypersurface determined by setting the eighth coordinate equal to any given Hamiltonian, we show the necessary existence of a preferred set of curves in the biconformal space satisfying the Hamiltonian equations of motion. This unique equivalence between the hypersurfaces in flat biconformal geometry and the Hamiltonian system provides a clear physical interpretation of the geometric variables of biconformal space.

2. Biconformal space

Recall the 15 transformations comprising the 4-dim conformal group:

1. Lorentz transformations (6)
2. Translations (4)
3. Inverse translations (4)
4. Dilation (1)

These preserve ratios of infinitesimal lengths and therefore permit global rescalings, which we identify as changes of units – an obvious symmetry of physical laws. It is important to clearly distinguish global or local changes of units, i.e., Weyl's original gauge transformations, from the *physical* size changes that are the consequence of the local dilational gauge vector, the Weyl vector, having nonvanishing curl. The clear imperative for any physical theory based on the gauging of the conformal group is that any such physical size changes be given an interpretation consistent with known measurements. These issues will be addressed further below, and in more detail in [2].

Extending the global conformal transformations to local symmetries gives conformal geometry. The extension is accomplished by introducing one gauge 1-form, ω_B^A, for each generator of the group. We choose the linear $O(4,2)$ representation for our notation, with $(A, B, \ldots) = (0, 1, \ldots, 5)$. Letting boldface or Greek symbols denote forms and $(a, b, \ldots) = (1, \ldots, 4)$, ω_B^A has four independent Lorentz invariant parts: the *spin connection*, ω_b^a, the *solder form*, ω_0^a, the *co-solder form*, ω_a^0, and the *Weyl vector*, ω_0^0. Orthonormality of the basis requires

$$\begin{aligned}
\omega_b^a &= -\eta_{bc}\eta^{ad}\omega_d^c \\
\omega_0^5 &= \omega_5^0 = 0 \\
\omega_5^a &= -\eta^{ab}\omega_b^0 \\
\omega_a^5 &= -\eta_{ab}\omega_0^b
\end{aligned} \tag{1}$$

The structure constants of the conformal Lie algebra now lead immediately to the Maurer-Cartan structure equations. These are simply

$$\mathbf{d}\omega_B^A = -\omega_B^C \wedge \omega_C^A + \mathbf{\Omega}_B^A \tag{2}$$

When broken into parts based on Lorentz transformation properties, eq.(2) gives:

$$\begin{aligned}
\mathbf{d}\omega_b^a &= -\omega_b^c \wedge \omega_c^a - \omega_b^0 \wedge \omega_0^a - \eta_{bc}\eta^{ad}\omega_0^c \wedge \omega_d^0 + \mathbf{\Omega}_b^a \\
\mathbf{d}\omega_0^a &= -\omega_0^0 \wedge \omega_0^a - \omega_0^b \wedge \omega_b^a + \mathbf{\Omega}_0^a \\
\mathbf{d}\omega_a^0 &= -\omega_a^0 \wedge \omega_0^0 - \omega_a^b \wedge \omega_b^0 + \mathbf{\Omega}_a^0 \\
\mathbf{d}\omega_0^0 &= -\omega_0^a \wedge \omega_a^0 + \mathbf{\Omega}_0^0
\end{aligned} \tag{3}$$

Notice that if we set $\omega_a^0 = \omega_0^0 = 0$, we recover the usual structure equations of 4-dim Riemannian geometry. The meaning of eqs.(2) depends on the choice of the base manifold. For a 4-dim base space the curvatures would be required to take the form

$$\mathbf{\Omega}_B^A = \mathbf{\Omega}_{Bcd}^A\, \omega_0^c \wedge \omega_0^d \qquad\qquad (A, B = 0, 1, 2, 3, 4) \tag{4}$$

The fibres would then span the inhomogeneous Weyl group consisting of Lorentz transformations, (co-) translations, and dilations. This gauge theory has been extensively researched [4]. Quite generally, the four 1-forms ω_a^0 are auxiliary and may be algebraically removed from the problem [5]. The resulting geometry is always a Weyl geometry (Riemannian plus dilations) and the action reduces to a linear combination of the square of the conformal curvature and the square of the field strength of the Weyl vector. Thus, no additional structure is gained in the 4-dim formulation by extending from the 11-parameter Weyl group to the 15-parameter conformal group.

