The components of fractional transport rate

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Abstract. Fractional transport rates are defined as the product of spatial grain entrainment, displacement length, and displacement frequency. Grain entrainment is defined relative to the population of grains on the bed surface at an initial time in order to account for partial transport, a condition in which only a portion of the exposed grains of a single size $D_i$ are mobilized over the duration of a transport event. A relation for grain displacement frequency is calculated from an assumed relation for displacement length and observed values of fractional transport rate, bed surface size distribution, and spatial entrainment in flume experiments. Fractional transport rates are independent of $D_i$ for fully mobilized fractions and decrease rapidly with $D_i$ for partially mobile fractions; the latter results from a decrease in both the displacement frequency and the mobilized proportion of each fraction. These two terms, which together determine the fractional entrainment rate, have different relations with flow strength and $D_i$ and should be independently included in a general fractional transport model. Grain-size similarity in fractional transport rates does not hold for conditions of partial transport. Partial transport is a grain-scale representation of incipient motion and can be directly related to the reference shear stress $\tau_{ri}$, a surrogate for the critical shear stress for incipient motion based on transport rate rather than grain mobility. Fractional mobilization at different flows may be predicted from $\tau_{ri}$, and partial transport appears to be the general condition at the reference transport rate. Using $\tau_{ri}$ as input, the transport component relations are used to predict the bed mobilization and fractional transport rates for a field case.

Introduction

The transport rate of riverbed material can be decomposed into the product of elementary terms, such as grain velocity, displacement distance, and the proportion or rate of grain entrainment. These components arise naturally from the considerations of grain motion used to develop theoretical models of sediment transport. Direct measurements of transport components are needed to extend our understanding of the physics of transport and support the development of general transport models. Such measurements are relatively rare because they require methods quite different from those typically used to measure transport rate. Grain displacements, entrainment rates, and velocities have been measured on motion pictures of the transport of unisize sediment [e.g., Fernandez Luque and van Beek, 1976; Nakagawa et al., 1978, 1980; Nelson et al., 1995] and of mixed-size sediment in the laboratory [Nakagawa et al., 1982] and the field [Drake et al., 1988]. Grain entrainment and displacement have also been measured in the field using tracer grains [e.g., Stelzner, 1981; Emmett et al., 1990; Church and Hassan, 1992; Schmidt and Ergenzinger, 1996].

Different sets of transport components can be defined. When observations are made on motion picture film, mass transport rate per unit width [$M L^{-1} T^{-1}$] can be decomposed as the product of the spatial concentration [$M L^{-2}$] and velocity [$L T^{-1}$] of moving grains or as the product of the mean length of individual grain displacements [$L$] and the mass of grains entrained or distrained per unit area and time [$M L^{-2} T^{-1}$] under steady state transport conditions. In an unusually complete film analysis of bed load transport, Drake et al. [1988] measured both sets of transport components, as well as the flux of grains crossing a flow-normal line, and found all three measures of transport to be consistent. When tracer gravels are used, unit transport rates are calculated as the product of the mean total displacement length [$L$] over the duration of a transport event [$T^{-1}$] (typically composed of many individual displacements), the depth [$L$] of active transport (as measured, for example, with scour chains), the bed porosity, and the grain density [$M L^{-3}$].

Here a different set of transport components is developed in order to account for partial transport, a condition in which only some of the grains exposed on the bed surface are entrained over the duration of a transport event. Partial transport may be defined for the entire population of grains on the bed surface, although it is the partial transport of individual size fractions that is directly relevant to predicting fractional transport rates. In a companion paper [Wilcock and McArdell, this issue] we demonstrate the nature and domain of partial transport using direct observations of the immobile proportion of each fraction over the duration of flume experiments with a widely sorted mixed-size sediment. Entrainment of the surface grains within an individual size fraction occurs over a range of bed shear stress $\tau_{ri}$. Within this range that fraction is in a state of partial transport; some grains are active (entrained at least once during a transport event), and the remainder are immobile. We also demonstrate that larger fractions are mobilized at larger flow strengths and that partial transport is associated with a rapid decrease in transport rate with grain size, a tendency noted in many gravel-bed rivers.

To account for partial transport, it is necessary that grain entrainment be specified relative to the grain population present on the bed surface at some initial time. An appropriate
term is the mass entrainment, over an entire transport event, of the grains found within a particular area at the start of the event. With this approach the proportion of a fraction that is active is considered separately from the frequency of entrainment of the active grains. The spatial entrainment used here may be measured in the field by replacing a volume of the bed with tracer gravels [Wilcock et al., 1996] and recording the proportion or mass of grains removed by the transport event. When this field measure of spatial entrainment is divided by the area of the sample and the duration of the event, the resulting mass entrainment rate will be much smaller than that obtained by directly observing the number of grains entrained per area and time (as with movies). The latter is larger by a factor equal to the mean number of displacements per grain that occur during the event.

In this paper, approximate relations for the individual transport components are developed using observations of fractional transport rate, bed-surface size distribution, and the mobilized proportion of individual size fractions over the duration of flume runs of different flow strength. These relations are used to illustrate the relative contribution of each transport component to the variation of fractional transport rate with grain size and flow strength. The unmodified component relations are then applied to independent transport observations from a gravel-bed river in order to evaluate their general applicability.

Partial transport is a measure of incipient motion based on the proportion of grains entrained at least once during a transport event. An entirely different measure of incipient motion conditions is the reference shear stress \( \tau_{u} \), which is defined as the shear stress that produces a small reference value of a dimensionless transport rate. Because \( \tau_{u} \) is defined using transport rates, without regard to the source of the grains, the proportion of the bed surface mobilized under reference transport conditions cannot be determined from the information used to calculate \( \tau_{u} \). One objective of this paper is to demonstrate the degree of bed mobilization associated with \( \tau_{u} \). Because \( \tau_{u} \) is easily and commonly determined from typical measurements of transport rate, a means of forecasting the mobilized proportion of the bed at \( \tau_{u} \) provides a broader physical interpretation of transport measurements and a basis for estimating the potential grain sorting and sediment exchange between the bed surface and subsurface.

