Asymptotic Multiphysics Modeling of Composite Beams

Qi Wang
Utah State University

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ASYMPTOTIC MULTIPHYSICS MODELING OF COMPOSITE BEAMS

by

Qi Wang

A dissertation submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY in

Mechanical Engineering

Approved:

Dr. Wenbin Yu
Major Professor

Dr. Steven L. Folkman
Committee Member

Dr. Thomas H. Fronk
Committee Member

Dr. Aaron Katz
Committee Member

Dr. Marvin W. Halling
Committee Member

Dr. Mark R. McLellan
Vice President for Research and Dean of the School of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah
2011
Abstract

Asymptotic Multiphysics Modeling of Composite Beams

by

Qi Wang, Doctor of Philosophy
Utah State University, 2011

Major Professor: Dr. Wenbin Yu
Department: Mechanical and Aerospace Engineering

A series of composite beam models are constructed for efficient high-fidelity beam analysis based on the variational-asymptotic method (VAM). Without invoking any a priori kinematic assumptions, the original three-dimensional, geometrically nonlinear beam problem is rigorously split into a two-dimensional cross-sectional analysis and a one-dimensional global beam analysis, taking advantage of the geometric small parameter that is an inherent property of the structure.

The thermal problem of composite beams is studied first. According to the quasisteady theory of thermoelasticity, two beam models are proposed: one for heat conduction analysis and the other for thermoelastic analysis. For heat conduction analysis, two different types of thermal loads are modeled: with and without prescribed temperatures over the cross-sections. Then a thermoelastic beam model is constructed under the previously solved thermal field. This model is also extended for composite materials, which removed the restriction on temperature variations and added the dependence of material properties with respect to temperature based on Kovalenko’s small-strain thermoelasticity theory.

Next the VAM is applied to model the multiphysics behavior of beam structure. A multiphysics beam model is proposed to capture the piezoelectric, piezomagnetic, pyroelectric, pyromagnetic, and hygrothermal effects. For the zeroth-order approximation, the classical
models are in the form of Euler-Bernoulli beam theory. In the refined theory, generalized Timoshenko models have been developed, including two transverse shear strain measures. In order to avoid ill-conditioned matrices, a scaling method for multiphysics modeling is also presented. Three-dimensional field quantities are recovered from the one-dimensional variables obtained from the global beam analysis.

A number of numerical examples of different beams are given to demonstrate the application and accuracy of the present theory. Excellent agreements between the results obtained by the current models and those obtained by three-dimensional finite element analysis, analytical solutions, and those available in the literature can be observed for all the cross-sectional variables. The present beam theory has been implemented into the computer program VABS (Variational Asymptotic Beam Sectional Analysis).
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Chapter 1

Introduction

1.1 Motivation

1.1.1 Beam Structure

Beam structure, or sometimes called slender structure, is defined as a structure having one of its dimensions much greater than the other two. Many engineering components can be idealized as beams. Typical applications of beam structures in civil engineering are bridges: an arch bridge is composed of both curved and prismatic beams; a truss bridge is mainly supported by trusses, which can be viewed as assembly of beams or beam girders. A large number of building and machine parts are beam-like structures: joists, lever arms, shafts, and turbine blades, etc. Examples of beam-like structures in aeronautics include helicopter rotor blades and high aspect-ratio wings. Beams can be classified in different ways: “T” or “H” beams according to the shapes of the cross-sections; prismatic, curved or twisted beams related to their initial geometry.

Structural analysis is concerned with calculating the deformation of and stress developed within a suitably constrained solid object under the action of applied loads. The three-dimensional (3D) formulation for linear static structural analysis includes fifteen equations: six strain-displacement relations, three equilibrium equations, and six stress-strain equations. Direct solutions to the fifteen governing equations for real engineering systems are not easy to obtain even with super computers nowadays. After studying these equations for more than two hundred years, mathematicians and engineers have proposed different simplified models for beam structures to deal with practical problems. Usually a theory of solid mechanics related to beam structures is called as “beam theory.” By using beam theory, the computational cost is typically several orders of magnitude less in comparison
A structural analysis of beam structure includes two parts: a one-dimensional (1D) global beam analysis and a two-dimensional (2D) cross-sectional analysis. The 2D cross-sectional analysis should be performed first since the 1D global analysis relies on the elastic constants, or “stiffness,” attained by the 2D cross-sectional analysis. Three of the better-known elastic constants in the textbooks are extension, bending, and torsion stiffness, which are usually denoted by the symbols $E_A$, $E_I$, and $G_J$, respectively. The 1D global beam analysis provides displacements and rotations, which are used to define the generalized strains. Once a beam problem has been solved, the stress resultants, including force and moment resultants, can be obtained by the 1D constitutive equations. It is pointed out that if the beam problem is defined in the weak form, the governing functional, strain energy can be expressed in the form of 1D variables; and the stress resultants are attained as the conjugate of the generalized strains. The stress resultants have a sense of “homogenization” in the conventional beam theories since they are calculated as the integration of stresses and moments along a certain direction at the section.

