2000

Actions for Biconformal Matter

andre wehner

James Thomas Wheeler
Utah State University

Recommended Citation

WehnerS, A. and Wheeler, J. T., Actions for Biconformal Matter, (Jan. 2000), 5 pp http://arxiv.org/pdf/hep-th/0001191

Actions for Biconformal Matter

André Wehner* and James T. Wheeler†

February 1, 2008

Abstract

We extend $2n$-dim biconformal gauge theory by including Lorentz-scalar matter fields of arbitrary conformal weight. For a massless scalar field of conformal weight zero in a torsion-free biconformal geometry, the solution is determined by the Einstein equation on an n-dim submanifold, with the stress-energy tensor of the scalar field as source. The matter field satisfies the n-dim Klein-Gordon equation.

1 Introduction

Recently, we developed a new gauge theory of the conformal group, which solved many of the problems typically associated with scale invariance [1]. In particular, this new class of *biconformal geometries* has been shown to resolve the problem of writing scale-invariant vacuum gravitational actions in arbitrary dimension without the use of compensating fields [2]. In the cited work, we wrote the most general linear vacuum action and completely solved the resulting field equations subject only to a minimal torsion assumption. We found that all such solutions were foliated by equivalent n-dimensional Ricci-flat Riemannian spacetimes.

Reference [2] left an open question: How are matter fields coupled to biconformal gravity? *A priori*, it is not at all obvious that any action for biconformal matter permits the same embedded n-dimensional Riemannian structure that occurs for the vacuum case since biconformal fields are functions of all $2n$-dimensions. Indeed, in the case of standard n-dimensional conformal gauging ([3]-[4]), we generally require compensating fields to recover the Einstein equation with matter (see, for example, [5]-[7]). To answer this question for biconformal space, we extend the results of [2] by introducing a set of Klein-Gordon-type fields ϕ^m of conformal weight m into the theory.[1] Using the Killing metric intrinsic to biconformal space, we write the natural kinetic term in the biconformally covariant derivatives of ϕ^m and find the resulting gravitationally

*Utah State University Department of Physics, Logan, UT 84322-4415, sllg7@cc.usu.edu

†Utah State University Department of Physics, Logan, UT 84322-4415, jwheeler@cc.usu.edu

[1] Details of the calculations described in this letter can be found at arXiv: hep-th/0001061.

coupled field equations. Then, for the case of one scalar field ϕ of conformal weight zero, we solve the new coupled gravitational and scalar field system, under the assumption of vanishing torsion. At the outset there are two likely outcomes: Either the presence of the scalar field will destroy the submanifold structure of the purely gravitational system, or the submanifold structure will be imposed on the fields. Which of these occurs is the central issue. We find that, as before, the solutions are foliated by equivalent n-dimensional Riemannian spacetime submanifolds whose curvatures now satisfy the usual Einstein equations with scalar matter. The field ϕ, which *a priori* depended on all $2n$ biconformal coordinates, is completely determined by the n coordinates of the submanifolds and satisfies the massless Klein-Gordon equation on each submanifold.

Thus, the new gauging establishes a clear connection between conformal gauge theory and general relativity with scalar matter, without the use of compensating fields.

2 The biconformal inner product and dual

Full detail on the new gauging of the conformal group $O(n, 2)$, $n > 2$, is available in [1]. We refer to the connection components associated with the Lorentz, dilation, translation, and special conformal transformation generators of $O(n, 2)$ as the spin connection, the Weyl vector, the solder form, and the co-solder form, respectively. We refer to the corresponding $O(n, 2)$ curvature components as the curvature, dilation, torsion, and co-torsion, respectively.

Biconformal space possesses a natural metric K^{AB} $(A, B = 1 \ldots n)$, which is obtained when the non-degenerate Killing form of $O(n, 2)$ is restricted to the $2n$-dimensional base space. The Killing metric provides a natural inner product between two r-forms $\mathbf{U}$ and $\mathbf{V}$ defined over biconformal space.