**Definition of Transport Components**

To account for partial transport, a set of transport components should include a term representing the mobile proportion of each fraction and must, therefore, be referenced to the population of grains on the bed surface at some initial time. To facilitate comparison among different size fractions, flows, and sediments, the individual components should also be defined so that they may be expected to take constant mean values under conditions of steady state transport, so that individual terms do not become arbitrarily large with increasing flow duration or period of observation. An appropriate set of components contains three terms:

\[
q_{bi} = M_{bi} (N/T) L_{1,i}
\]

where the equation is in units of \([ML^{-1}T^{-1}]\), \(M_{bi}\) is the mass of fraction \(i\) entrained per unit bed area over the time period \(T\) \([ML^{-2}T^{-1}]\), \(N_i\) is the number of times an individual grain of fraction \(i\) is entrained during \(T\), and \(L_{1,i}\) is the mean length of a single displacement. \(M_{bi}\) is defined to include only grains found within an area of the bed at the start of \(T\). Other grains, with an initial position outside of the area, will be deposited and reentrained during \(T\). The total mass of sediment, regardless of its initial position, that is entrained from unit bed area over \(T\) will be larger than \(M_{bi}\) by a factor of \(N_i\).

Although all variables in (1) can be represented by frequency distributions, the focus here is their mean values, as defined in (1). The product of means given in (1) is equivalent to the ensemble average product for some plausible frequency distributions of the transport components (e.g., an exponential or gamma distribution for \(L_{1,i}\), as suggested by Einstein [1937], and recently evaluated using tracer gravels by Hassan and Church [1992] and Schmidt and Ergenzinger [1992]). For other possible frequency distributions for the transport components the transport components in (1) must be regarded as "effective" averages, in the sense that they are defined such that their product gives the mean \(q_{bi}\) (D. Furbish, personal communication, 1996).

Under steady state transport conditions, \(q_{bi}\) fluctuates about a constant mean. The transport components in (1) are defined so that the time average of all terms on the right-hand side approaches a constant value under steady state conditions. \(M_{bi}\) may be expected to increase with time over an initial period during which the active grains are entrained and then approach a constant value as the number of remaining entrainable grains approaches zero. The duration of this initial period presumably corresponds to both the duration and strength of the flow, suggesting that it should scale with the cumulative mass of transported sediment. The rate at which additional grains are entrained from an initial bed surface becomes approximately zero after the cumulative transport of approximately two to four surface layers [Wilcock and McArdell, 1993; this issue]. The time average of \(L_{1,i}\) may be expected to approach a constant value once the bed and transport reach a mutual adjustment under steady state conditions. In contrast, \(N_i\) may be expected to increase directly with time, so it is the appropriate term in (1) to be scaled by \(T\).

**Experimental Observations**

The experimental observations used here to develop specific functional relations for the components in (1) were collected from flume runs using a range of flow strengths and a widely sorted sand/gravel mixture [Wilcock and McArdell, 1993; this issue]. The sediment has a bimodal grain-size distribution with a minimum size of 0.21 mm, median size \(D_{50} = 5.3 \) mm, and maximum size of 64 mm. The experimental runs span a range in mean bed shear stress \(\tau_{0}\) from 2.0 to 7.3 Pa and a range in transport rate \(q_{bi}\) from 7.5 to 570 g m\(^{-2}\) s\(^{-1}\).

The relevant measurements include the transport rate of individual size fractions \(q_{bi}\), the proportion of each fraction on the bed surface \(P_{bi}\), and the proportion of each fraction mobilized over the duration of each run \(Y_i\). The latter was determined from time series of photographs of the bed surface. Grain immobility observations were made for all size fractions coarser than 4.0 mm, which includes the coarsest 58% of the sediment. The results of four of five runs with grain immobility observations are presented here; the omitted run has a shear stress and transport rate similar to one of the included runs but a shorter duration, so that the estimates of \(Y_i\) are less certain. Detail regarding the experimental methods and the variation
Figure 1. Summary of grain immobility and fractional transport observations, as a function of grain size $D_i$, for the BOMC sediment [Wilcock and McArdell, this issue]. (a) Mobile proportion $Y_i$ of individual size fractions. (b) Reference shear stress $\tau_{ri}$, shear stress ($\tau[50]$) that mobilizes 50% of individual size fractions, and shear stress producing equal mobility ($p_i = F_i$). (c) Fractional transport rates $q_{bi}/F_i$, with contours of $Y_i = 0.1, 0.5,$ and 0.9 superimposed. Values of mean bed shear stress $\tau_0$ and total transport rate $q_b$ for each run are BOMC18: 2.0 Pa, 7.5 g m$^{-1}$ s$^{-1}$; BOMC19: 3.2 Pa, 25.7 g m$^{-1}$ s$^{-1}$; BOMC4: 5.0 Pa, 157 g m$^{-1}$ s$^{-1}$; and BOMC5: 7.3 Pa, 572 g m$^{-1}$ s$^{-1}$.
Approximation of Transport Components

In the following, $M_{ai}$ is estimated using experimental observations of $Y_i$ and the bed surface size distribution $F_i$. Of the two remaining transport components, neither $N_i/T$ nor $L_{i1}$ were measured, but one of these may be calculated using (1) and measured values of $q_{ai}/F_i$, if a function of $\tau_0$ and $D_i$ can be specified for the other. Because some independent observations of $L_{i1}$ are available, a function for $L_{i1}$ is assumed and $N_i/T$ is calculated from (1).