However, the approach to be followed here differs strikingly from this standard picture. The conformal group, $\mathcal{C}$, posesses eight translational generators, distinguishable from the remaining generators by their fixed points. These remaining generators form the isotropy subgroup, $\mathcal{C}_0$, and the natural base space $\mathcal{C}/\mathcal{C}_0$ is 8-dimensional. The curvatures therefore take the form

$$\mathbf{\Omega}_B^A = \mathbf{\Omega}_{Bcd}^A\, \omega_0^c \wedge \omega_0^d + \mathbf{\Omega}_{Bd}^{Ac}\, \omega_c^0 \wedge \omega_0^d + \mathbf{\Omega}_B^{Acd}\, \omega_c^0 \wedge \omega_d^0 \tag{5}$$

with Lorentz transformations and dilations as the fibre symmetry. The 8-dim base manifold spanned by ω_0^a and ω_a^0 will be called *biconformal space* and the full fiber bundle the biconformal bundle.

Notice that the 8-dim form of the curvature allows some of the terms in the structure equations to be incorporated into the curvature. Specifically, we define

$$\begin{aligned}
\overline{\mathbf{\Omega}}_0^0 &\equiv -\omega_0^a \wedge \omega_a^0 + \mathbf{\Omega}_0^0 \\
\overline{\mathbf{\Omega}}_b^a &\equiv -\omega_b^0 \wedge \omega_0^a - \eta_{bc}\eta^{ad}\omega_0^c \wedge \omega_d^0 + \mathbf{\Omega}_b^a
\end{aligned} \tag{6}$$

so that the corresponding structure equations reduce to

$$\begin{aligned}\overline{\mathbf{\Omega}}_0^0 &= \mathbf{d}\omega_0^0 \\ \overline{\mathbf{\Omega}}_b^a &= \mathbf{d}\omega_b^a + \omega_b^c \wedge \omega_c^a\end{aligned} \tag{7}$$

If the curvatures of the solder and co-solder forms (i.e., the torsion and co-torsion) leave the remaining two structure equations in involution then the full set of structure equations are those of a pair of scale- and Lorentz-conjugate 4-dim Weyl geometries. The scale and Lorentz transformations have inverse effects on the two 4-dim subspaces, with covariant, weight +1 vectors in one space corresponding to contravariant, weight −1 vectors in the other. These transformations are not independent, e.g., if a given Lorentz transformation acts on one of the 4-dim subspaces, the inverse of the *same* Lorentz transformation acts on the conjugate space.

It can be shown [1] that for vanishing curvature, $\mathbf{\Omega}_B^A = 0$, there locally exist coordinates x^a, y_a such that

$$\begin{aligned}\omega_0^0 &= \alpha_a(x)\mathbf{d}x^a - y_a\mathbf{d}x^a \equiv W_a\mathbf{d}x^a \\ \omega_0^a &= \mathbf{d}x^a \\ \omega_a^0 &= \mathbf{d}y_a - (\alpha_{a,b} + W_aW_b - \tfrac{1}{2}W^2\eta_{ab})\mathbf{d}x^b \\ \omega_b^a &= (\delta_d^a\delta_b^c - \eta^{ac}\eta_{bd})W_c\mathbf{d}x^d\end{aligned} \tag{8}$$

where $\alpha_{a,b}$ denotes the partial of α_a with respect to x^b. There are several points to note about these solutions:

(1) The Weyl vector, W_a, while having only four nonvanishing components in this basis, is actually a function of y_a as well. This necessary dependence is the reason for the failure of Weyl's original scale-invariant theory of electromagnetism [2].

(2) The solutions hold for vanishing conformal curvatures, not for the barred curvatures. Vanishing barred curvatures would lead to a flat pair of conjugate Riemannian spacetimes, while vanishing conformal curvatures correspond to constant curvature Weyl spacetimes.

(3) This form of the solution is preserved by 4-dim gauge transformations, $\phi(x)$. The gauge transformation must be associated with the undetermined vector field $\alpha_a(x)$. For later use we note that $\omega_0^a \wedge \omega_a^0 = \mathbf{d}x^a \wedge \mathbf{d}y_a - \alpha_{a,b}\mathbf{d}x^a \wedge \mathbf{d}x^b$ reduces to $\omega_0^a \wedge \omega_a^0 = \mathbf{d}x^a \wedge \mathbf{d}y_a$ if and only if α_a is pure gauge.