It needs to be pointed out that sometimes information from 1D global analysis is not enough for design or further analysis. For example, to analyze helicopter rotor blades having a cross-section shown in Fig. 1.1, more information, to be specific, 3D strain and stress fields, is needed over the whole section rather than only “homogenized” stress resultants and generalized strains. Thus one should be able to recover the 3D field through the cross-sectional analysis.
1.1.2 Thermoelastic and Multiphysics Analysis

Thermoelastic analysis is meant to describe the thermal and mechanical behaviors of the engineering structures subject to combined loads. The significance of thermoelastic analysis of beam structure can be justified from the following two aspects. Firstly, a beam is a commonly used structure which often works under extreme conditions in aerospace systems. One example is the wing of solar array. Without the protection of atmosphere, the temperature difference between the two surfaces of the wing, one exposed to the sun and the other in the shade, could be several hundred degrees. The difficulty in this case is compounded by the fact that operating conditions involve not only extreme temperature levels but also severe temperature gradients. Secondly, most of these aerospace structures are made of composite materials which are more sensitive and vulnerable to temperature change than their isotropic counterparts. For composites, the thermal expansion coefficients of different constituents of the material are usually dramatically different from each other resulting in high stresses due to temperature changes from a stress free environment. Therefore, efficient high-fidelity thermal-mechanical models are needed to efficiently yet accurately predict the thermal and mechanical behaviors of the structures.

Multiphysics analysis mainly deals with the behavior of engineering structures composed of smart materials working under multiple physical fields: mechanical, thermal, electric, and magnetic. Smart materials are defined as those that exhibit coupling between multiple physical domains. A well-known smart material is piezoelectric, which creates conversion interface between electrical energy and mechanical energy. As an analogy with the exhibition of electromechanical coupling of piezoelectric materials, magnetic materials respond to an externally applied magnetic field by exhibiting a shape change which is known as magnetostriction, demonstrating the Joule effect. On the other hand, these magnetic materials also demonstrate the Villari effect indicated by changing their magnetization and consequently the magnetic induction in response to the applied stress. Recently, the newly developed composites contain both piezoelectric phase and piezomagnetic phase exhibit a magnetoelectric coupling effect which does not exist in either of the two constitutive phases.
Structures composed of these smart materials, often function as actuators and sensors, are usually called “smart structures.” Applications of the smart structure include vibration suppression [1], shape control of composite plates [2], and aeroelastic stability augmentation [3]. For example, embedded or surface-bonded smart actuators on a helicopter rotor blade can induce airfoil camber change that in turn can cause a variation of lift distribution. However, as noted in Ref. [4], one of the major barriers of smart structure technology is the lack of reliable smart systems mathematical modeling and analysis. This barrier is caused by (1) the difficulty to model coupling effects, including thermo-elastic, electro-elastic, magneto-elastic or any combination of them, and (2) anisotropic and heterogeneous nature of composite materials.

1.2 Previous Work

1.2.1 Thermoelastic Beam Modeling

Problem of thermal stresses has long been a subject of interest. Duhamel studied the formulation of elasticity problems including the effect of temperature variation in 1837, shortly after the basic formulations of elasticity theory itself [5]. However, it seems that not many thermoelastic models for beams, especially composite beams, have been developed. This section will attempt to mention those with most influence and relevance with respect to the present work.