Entrainment

The number of grains of size $i$ per unit bed area is approximately $F_i/D_i^3$. Multiplying this by $Y_i$ and the mass of an individual grain $m_i$ gives the mass entrainment, per unit area, from the bed surface:

$$M_{ai} = (m_iF_iY_i/D_i^3)$$

where the equation is in units of $[ML^{-2}]$. Because it represents entrainment from only the bed surface, (3) is likely to underestimate the entrainment of finer fractions at flows larger than those causing full mobilization ($Y_i = 1$). In this case, subsurface grains will also be entrained and $M_{ai}$ becomes proportional to an exchange depth $d_x$. For $Y_i = 1.0$, $M_{ai}$ is given by

$$M_{ai} = (m_iF_i/D_i^3) d_x$$

where the term in parentheses represents the mass of fraction $i$ per unit volume of bed which when multiplied by $d_x$, gives the mass of fraction $i$ entrained per unit bed area. Here $d_x$ is approximated as the size of the fraction $D[50]_i$ that is 50% mobile at a given $\tau_0$. For the experimental sediment a suitable dimensionless form for this relation [Wilcock and McArdell, 1993] is

$$d_x/D_{50} = 397(\tau_0[D_{50}]_i)^{1.5}$$

where $\tau_0$ is the Shields parameter $\tau_0/[\sigma (s - 1) \rho g D_{50}]$. Because $D[50]_i$ and $d_x$ depend on the incipient motion conditions for individual size fractions, which can vary with sediment properties [e.g., Wilcock, 1993], the form of (5) is likely to be particular to our experimental sediment. The basis for a more general calculation of $d_x$ from observed values of $\tau_0$ is developed later in this paper.

For $d_x \leq D_i$, a condition of partial transport may be assumed to exist and $M_{ai}$ is given by (3). For $d_x$ much greater than $D_i$, a fraction may be assumed to be fully mobilized ($Y_i = 1$) and $M_{ai}$ is given by (4). A smooth transition between the two is achieved using the fact that $d_x = D[50]_i = D_i$ at $Y_i = 0.5$ and specifying $M_{ai}$ as

$$M_{ai} = (\pi \rho_s / 6) F_i D_i \Delta_i Y_i$$

(6)

where $\Delta_i$ is given by

$$\Delta_i = \begin{cases} 1 & Y_i \leq 0.5 \\ d_x/D_i & 0.5 < Y_i \leq 1.0 \end{cases}$$

(7)

and a spherical grain volume is assumed, giving $m_i = (\pi / 6) \rho_s D_i^3$. The variation with $\tau_0/D[50]_i$ of $Y_i$, $\Delta_i$, and their product is shown in Figure 2. $Y_i$ is given by a lognormal distribution of $\tau_0/D[50]_i$, with mean zero and standard deviation 0.2; $d_x/D_i$ is given by (5), which, using the fact that $d_x = D_i$ at $\tau_0 = \tau[50]_i$, may be expressed as

$$d_x/D_i = (\tau_0/\tau[50]_i)^{1.5}$$

(8)

For later purposes we note that $\Delta_i$ differs from $d_x/D_i$ by no more than a factor of 2 for $Y_i > 0.01$, so that $\Delta_i \sim d_x/D_i$ provides a useful approximation for all measurable grain entrainment.

Displacement Length

The mean length of individual grain displacements $L_{i1}$ may be expected to depend on both $\tau_0$ and $D_i$. Schmidt and Ergen-
zinger [1992] used radio tracers in a steep mountain stream and found that \( L_{11} \) for discharges greater than the mean observed during their field work were nearly twice those for discharges smaller than the mean. Nakagawa et al. [1982] calculated \( L_{11} \) for individual fractions in mixed-size sands using observed \( q_{bi} \) and displacement frequencies (comparable to \( N_{i}/T \)) measured from motion picture film. They proposed a relation for displacement length in which \( L_{11} \) is proportional to \( (\tau_{0})^{0.093} \). Although their calculated values of \( L_{11} \) clearly increase with \( \tau_{0} \), the scatter in the data and the limited range of \( \tau_{0} \) used in their experiments prevented a general confirmation of this relation. Nakagawa et al. also found that \( L_{11} \) increased directly with \( D_{i} \), so that \( L_{11}/D_{i} \) took values between 6 and 30, where some of the variation is explained by \( \tau_{0} \). Visual observations of \( L_{11} \) in a laboratory flume were made for a range of flows and a single grain size \((D = 33 \text{ mm})\) by Stelczer [1981, p. 194], who found that \( L_{11} \) varied directly with the near-bed flow velocity \( u_{c} \) over a 5-fold range in \( u_{c} \), \( 0.2-1.0 \text{ m} \text{ s}^{-1} \). Assuming a standard logarithmic velocity profile in the near-bed region, \( L_{11} \) is then proportional to the square root of \( \tau_{0} \), as proposed by Nakagawa et al. [1982].

The most detailed observations of displacement length in a mixed-size sediment have been made by Drake et al. [1988], who directly measured \( L_{11} \) on motion picture film of transport on Duck Creek, a gravel-bed stream with \( D_{so} = 4.0 \text{ mm} \). They measured \( L_{11} \) for three sizes, \( 2-4 \), \( 4-8 \), and \( 8-16 \text{ mm} \), and found \( L_{11} \) to be given approximately by \( 15 \times D_{i} \). The films were made at a constant flow strength corresponding to \( *_{so} \), so the variation of \( L_{11} \) with \( \tau_{0} \) could not be determined. This value of \( *_{so} \) is bracketed by the values used in the experiments of Nakagawa et al. [1982] \((0.032 < *_{so} < 0.15) \) for runs with five different sediments. The result of Drake et al. may be combined with a square root dependence on \( \tau_{0} \) in the dimensionless form

\[
L_{11}/D_{i} = 15(\tau_{0}/0.093)^{0.05} \tag{9}
\]

A larger coefficient in (9), on the order of 60, is needed to fit the Stelczer [1981] data, although a direct comparison cannot be made because Stelczer's results are given in terms of the bottom velocity \( u_{c} \), rather than \( \tau_{0} \). The difference between the two constants may be attributed in part to a decrease in \( L_{11} \) for the widely sorted mixture in the field case. Further, it is likely that real-time observation of tracer displacement lengths will be biased toward longer displacements, whereas film analysis facilitates the measurement of displacements of short length. Because of the greater accuracy of film observations and the widely sorted nature of the field sediment, the value of 15 is used in (9) and the absolute error in this estimate of \( L_{11} \) is assumed to be no larger than a factor of 4. The relative error in \( L_{11} \) from size to size should be smaller.