3. A geometry for Hamiltonian mechanics.

Leaving biconformal geometry for the moment, we turn to a geometric approach to classical Hamiltonian dynamics. We first show that the action of a classical system may be used to generate biconformal spaces. Then, we give a unique prescription for generating a flat biconformal space. Begin with the Hilbert form

$$\omega = H\mathbf{d}t - p_i\mathbf{d}x^i \tag{9}$$

where $H = H(p_i, x^i, t)$. The integral of ω is the action functional. The exterior derivative of ω may always be factored:

$$\begin{aligned}\mathbf{d}\omega &= -\frac{\partial H}{\partial x^i}\mathbf{d}x^i \wedge \mathbf{d}t - \frac{\partial H}{\partial p_i}\mathbf{d}p_i \wedge \mathbf{d}t - \mathbf{d}x^i \wedge \mathbf{d}p_i \\ &= -(\mathbf{d}x^i - \frac{\partial H}{\partial p_i}\mathbf{d}t) \wedge (\mathbf{d}p_i + \frac{\partial H}{\partial x^i}\mathbf{d}t)\end{aligned} \tag{10}$$

Therefore, if we define

$$\begin{aligned}\omega^i &\equiv (\mathbf{d}x^i - \frac{\partial H}{\partial p_i}\mathbf{d}t) \\ \omega_i &\equiv (\mathbf{d}p_i + \frac{\partial H}{\partial x^i}\mathbf{d}t)\end{aligned} \tag{11}$$

then we can write

$$\mathbf{d}\omega = -\omega^i \wedge \omega_i \tag{12}$$

This factoring is clearly preserved by local symplectic transformations of the 6-basis (ω^i, ω_i), as well as reparameterizations of the time. Obviously these transformations include the usual canonical transformations of coordinates as a special case. One class of such allowed transformations of basis is achieved by the addition of $c_{ij}\mathbf{d}x^j$, with $c_{ij} = c_{ji}$, to ω_i Here we take $c_{ij} = 0$, but in section 4 we show the existence of a unique choice of the 4-dim extension of c_{ij} that leads to flat biconformal space.

Continuing, we wish to show that any choice for ω^i and ω_i leads to a *dilationally* flat biconformal space. Taking the exterior derivatives of ω^i and ω_i we find:

$$\begin{aligned}
\mathbf{d}\omega^i &= \frac{\partial^2 H}{\partial x^j \partial p_i}\mathbf{d}x^j \wedge \mathbf{d}t + \frac{\partial^2 H}{\partial p_j \partial p_i}\mathbf{d}p_j \wedge \mathbf{d}t \\
\mathbf{d}\omega_i &= -\frac{\partial^2 H}{\partial x^i \partial p_j}\mathbf{d}p_j \wedge \mathbf{d}t - \frac{\partial^2 H}{\partial x^j \partial x^i}\mathbf{d}x^j \wedge \mathbf{d}t.
\end{aligned} \tag{13}$$

If we define

$$\begin{aligned}
\omega^i_j &\equiv -\frac{\partial^2 H}{\partial x^j \partial p_i}\mathbf{d}t \\
\mathbf{\Omega}^i &\equiv \frac{\partial^2 H}{\partial p_j \partial p_i}\mathbf{d}p_j \wedge \mathbf{d}t \\
\mathbf{\Omega}_i &\equiv -\frac{\partial^2 H}{\partial x^j \partial x^i}\mathbf{d}x^j \wedge \mathbf{d}t
\end{aligned} \tag{14}$$

then eqs.(13) are simply

$$\begin{aligned}
\mathbf{d}\omega^i &= -\omega^j \wedge \omega^i_j + \mathbf{\Omega}^i \\
\mathbf{d}\omega_i &= -\omega^j_i \wedge \omega_j + \mathbf{\Omega}_i
\end{aligned} \tag{15}$$

Finally differentiating $\mathbf{d}\omega^i_j$, using the fact that $\mathbf{d}\omega^k_j \wedge \mathbf{d}\omega^i_k = 0$ and defining the final curvature as

$$\mathbf{\Omega}^i_j \equiv \mathbf{d}\omega^i_j + (\delta^i_k \delta^l_j - \delta_{jk}\delta^{il})\omega^k \wedge \omega_l \tag{16}$$

gives the complete set of structure equations:

$$\begin{aligned}
\mathbf{d}\omega^i_j &= -\omega^k_j \wedge \omega^i_k - \omega^i \wedge \omega_j + \delta_{jk}\delta^{il}\omega^k \wedge \omega_l + \mathbf{\Omega}^i_j \\
\mathbf{d}\omega^i &= -\omega^j \wedge \omega^i_j + \mathbf{\Omega}^i \\
\mathbf{d}\omega_i &= -\omega^j_i \wedge \omega_j + \mathbf{\Omega}_i \\
\mathbf{d}\omega &= -\omega^i \wedge \omega_i
\end{aligned} \tag{17}$$