It is also possible to define the biconformal dual of a general r-form, ${}^*\mathbf{V}$, which is a $2n-r$-form of the same weight as $\mathbf{V}$. Then the term $\mathbf{U}\wedge{}^*\mathbf{V} = \mathbf{V}\wedge{}^*\mathbf{U}$ is proportional to both the inner product of $\mathbf{U}$ and $\mathbf{V}$ and the scale-invariant volume form of biconformal space, $\mathbf{\Phi}$, whose existence we demonstrated in [2].

In constructing the biconformal theory of scalar matter we let ϕ^m be a set of massless Lorentz-scalar fields of conformal weight $m \in \mathbf{Z}$ [8], which depend on all $2n$ coordinates. Then $\mathbf{D}\phi^m$ denotes their biconformally covariant derivatives. Each such covariant derivative is a one-form which is also of weight m. Since the dual operator preserves the conformal weight, the term ${}^*\mathbf{D}\phi^{-m}$ must be of weight $-m$. With appropriate conventions for signs and combinatorial factors we arrive at the Weyl-scalar-valued action

$$S_{Matter} = \frac{1}{2}\lambda \sum_m \mathbf{D}\phi^m \wedge^* \mathbf{D}\phi^{-m} = \frac{1}{2}\lambda \frac{(-1)^n}{n!^2} \sum_m K^{AB} D_A\phi^m D_B\phi^{-m}\Phi$$

for some constant λ. Here $D_A\phi^m$ denotes the components of $\mathbf{D}\phi^m$ in the biconformal basis. We make use of the 'dual' form of S_{Matter} when we vary the action with respect to the field and the Weyl vector, whereas the form of S_{Matter} that explicitly displays the dependence on the Killing metric proves more useful in varying the solder and co-solder forms.

3 The linear scalar action and solution of the field equations

In a $2n$-dimensional biconformal space the most general Lorentz and scale-invariant action $S_{Gravity}$ which is linear in the biconformal curvatures and structural invariants is given in [2]. For a set of massless Lorentz scalar fields ϕ^m of weight m, we now have

$$S = S_{Matter} + S_{Gravity}.$$

Variation of this action with respect to the scalar fields yields the wave equation for each ϕ^m. Variation with respect to the connection one-forms gives rise to the field equations for the various components of the biconformal curvatures. Their form is essentially the same as in [2], except that six of the eight gravitational field equations are now coupled to matter sources. Due to the presence of these sources, it not obvious that the submanifold character of the general solution for the curvatures and connections found in [2] is still valid. We find, however, that we can reproduce and extend our results from the vacuum case such that the usual Einstein theory with scalar matter emerges. This is the central focus of this letter. In the remainder of this section, we review the main features of the solution to a sufficient degree that we can highlight those aspects of the solution that are novel. There are four principal parts to the solution which we discuss in turn.

First, the field equations for the torsion and co-torsion are solved algebraically, relating the torsion, co-torsion, Weyl vector and now the matter fields. Here we find that the presence of the matter fields ϕ^m influences only the form of the Weyl vector. Even a weak constraint on the traces of the torsion and co-torsion determines the Weyl vector completely in terms of the ϕ^m and their covariant derivatives, which couple in such a way that the Weyl vector vanishes unless conjugate weights, $+m$ and $-m$, are both present. This result is in contrast with [2], where it was observed that constraining the torsion to vanish also forces the Weyl vector to vanish. Thus, while we assumed in [2] that the torsion had to be at least "minimal" in order not to constrain the Weyl vector unduly, it now appears that the Weyl vector vanishes unless there are appropriate matter fields present. Taking this view, we are free to assume vanishing torsion, although we note that the minimal torsion assumption would also give the Weyl vector in terms of the matter.

Vanishing torsion implies the above-mentioned weak trace condition on the co-torsion. Hence, there exists a gauge in which the Weyl vector is given in terms of covariant derivatives of the fields ϕ^m. We will explore such dilational sources further elsewhere. Here it is sufficient to restrict our attention to the case $m = 0$. Thus, for the remainder of this paper we restrict ourselves to the case of one scalar field ϕ of conformal weight zero.