Most observations and estimates of \( L_{11} \) are for sizes in the coarser half of the grain-size distribution, so little information is available to estimate \( L_{11} \) for \( D_{i} \) much smaller than \( D_{so} \). Although the focus here is also on the coarser half of the size distribution, a means of estimating \( L_{11} \) for \( D_{i} < D_{so} \) is useful in order to estimate fractional transport rates for the entire bed size distribution. If \( L_{11} \) for the finer fractions is influenced by both the absolute grain size \( D_{i} \) and the general roughness field of the bed surface, a plausible modification of (9) is

\[
L_{11}/D_{i} = 15(D_{i}/D_{so})^{1/3}(\tau_{0}/0.093)^{0.05} \tag{10}
\]

where \((D_{i}/D_{so})^{1/3}\) accounts for the relative size effect on \( L_{11} \).

The range of \( \xi \) may fall between zero \((L_{11} \approx D_{i})\) and \(-1\) (for which \( L_{11} \) is entirely determined by bed roughness, represented by \( D_{so} \)). \((D_{i}/D_{so})^{1/3}\) adds a parameter to the formulation that can only be adjusted by comparing predicted and observed fractional transport rates, although it will be shown later that its effects on the variation of \( q_{bi}/F_{i} \) with \( D_{i} \) are relatively limited. Equation (10) is applied only to fractions finer than \( D_{so} \); (9) is used for the coarser fractions.

**Displacement Frequency**

\( N_{i}/T \) may be calculated from (1) using measured values of \( q_{bi}, F_{i}, \tau_{0}, \) and \( Y_{r} \) with (5), (6), (7), and (9). Calculated values of \( N_{i}/T \) are plotted in Figure 3a as a function of \( \tau_{0}/\tau_{ri} \). The narrow scatter of the data suggests that \( N_{i}/T \) is independent of grain size beyond its effect on \( \tau_{ri} \). The collapse produced by \( \tau_{0}/\tau_{ri} \) is superior to that produced by \( \tau_{0}/\tau_{ri}(50) \), which may be expected because \( \tau_{ri} \) is determined as the value of \( \tau_{0} \) associated with a particular transport rate and \( q_{bi} \) is the dominant term producing the variation in the \( N_{i}/T \) estimated from (1). The absence of size dependence in Figure 3a suggests that a suitable dimensionless form for \( N_{i}/T \) is

\[
N_{i}^{*} = \frac{N_{i}}{T} \left( \frac{D_{so}}{g} \right)^{1/2} \tag{11}
\]

which is plotted as a function of \( \tau_{0}/\tau_{ri} \) in Figure 3b. It is useful to represent the trend between \( N_{i}^{*} \) and \( \tau_{0}/\tau_{ri} \) with

![Figure 3](image-url)
a closed-form function in order to illustrate the relations among the different components of fractional transport and to compare the transport components with existing transport relations. If there is no size dependence in this relation (i.e., all size dependence is accounted for by $\tau_0$), a single relation may be fitted to all grain sizes on Figure 3. The relation

$$N_t^* = K \left[ 1 - 0.85 \frac{\tau_n}{\tau_0} \right]^{4.5}$$

(12)

with $K = 0.01$ is shown in Figure 3b. The relation shown in Figure 3a differs from (12) by the factor $(g/D_{50})^{1/2}$. The form of (12) is suggested by the transport relation of Parker [1979]:

$$\frac{W_r^*}{W_t^*} = 5600 \left[ 1 - 0.85 \frac{\tau_n}{\tau_0} \right]^{4.5}$$

(13)

where $W_r^*$ is the dimensionless transport rate defined in (2) and $W_t^*$ is the reference transport rate, taken to be 0.002. Equation (13) is a power approximation of the Einstein transport relation at small transport rates and has been found to represent gravel-bed transport rates measured under a variety of conditions. Using (12) to represent $N_t^*$ facilitates a comparison in a later section of the transport components developed here with those implied by a similarity collapse of fractional transport rates, of which (13) is one example.

Comparison of Transport Components

The components of fractional transport rate are presented in Figure 4 as a function of grain size, which provides the most direct illustration of partial transport and the decrease in transport rate with grain size. $M_{ai}/F_i$ is found from (5), (6), and (7), with $Y_i$ given by a lognormal distribution of $\tau_0/\tau_{50}$, and the values of $\tau_{50}$ in Figure 1b. $N_i/T$ is found from (12), and $L_{11}$ is found from (9) for $D_i > D_{50}$ and from (10) for $D_i < D_{50}$. Observed $q_{bi}/F_i$ is shown together with the values calculated from (1) using the plotted transport components. Values are shown for all size fractions, even though the immobility observations and the relation for $N_i/T$ are developed for sizes coarser than 4.0 mm.

Observed and predicted values of $q_{bi}/F_i$ match closely, which is no real measure of predictive capability because $N_i/T$ has been calculated from the observed transport, but does indicate that the component functions provide a consistent representation for all $D_i$ and $\tau_0$. For $D_i < D_{50}$, two values of the parameter $\xi$ in (10) are shown: $\xi = 0$, making (10) identical to (9); and $\xi = -0.3$, causing $L_{11}$ to decrease less rapidly with decreasing $D_i$. The differences in $L_{11}$ and $q_{bi}/F_i$ for these two estimates are relatively minor; $\xi = -0.3$ causes $q_{bi}/F_i$ to decrease less rapidly with $D_i$ and therefore may provide a slightly better representation of $L_{11}$ for smaller $D_i$.

Figure 4 illustrates the effect of partial transport on fractional transport rates. $M_{ai}/F_i$ is independent of $D_i$ for $Y_i = 1$ and decreases rapidly over the partial transport range as $Y_i$ decreases to zero over a range in grain size of approximately a factor of 4. $N_i/T$ increases with decreasing grain size, driven by the decrease in $\tau_0/\tau_{50}$ and corresponding increase in $L_{11}$. $L_{11}$ increases with $D_i$ and increases slightly with $\tau_0$. In the fully mobilized region of transport the variation with $D_i$ of $L_{11}$ and $N_i/T$ roughly balance to produce approximate equal mobility, shown as a weak variation of $q_{bi}/F_i$ with $D_i$. In the partial transport region, $M_{ai}/F_i$ decreases rapidly with grain size, taking the same sign as $N_i/T$, and the fractional transport rates decrease rapidly to vanishingly small values. This suggests that the characteristic decrease in transport rate with grain size in gravel-bed rivers results from a corresponding decrease in both the proportion of active grains and their displacement frequency.