The final equation, for $\mathbf{d}\omega$, and the dependence of the curvatures on *both* ω^i and ω_i clearly show this to be a subspace of a biconformal space. Since we may expand

$$\omega = H\mathbf{d}t - p_i \mathbf{d}x^i = (H - p_i \frac{\partial H}{\partial p_i})\mathbf{d}t - p_i \omega^i \tag{18}$$

we could also have included the Weyl vector terms present in eqs.(3) in the definitions of the torsion and co-torsion. Therefore, since ω_i depends on $\mathbf{d}p_i$, biconformal space may be viewed as a form of scale-invariant phase space. We note that this interpretation agrees in some ways with earlier proposals [6] relating phase space and Weyl geometry. These proposals, however, lack the full geometric structure of conformal gauge theory, do not demonstrate the intrinsically biconformal structure of Hamiltonian systems, and use a different inner product than that proposed in [2].

We next show the relationship of the geometry above to classical mechanics.

In the 7-dim geometry defined above, the combined involution of ω^i and ω_i in eqs.(14, 15) allows us to set $\omega^i = \omega_i = 0$, thereby singling out a fibration of the bundle by 1-dim subspaces, i.e., the classical paths of motion. These simply give Hamilton' s equations of motion:

$$\begin{aligned} \mathbf{d}x^i &= \frac{\partial H}{\partial p_i}\mathbf{d}t \\ \mathbf{d}p_i &= -\frac{\partial H}{\partial x^i}\mathbf{d}t \end{aligned} \tag{19}$$

The Frobenius theorem guarantees the existence of solutions to these equations for the paths. The structure equations then reduce to

$$\mathbf{d}\omega^i_j = \mathbf{d}\omega = 0 \tag{20}$$

which are identically satisfied on curves.

On the full bundle, the condition that ω be exact is the Hamiltonian-Jacobi equation, since we may write $\omega = \mathbf{d}S$. Substituting for ω and expanding $\mathbf{d}S$ then gives

$$-H\mathbf{d}t + p_i\mathbf{d}x^i = \frac{\partial S}{\partial x^i}\mathbf{d}x^i + \frac{\partial S}{\partial p_i}\mathbf{d}p_i + \frac{\partial S}{\partial t}\mathbf{d}t \tag{21}$$

so that

$$\frac{\partial S}{\partial p_i} = 0 \tag{22}$$

$$\frac{\partial S}{\partial x^i} = p_i \tag{23}$$

and

$$H(\frac{\partial S}{\partial p_i}, x^i, t) = -\frac{\partial S}{\partial t}. \tag{24}$$

Therefore, since ω is the Weyl vector of the generated biconformal space, the Hamilton-Jacobi equation holds if and only if the Weyl vector is pure gauge, $\omega = \mathbf{d}S$. A gauge transformation reduces the Weyl vector to zero. Since the dilational curvature is also always zero for the geometry built from a Hamiltonian system, we also have

$$\omega^i \wedge \omega_i = 0, \tag{25}$$

implying three linear dependences between these six solder forms. Together with the linear dependence of $\mathbf{d}H$, the Hamilton-Jacobi equation therefore specifies a 4-dim subspace of the full biconformal space.

4. Flat biconformal space and the Hamiltonian geometry

Next, we show how to specify a unique *flat* biconformal space for a given Hamiltonian system. This is achieved by first placing the Hamiltonian geometry in an 8-dim setting, then by being careful in the choice of the solder and co-solder forms.

First, replace $H = H(p_i, x^i, t)$ by an independent variable, $-p_4$. Then

$$\omega^{\mathbf{0}}_{\mathbf{0}} = -p_a\mathbf{d}x^a \tag{26}$$

and

$$\mathbf{d}\omega^{\mathbf{0}}_{\mathbf{0}} = -\mathbf{d}x^a \wedge \mathbf{d}p_a, \tag{27}$$

where we have reserved the symbols ω, ω_i and ω^i for the special case when $p_4 = -H(p_i, x^i, t)$. This time, we make our identification of the solder and co-solder forms ω^a_0 and ω^0_a by comparing the expression for $\mathbf{d}\omega^0_0$ to the flat biconformal solution. There is clearly a *unique* extension to a flat biconformal space, given by setting $\alpha_{[a,b]} = 0$ in the general flat solution, as noted above. A scale transformation then removes α_a altogether so that, in terms of the usual phase-space coordinates, the frame field of the flat biconformal extension becomes

$$\begin{aligned} \omega^0_0 &= -p_a\mathbf{d}x^a \\ \omega^a_0 &= \mathbf{d}x^a \\ \omega^0_a &= \mathbf{d}p_a - (p_a p_b - \tfrac{1}{2}p^2\eta_{ab})\mathbf{d}x^b \\ \omega^a_b &= (\delta^a_d\delta^c_b - \eta^{ac}\eta_{bd})p_c\mathbf{d}x^d \end{aligned} \tag{28}$$