Second, with vanishing torsion, we deduce a reduced form for the curvatures by combining the remaining four field equations with the Bianchi identities involving the torsion. Each of these curvature equations now carries a source. It is here that we begin to see that the submanifold structure predominates even in the presence of matter. Despite the full stress-energy sources, we find that the curvatures reduce much as in [2], with many of the components necessarily vanishing. As a result, the matter field ϕ, which *a priori* depended on all $2n$ biconformal coordinates, has vanishing derivatives in half of the $2n$ directions.

Third, we move to the details of the dimensional reduction. Due to the vanishing torsion, one of the structure equations is in involution in the n solder forms $\mathbf{e}^a$. By the Frobenius theorem, we can consistently set the $\mathbf{e}^a$ to zero and obtain a foliation by n-dimensional submanifolds. Because of the reductions in the form of the curvatures described above, these submanifolds are flat, with each submanifold spanned by n one-forms, $\mathbf{f}_a$. With appropriate gauge choices the connection for the subsystem can be found in the usual way in terms of the $\mathbf{f}_a$ and their derivatives $\mathbf{df}_a$.

Returning to the full biconformal space, we find that the $\mathbf{df}_a$ remain at least linear in $\mathbf{f}_a$, so that the $\mathbf{f}_a$ are also involute. We can therefore set $\mathbf{f}_a$ to zero to obtain a second foliation by n-dimensional submanifolds with coordinates x^μ. These submanifolds are spanned by the $\mathbf{e}^a$ and have nonvanishing curvature.

This brings us to our central result: When the field equations for the curvature are projected to the $\mathbf{f}_a = 0$ submanifolds, we obtain the Einstein equations on each submanifold with the usual stress-energy tensor, given by x-derivatives of the matter field. The solution of the n-dimensional Einstein equation for the solder form and the n-dimensional wave equation for the scalar field (implicit in the Einstein equation) determine the full $2n$-dimensional biconformal solution. This establishes a direct connection between general relativity with scalar matter and the more general structure of biconformal gauge theory with scalar matter.

Finally, we examine the field equation for $\phi(x^\mu)$, which is the $2n$-dimensional wave equation in a null basis, *not* the n-dimensional wave equation. It is straightforward to check that this equation is identically satisfied. Thus, ϕ is constraint only by the n-dimensional wave (Klein-Gordon) equation, which emerges in the usual way as a consequence of the vanishing divergence of the stress-energy tensor.

4 Conclusions

We have developed aspects of the theory of scalar matter in biconformal space. Using the existence of an inner product of r-forms and a dual operator, we constructed an action for a scalar matter field ϕ^m coupled to gravity and found the field equations. We solved them for the case of a scalar field of conformal weight zero in a torsion-free biconformal geometry. As in the vacuum case, the generic solutions are foliated by equivalent n-dimensional Riemannian spacetime manifolds. The curvature of each submanifold satisfies the usual Einstein equations with scalar matter. The scalar field is entirely defined on the submanifold and satisfies the n-dimensional massless Klein-Gordon equation. Together these two fields determine the entire $2n$-dimensional biconformal space.

References

[1] J.T. Wheeler, J. Math. Phys. **39** (1998) 299.

[2] A. Wehner and J. T. Wheeler, Nucl. Phys. B 557 (1-2) (1999) 380.

[3] J. Crispim-Romao, A. Ferber and P.G.O. Freund, Nucl. Phys. B**126** (1977) 429.

[4] M. Kaku, P.K. Townsend and P. Van Nieuwenhuizen, Phys. Lett. B **69** (1977) 304; M. Kaku, P.K. Townsend and P. Van Nieuwenhuizen, Phys. Rev. Lett. **39** (1977) 1109; M. Kaku, P.K. Townsend and P. Van Nieuwenhuizen, Phys. Rev. D**17** (1978) 3179.

[5] S. Deser, Ann. Phys. 59 (1970) 248.

[6] E. Bergshoeff, M. de Roo and B. de Wit, Nucl Phys. B**217** (1983) 489.

[7] P. van Nieuwenhuizen, Phys. Rev. D **32** (1985) 872.

[8] J.T. Wheeler, in preparation.