Exchange Rates

The mean rate of grain entrainment per unit bed area (regardless of grain position at some initial time) may be directly measured on motion picture film [e.g., Fernandez Luque and van Beek, 1976; Drake et al., 1988]. This may be called the exchange rate $E_{ai}$, to distinguish it from the entrainment rate $M_{ai}/T$. Drake et al. measured $E_{ai}$ from motion picture film for three size fractions and a constant flow strength. Values of $E_{ai}$ may be calculated for the BOMC runs as $M_{ai}(N_i/T)$, providing a comparison with the results of Drake et al. and (because $\tau_0$ is varied in the BOMC runs) an illustration of the possible variation of $E_{ai}$ with both $D_i$ and $\tau_0$.

Because $E_{ai}$ will depend on the proportion of each fraction available for entrainment, comparison among different sediments requires scaling by $F_i$. Drake et al. [1988] measured the bed surface concentration as a mass concentration $C_{bi}[M L^{-2}]$ rather than as a proportion, because the proportion of the finest fractions could not be reliably determined on the film. The scaled exchange rate is therefore the proportion of surface grains entrained per unit time $E_{pi} = E_{ai}/C_{bi}[T^{-1}]$. This is equivalent to the product $Y_i(N_i/T)$ and values of $E_{pi}$ calculated for BOMC are compared to those observed for Duck Creek in Figure 5.

To facilitate the comparison, $E_{pi}$ is plotted as a function of relative grain size $D_i/D_{50}$ and values of the dimensionless transport rate $q_{50}^* = q_{bi}/(\rho_i(g - 1)D_{50}^3)^{0.5}$ are indicated for each BOMC run. The value of $q_{50}^*$ for the Duck Creek data is 0.003 for the mean transport rate of approximately 8 g m$^{-1}$ s$^{-1}$. The transport rate (calculated as $E_{pi}L_{11}$) in the frames used for measuring $E_{ai}$ is 50–100% larger, so the BOMC run with comparable transport rates is BOMC19, for which $q_{50}^* = 0.0062$. Values of $E_{pi}$ for Duck Creek are also calculated using observed $q_{bi}$ and $L_{11}$ and $E_{pi} = q_{bi}/(L_{11}C_{bi})$. These values are smaller than those found directly from the measured $E_{ai}$ because the mean transport rate is smaller than that in the portion of film used to measure $E_{ai}$. Values of $E_{pi}$ agree relatively closely for the two finer sizes at Duck Creek; and the largest size deviates somewhat (Figure 5). The similar order of magnitude for the Duck Creek and BOMC $E_{pi}$ is significant, given the very different means by which $E_{pi}$ is estimated. Detail of the variation of $E_{pi}$ with $D_i$ is likely to be influenced by the size distribution in each case. In particular, a weaker variation of $E_{pi}$ with $D_i$ may be expected for the narrower size distribution in the field case.

For the BOMC values, $E_{pi}$ decreases rapidly with grain size, primarily as a function of decreasing $N_i/T$ for all sizes and decreasing $Y_i$ for the coarsest few fractions (Figure 4). For

Figure 4. (opposite) Components of fractional transport rates for runs (a) BOMC18, (b) BOMC19, (c) BOMC4, and (d) BOMC5. Method of calculation explained in text. The grain size $D_{ref}$ that is just at reference transport conditions for each run is found from $\tau_0 = 0.68D_{ref}^{0.52}$. Calculated values shown for $L_{11} \propto D_i$ for all $D_i(9)$ and for $L_{11} \propto D_i^{0.7}$ for $D_i < D_{50}$ ($\xi = -0.3$ in (10)). The latter provide a slightly better fit to $q_{bi}/F_i$. Calculated and observed transport rates match closely for sizes smaller than 4.0 mm, for which no immobility observations were made and which were not used to fit (12).
Figure 5. Proportional exchange rate $E_{p,i}$ as a function of grain size $D_i$. BOMC values calculated as $Y_ri(Nf/T)$. Duck Creek values obtained from measurements on motion picture film of (1) exchange rate $E_{a,i}$ and bed concentration $C_{bi}$ (“obs.”) and (2) fractional transport rates and displacement lengths $L_{1,i}$ (“transp.”). The BOMC run with the transport conditions closest to those of Duck Creek is BOMC19, for which $q_{s0} = 0.0062$.

Fully mobilized grains, $E_{p,i}$, increases with flow strength from 0.2 to 0.35 s⁻¹ over the range of transport rates measured with BOMC.

Similarity of Fractional Transport Rates

In the Parker [1979] transport relation (13), $W_{*,i}$ is independent of $D_i$ beyond its influence on $\tau_{ri}$. This is the classic similarity transformation for fractional transport rates found in earlier mixed-size sediment transport models [e.g., Ashida and Michue, 1972; Parker et al., 1982]. In this approach the form of the transport relation (such as (13)) is taken to be independent of grain size, because size-dependent differences in fractional transport rate are assumed to be entirely accounted for by the variation of $\tau_{ri}$ with $D_i$.

The transport component relations developed here may be compared to the similarity hypothesis by using (1), (6), (9), and (12) in the definition of $W_{*,i}$ (2) to develop an alternative expression for $W_{*,i}$:

$$W_{*,i} = \frac{1}{327(D/D_{50})(\tau_{*,i})^{4.5}} \left[ 1 - 0.85 \frac{\tau_{ri}}{\tau_{0}} \right]$$

(15)

where the constant 327 is chosen to make the resulting fractional entrainment curves fall about the data in Figure 3b. This constant corresponds to a constant in (13) of $12,150Y_{ri}$, rather than 5600. If $W_{*,i} = W_{*,i}$ is taken to be independent of $D_i$ (transport similarity), then a size dependence in $N_{*,i}$ is implied by the component relations for $M_{*,i}$ and $L_{1,i}$. This size dependence is not evident in the calculated values of $N_{*,i}$ (Figure 3b). If $N_{*,i}$ is taken to be independent of grain size, then $W_{*,i} = W_{*,i}$ will have a size dependence for partially mobile fractions, as in (14).