If we now restrict to the hypersurface $p_4 = -H(p_i, x^i, t)$ we recover the Hamiltonian system, and have shown it to lie in a unique flat biconformal space. Note that the symmetric coefficient

$$c_{ab} = p_a p_b - \tfrac{1}{2} p^2 \eta_{ab} \tag{29}$$

provides a symplectic change of basis with respect to the manifestly closed and nondegenerate 2-form, $\mathbf{d}\omega_0^0 = -\omega_0^a \wedge \omega_a^0 = -\mathbf{d}x^a \wedge \mathbf{d}p_a$. The differential of ω_0^0 is still seen to factor into the form of eq.(10) either directly by differentiation or by substitution of $p_4 = -H(p_i, x^i, t)$ into $-\omega_0^a \wedge \omega_a^0$. However, while the involution of eq.(13) for ω_i and ω^i still holds it does not mean that ω_0^a and ω_a^0 vanish. Instead, the classical curves are found by first writing the frame field in terms of ω^i and ω_i:

$$\begin{aligned}
\omega_0^0 &= (H - p_i \frac{\partial H}{\partial p_i})\mathbf{d}t - p_i \omega^i \\
\omega_0^i &= \omega^i + \frac{\partial H}{\partial p_i}\mathbf{d}t \\
\omega_0^4 &= \mathbf{d}t \\
\omega_i^0 &= \omega_i - (p_i p_j + \tfrac{1}{2}(H^2 - p^2)\delta_{ij})\omega^j + (p_i H - \frac{\partial H}{\partial x^i} - \tfrac{1}{2}(H^2 - p^2)\delta_{ij}\frac{\partial H}{\partial p_j})\mathbf{d}t \\
\omega_4^0 &= (H p_i - \frac{\partial H}{\partial x^i})\omega^i + (H \frac{\partial H}{\partial p_i} p_i - \tfrac{1}{2}(H^2 - p^2) - \frac{\partial H}{\partial x^i}\frac{\partial H}{\partial p_i})\mathbf{d}t \\
\omega_j^i &= (\delta_n^i \delta_j^m - \delta^{im}\eta_{jn}) p_m (\omega^n + \frac{\partial H}{\partial p_n}\mathbf{d}t) \\
\omega_4^i &= -H\omega^i + (\delta^{ij} p_j - H \frac{\partial H}{\partial p_i})\mathbf{d}t \\
\omega_i^4 &= \delta_{ij}\omega_4^j
\end{aligned} \tag{30}$$

The simple example of a free particle is instructive. Setting $\omega^i = \omega_i = 0$ and $H^2 = p^2 + m^2 \neq 0$, the equations above reduce to:

$$\begin{aligned}
\omega_0^0 &= \frac{m^2}{H}\,\mathbf{d}t \\
\omega_0^a &= \frac{\eta^{ab} p_b}{m^2}\,\omega_0^0 \\
\omega_a^0 &= \tfrac{1}{2} p_a\,\omega_0^0 \\
\omega_b^a &= 0
\end{aligned}$$

where use of ω_0^0 in place of $\mathbf{d}t$ simplifies the expressions. The solder and co-solder forms are proportional to the displacement and momentum, respectively.

Also, we see that the involution required to specify the classical paths necessarily exists. Since the biconformal base space is spanned by the eight forms $\mathbf{d}x^a$ and $\mathbf{d}p_a$, the only way the involution could fail is if there was an independent $\mathbf{d}t \wedge \mathbf{d}p_4$ term in the torsion or co-torsion of the $p_4 = -H$ hypersurface. But since $\mathbf{d}H$ is given *a priori* in terms of the other seven forms, this cannot happen.

We conclude: *the Hamiltonian dynamics of a point particle is equivalent to the specification of a hypersurface, $y_4 = y_4(y_i, x^a)$, in a flat biconformal space, and the consequent necessary existence of preferred curves in the hypersurface.*

The principal significance of this result is the unambiguous identification of the geometric quantities which arise in the gauging of the 15-dim conformal group. While previous treatments of this gauge theory always lead to a 4-dim Weyl geometry in which the special conformal transformations (i.e., the inverse translations) are auxiliary, the current approach includes additional freedom beyond that of a Weyl geometry. This extra freedom is now seen to correspond to the inclusion of momentum variables in the physical description.

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