Relation Between Partial Transport and the Reference Transport Rate

The calculated transport components in Figure 4 depend on two essential empirical results: the variation with $D_i$ of $r[50]$ and $\tau_{ri}$. The first determines the degree of partial transport and the entrainment per unit area of each fraction; the second is used to scale $\tau_{0}$ in the prediction of the displacement frequency. Although both measures of incipient motion show a similar variation with $D_i$, each was measured independently and by different methods. Because $r[50]$, is not generally known, whereas $\tau_{ri}$ may be measured or estimated, the goal here is to develop a relation between the two, so that $r[50]$, may be predicted as a function of $\tau_{ri}$. This provides a basis for not only predicting the degree of partial transport at the reference transport rate but also for calculating the transport components for other sediments with known $\tau_{ri}$.

The transport component relation (1) may be substituted in the definition of $W_{*,i}$ (2) so that the reference transport definition becomes

$$\left( \frac{s - 1}{\rho \mu} \right) \frac{M_{*,i}}{F_{1,i}} \left( \frac{N_{*,i}}{T} \right) (L_{1,i}) = 0.002$$

(16)

where the $r$ subscript denotes values at reference transport conditions. If $N_{*,i}/T$ is assumed to be a function of only $\tau_{0}/\tau_{ri}$, as suggested by (12) and Figure 3b, $N_{*,i}/T$ will be a constant at the reference transport rate, for which $\tau_{0}/\tau_{ri} = 1$. With this assumption and using (6) to represent $M_{*,i}/F_{1,i}$ (16) may be solved for $Y_{ri}$ at the reference transport rate, $Y_{ri}$, giving

$$Y_{n} = \text{const} \left[ \frac{\tau_{ri}^{\delta}/D_{i}}{(\Delta_{i})(L_{1,i})} \right]$$

(17)

where const includes all numerical constants and terms related to $\rho$, $\rho_o$, and $D_i$. The variation of $Y_{ri}$ with $D_i$ can be approximated by assuming that $\tau_{ri} \propto D_i^\theta$ (where $\theta$ is a constant) and that the denominator in (17) is proportional to $D_i^\gamma$ ($L_{1,i} \propto D_i$ from (9) and, at the reference transport rate, $(\Delta_{i})(r = 1.0)$). For $0 < \theta < 0.67$, which should represent nearly all sediments, (17) becomes $Y_{ri} \propto D_i^\gamma$, with $-1 < \gamma < -2$ and $\gamma$ becoming more negative as $\theta$ approaches zero. Thus $Y_{ri}$ is found to decrease with grain size, using no result specific to the BOMC sediment except that $N_{*,i}/T$ is assumed to be a function of only $\tau_{0}/\tau_{ri}$. Because $Y_{ri}$ must be less than or equal to 1, a decrease of $Y_{ri}$ with grain size implies that partial transport exists at reference transport conditions.
For the specific functions (6, 9, and 12) used here to approximate the transport components, solving (16) for \( Y_{ri} \) gives

\[
Y_n = 39.6(s - 1)^{1/2} \frac{\tau_{ri}^*}{(\Delta_d_d)(D/D_{50})}
\]

where \( \tau_{ri}^* = \tau_{ri}/[(s - 1) \rho g D_i] \). Because \( \tau_{ri}^* = D_i^{0.55} \) for the BOMC sediment, (18) shows that \( Y_{ri} \) varies with \( D_i \) to approximately the \(-3/2\) power. Values of \( Y_{ri} \) calculated using observed \( \tau_{ri} \) in (18) are shown in Figure 6a together with values determined directly from the grain immobility observations [Wilcock and McArdell, this issue, Figure 3b]. The agreement between the two is quite good. For the BOMC sediment, (18) may be used to predict \( Y_{ri} \) from specified values of \( \tau_{ri} \) and \( D_i/D_{50} \). It remains to be demonstrated that this relation is accurate for other sediments.

A relation between \( \tau_{ri} \) and \( Y_{ri} \) may also be developed from the Parker [1979] transport relation. At \( \tau_0 = \tau_{ri} \), the ratio \( W_{ri}^p/W_{ri}^* \) on the left side of (13) is equal to 1. If the constant 5600 in (13) is replaced by 12,150 \( \tau_r \) (as suggested by plotting \( N_d^p \) derived from the Parker relation on Figure 3b), (13) may be solved for \( Y_{ri} \), giving

\[
Y_n = 0.42
\]

This result may also be obtained by using (15) to replace \( (N_d/T) \), in (16) and solving for \( Y_{ri} \). In both (18) and (19) a condition of partial transport is indicated at reference transport conditions.

If \( Y_{ri} \) is known, or predicted from \( \tau_{ri} \) and \( D_i/D_{50} \), \( \tau_r[50] \), may be determined if the form of the \( Y_r \) distribution is assumed. In Figure 6b, \( \tau_r[50] \), is predicted from the values of \( Y_{ri} \) in Figure 6a using a lognormal distribution of \( \tau_r/[\tau_r] \), with mean zero and standard deviation 0.2. For \( Y_{ri} \) calculated from (18) the variation of \( \tau_r[50] \), with \( D_i \) closely follows the fitted values from Figure 1b. The dependence of \( \tau_r[50] \), on the assumed shape of the entrainment distribution is not particularly strong; the distribution is merely used to locate \( \tau_r[50] \), from a nearby value of \( \tau_r \).

Both (17) and (18) (with the former depending only on the assumption that \( N_d/T \) is a function of only \( \tau/\tau_{ri} \)) suggest that partial transport prevails at reference transport conditions and that the variation with \( D_i \) of \( Y_{ri} \) and \( \tau_{ri} \), may be predicted as a function of \( \tau_{ri} \). Equation (18) provides a specific relation for calculating the degree of partial transport at the reference transport condition. Because observations of \( q_{bi} \), but not of \( Y_{ri} \), are available for other sediments, (18) can be evaluated only in combination with the other transport component relations.

Comparison With the Oak Creek Transport Data

An evaluation of the general applicability of the fractional transport component relations can be made by comparing predicted and observed fractional transport rates for an independent case. The data used for this comparison were collected by Milhous [1973] using a vortex sampler on Oak Creek, a small gravel-bed stream in the Oregon coast range. These data provide a useful comparison because of their unusually high quality and because they are well known, having been used by Parker et al. [1982] to develop the concept of equal mobility in bed load transport. The Oak Creek data are of particular interest here because, in contrast to BOMC, \( \tau_{ri} \), varies little with \( D_i \). This provides an opportunity to evaluate the contribution of partial transport to the fractional transport of a sediment for which all sizes are entrained over a narrow range of \( \tau_r \).

The data used were collected during the winter of 1971, which Milhous [1973] indicates are of the best quality. The 22 measurements for discharge \( Q > 1 \text{ m}^3 \text{s}^{-1} \) are used, which correspond to flow sufficient to break up the bed surface layer and are the same data used by Parker et al. [1982]. To facilitate comparison among different flows and provide larger sample sizes, the 22 observations are grouped into six samples with similar \( q_b \) (all \( q_b \) within a group fall within a factor of 2). Skin friction \( \tau_0 \) is calculated using the method of Einstein [1950]: \( \tau_0 \) for each group is calculated as a mean weighted by individual sample size.

Transport components and transport rates for Oak Creek data are predicted using, unmodified, the transport component relations developed for BOMC. For each grain size and flow, values of \( \tau_{ri} \), \( \tau_r \), \( D_i \), and \( D_{50} \) must be specified. \( M_{ai}/F \), is predicted using (6). \( L_i \) is predicted using (9) for \( D_i > D_{50} \) and using (10) with \( \xi = -0.3 \) for \( D_i < D_{50} \). \( N_d/T \) is predicted using (12) with \( K = 0.01 \). To complete the calculation, values of \( Y_r \) and \( d_i \) must be predicted. First, \( Y_r \), is predicted as a function of \( \tau_{ri} \) and \( D_i/D_{50} \) using (18). The value of \( \tau_{ri}/[\tau_r] \), is found for each \( Y_r \), using a lognormal distribution of \( \tau_{ri}/[\tau_r] \), with mean zero and standard deviation 0.2. For \( Y_r < 0.5 \) the calculation of \( Y_{ri} \), is direct. For \( Y_r > 0.5 \), \( \Delta_i = d_i/D_i \), and depends on \( D_{50} \), and therefore on the \( Y_r \) distribution, so the solution becomes iterative, although it converges.
Figure 7. Calculated and observed fractional transport rates for Oak Creek (Winter 1971 data of Mithous [1973] grouped into six samples). Calculated values use the transport component relations developed for BOMC, as explained in the text. (a) Reference shear stress $\tau_{ri}$, proportion mobilized $Y_{ri}$ at $\tau_{ri}$, and mean entrainment stress $\tau_{50,i}$. $Y_{ri}$ is calculated from $\tau_{ri}$ using (18); $\tau_{50,i}$ is calculated from $\tau_{ri}$ and $Y_{ri}$ using a lognormal distribution of $\tau_{ri}/\tau_{50,i}$. The variation of $\tau_{50,i}$ with $D_i$ is used to find $d_0$ for each $\tau_i$. (b) Mobile proportion $Y_i$ for each sample. (c) Observed and calculated fractional transport rates. Method of calculation explained in the text. The break in slope of predicted and observed trends match closely, suggesting that partial transport exists even for sediments such as Oak Creek, for which $\tau_{ri}$ varies little with grain size. Differences between observed and calculated values may result largely from uncertainty in $\tau_{50,i}$; values of $\tau_0$ within 5% of those used give a close match between predicted and observed for all samples.

Values of $\tau_{ri}$ for Oak Creek (Figure 7a) were determined by interpolation on plots of $q_{bi}/F_i$ versus $\tau_0$ and may be approximated as $\tau_{ri}/\tau_{50,i} = (D_i/D_{50})^{0.057}$ with $\tau_{50,i} = 0.040$ [Wilcock, 1993]. The corresponding values given by Parker et al. [1982] are $\tau_{ri}/\tau_{50,i} = (D_i/D_{50})^{0.018}$ with $\tau_{50,i} = 0.088$. The difference in $\tau_{50,i}$ arises because $\tau_0$ has been calculated here.
using the form-drag correction of Einstein [1950], giving smaller values of \( \tau_0 \). Because the same values of \( \tau_0 \) are used to find \( \tau_i \) and to predict the transport components, the difference is of no immediate significance. The exponent on \( (D_i/D_{50}) \) is slightly different because \( \tau_i \) has been determined by interpolation using additional transport data that extend to values of \( W^2 \) smaller than 0.002 rather than by fitting a loglinear relation of fixed slope through each fraction, as done by Parker et al. [1982].

Predicted values of \( Y_{ri} \) decrease with grain size for all sizes larger than 25 mm (\( D_{50} \)) (Figure 7a). Values of \( \tau_i \) are calculated only for fractions with \( Y_{ri} < 1.0 \). For \( Y_{ri} = 1 \), \( \tau_i \) cannot be reliably found from a probability distribution for \( Y_i \). Values of \( \tau_i \), for fractions with \( Y_{ri} < 1.0 \) span the range of \( \tau_i \) for the transport samples, as should typically be the case, so that \( D_i[50] \) and \( d_i \) for each sample may be found; \( Y_i \) increases more steeply with \( D_i \) than does \( \tau_i \), following the relation \( \tau_i \sim D_i^{-0.38} \). The inverse of this relation is used to estimate \( d_i \), which in the form of (5), is

\[ d_i/D_{50} = 7968 (\tau_i)^{2.61} \]

where the exponent 2.61 is greater than the value 1.5 in (5), as should be the case when \( \tau_i \) increases only slightly with \( D_i \). Values of \( Y_i \) for each fraction and sample are given in Figure 7b. Partial transport is predicted to exist for all fractions larger than 25 mm (\( D_{50} \)) and the magnitude of \( Y_i \) increases with \( \tau_i \).

The agreement between predicted and observed fractional transport rates (Figure 7c) is remarkably good, considering that the component relations developed for BOMP are applied unmodified to the Oak Creek data. The most significant match is in the shape of the fractional transport curves: the break in slope between fully mobilized and partial transport matches closely for both sets of curves, suggesting the presence of partial transport even though \( \tau_i \) varies little with grain size.

The agreement between predicted and observed \( q_h/F_i \) is best for fractions in the coarse half of the mix (\( D_{50} = 20 \) mm). These are the fractions for which the transport component relations have been developed and for which partial transport is predicted. For the largest five samples, 41 of the 45 predicted fractional transport rates are within a factor of 2 of the observed values, as are 23 of 25 values for \( D_i > 19 \) mm. Additional plausible adjustments, particularly in the coefficient and exponent of (12) for \( D_i < D_{60} \), could be made to cause the observed and predicted transport rates to match more closely, although such adjustments are beyond the intention of this paper. The separation between predicted and observed fractional transport rates may be attributed to uncertainty in the values of \( \tau_i \) used to predict \( q_h/F_i \). Values of \( \tau_i \) within 5% of those used cause the predicted and observed trends to overlap for all samples. The influence of uncertainty in \( \tau_i \) increases as \( \tau_i \) approaches \( \tau_0 \), because of the rapid decrease with \( \tau_i \) of both \( Y_i \) and \( N_i/T \) at small values of \( \tau_i/\tau_0 \).

Conclusions

Fractional transport rate can be expressed as the product of the mass entrainment per unit area, displacement frequency, and displacement length. To account for partial transport of individual size fractions, spatial entrainment is defined relative to the population of grains on the bed surface at some initial time. Partial transport is a condition in which the transport of a fraction is composed of only a portion the surface grains in that fraction, with the remaining grains on the bed surface remaining immobile over the duration of the transport event. Partial transport directly affects the population of grains available for transport and is associated with a rapid decrease in transport rate with increasing grain size, a common feature in gravel-bed rivers.

Approximate relations are developed for each transport component. Observations of the surface concentration and mobile proportion of each fraction are used to estimate spatial entrainment. Displacement length is assumed to increase directly with grain size, as has been found in other studies. Displacement frequency is calculated using the transport component relation with measured values of fractional transport rate and bed surface size distribution. Displacement frequency appears to depend entirely on the bed shear stress \( \tau_i \) and the reference shear stress \( \tau_0 \), which is the value of \( \tau_0 \) producing a small dimensionless reference transport rate and serves as a surrogate for incipient motion of each fraction. The ratio \( \tau_0/\tau_i \) provides a similarity collapse for displacement frequency for all sizes.

The transport component relations may be used to illustrate the relation between partial transport and fractional transport rates. For fully mobilized fractions (all surface grains entrained at least once), spatial entrainment, scaled by the proportion of each fraction on the bed surface, is independent of grain size \( D_i \). Displacement frequency decreases with \( D_i \) and displacement length increases with \( D_i \). The magnitude of these last two components balance, producing fractional transport rates that are independent of \( D_i \). For coarser sizes in the partial transport region (some grains remain immobile over the duration of the run), spatial entrainment decreases rapidly with \( D_i \), taking the same sign as the variation with \( D_i \) of displacement frequency, causing the fractional transport rates to decrease rapidly with \( D_i \).

Many models of mixed-size sediment transport are based on a similarity transformation in which the transport relations for each grain size are collapsed to a single curve. The basis for the transformation is the assumption that all size dependence in the fractional transport rates is accounted for by the variation with grain size of the critical shear stress for incipient grain motion. When the transport component relations developed here are combined into such a form, the dimensionless transport is seen to depend on the active proportion of the bed surface \( Y_i \). For sizes in a state of partial transport, \( Y_i \) varies with \( D_i \), suggesting that transport similarity does not exist under conditions of partial transport.

Partial transport is a measure of incipient motion based on the proportion of surface grains entrained at least once during a transport event, whereas the reference shear stress \( \tau_i \) is defined using transport rates, without regard to the source of the grains. Both measures of incipient motion are used in the transport component relations developed here. Because \( \tau_i \), is not generally known, whereas \( \tau_i \) may be measured or estimated, a relation between the two provides a basis for not only predicting the degree of partial transport at the reference transport rate but for calculating the transport components for other sediments. An approximate case is made that partial transport is the general condition at the reference transport rate and that the value of \( Y_i \) at \( \tau_i \) decreases with \( D_i \), even for sediments with a narrow range of \( \tau_i \). An explicit relation between \( Y_i \), \( D_i \), and \( \tau_i \) is developed for the experimental sediment, thereby allowing prediction of \( Y_i \) from \( \tau_i \) and the bed size distribution.
The transport component relations developed for the laboratory case are used, unmodified, to predict the fractional transport rates for Oak Creek, a small gravel-bed river. A reasonably good fit is found between predicted and observed fractional transport rates. Partial transport is predicted to occur for the coarsest half of the sediment, even though all sizes are entrained over a narrow range of $\tau_w$. When plotted as a function of grain size, both predicted and observed fractional transport rates show a similar decrease with grain size, further supporting the conclusion that partial transport occurs for the coarser sizes, even though $\tau_w$ varies little with grain size.

The fractional transport component relations presented here do not comprise a complete model for predicting fractional transport rates, primarily because an independent estimate of $\tau_w$ is needed to determine the displacement frequency and the mobile proportion of each fraction. Although some empirical [e.g., Wilcock, 1993] and theoretical [e.g., Wiberg and Smith, 1987] basis exists for estimating $\tau_w$, spatial entrainment, displacement frequency, and transport rate are all sensitive to $\tau_w$; at values approaching 1 so that uncertainty in either $\tau_w$ or $\tau_w$ can produce a large predictive error.

The mobilized proportion of the bed surface is of interest in any problem involving grain sorting between the transport and the bed or size-dependent dispersion of sediment through a river system. With specified values of $\tau_w$, as may be determined with a few observations of flow and transport at small transport rates, the entrainment function proposed here provides a basis for predicting the immobile proportion of the bed surface and its variation with $\tau_w$.

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References


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