A recent book [6] on thermal stresses includes a historic note on the developments in its field up to the present day, so interested readers can find much more details there. The creators of theory of elasticity, B. de Saint-Venant, G. Lamé, and P. S. Laplace, reduced thermoelasticity problems to elasticity problems by considering thermal loads as body forces. The formulation of thermoelasticity equations at the early stage can be found from Refs. [7, 8]. A lot of effort had been invested in the research on thermal stress during and after World War II. The active topics include both theoretical and experimental research on attaining temperature distributions under different boundary conditions, finding thermal stresses in complex structures and thermal stability, etc. [9–11]. An important monograph
written by Boley and Weiner [12] should be mentioned here. All the important topics of thermal problem ranging from the fundamentals of thermoelasticity to practical problems like thermal stresses in engineering structures and some non-linear thermal problems were thoroughly studied. Based on the thermoelastic theories, some specific problems were solved by scientists and researchers. S. Timoshenko [13] proposed a general theory of bending of a bi-metal strip submitted to a uniform heating in 1925. J. Lighthill and J. Bradshaw [14] developed a model for thermal stresses problem of turbine blades. Aleck [15] studied the thermal stresses in a clamped rectangular plate. Different types of beams including prismatic beams, curved beams, and thin-walled beams subject to temperature loads were studied in Ref. [12]. In order to simplify the original 3D problems, especially during the years without powerful computers, some assumptions were used to attain the analytical solutions. Euler-Bernoulli beam theory, also known as classical beam theory, is the most widely taught and used one for its simplicity. The primary kinematic assumptions of this theory are that the cross-section of the beam is infinitely rigid in its own plane and remains plane and normal to the deformed axis after deformation. For thin beams made of isotropic materials it works fairly well. Boley and Testa [16] extended the classical beam theory including the temperature effects to composite beams with rectangular cross-section. Boley [17] also discussed the limitations of the classical beam theory on thermal stresses. As a refinement to the classical beam theory, S. Timoshenko [18] developed a new model that allowed for the possibility that normal cross-section from the undeformed configuration could become oblique to the deformed reference line by introducing two transverse shear modes of deformation. Ochoa and Marcano [19] developed a Timoshenko-like laminated beam model incorporating transverse shear effects for thermoelastic analysis. The results showed that there is a substantial difference in the stress field obtained from refined and classical beam theory for the cases of nonuniform temperature fields.

The invention of computer made it possible to accurately predict the behavior of structures using 3D finite element analysis (FEA). However, people still need to rely on the simplified models due to the widespread use of composite materials during the past several
decades. For example, a real rotor blade could be made of hundreds of composite layers. A 3D finite element model for this blade could easily exceed $10^9$ degrees of freedom because at least one element is needed per layer thickness. A routine aeroelastic analysis of rotorcraft with four such blades cannot be solved using 3D FEA on any computer now. To avoid the expensive computational cost of the original 3D models, researchers are striving to simplify the analysis of composite beam structures. Theories of composite beam structure, including classical laminate theory [20] and refined laminate theory [21–24], have been reviewed in detail in Ref. [25]. Some of these theories, both classical and refined theories, have been extended to deal with the thermal problem of composite beams [26,27]. Copper and Pilkey [28] developed an analytical thermoelastic solution for beams with arbitrary temperature distribution. The problem is considered as a plane strain problem and the maximum stresses on a certain cross-section are validated with 3D solution and strength of materials solution. Huang et al. [29] investigated a functionally graded anisotropic cantilever beam subject to thermal and mechanical loads. The problem is solved analytically based on the plane stress assumption. Rao and Sinha [30] proposed a finite element model to deal with the coupled thermostructural analysis of composite beams. This model is based on Timoshenko beam theory and plane stress assumption, and the temperature is also assumed to be uniform through the thickness of the beam. It cannot yield the thermal and mechanical field over the cross-section but only an averaged temperature and the stress resultants. Vidal and Polit [31] developed a three-noded thermomechanical beam element for composite beam analysis. The thermal problem separates into two problems to be solved consecutively: the heat conduction problem to solve for the thermal field and the one-way coupled thermoelastic problem for the structure under a prescribed thermal field. This work allowed the variations of thermal and mechanical fields along the thickness of cross-section. Trigonometric functions are used in the assumed displacement field to avoid shear correction factors. But one dimension over the cross-section, say the width, is neglected in the solution. Kapuria et al. [32] reported a beam model based on zigzag theory for thermal analysis. By modifying the third order zigzag model, the contribution of
thermal expansion coefficient along thickness of the beam is considered while this model still neglects the variations along width on the cross-section. Another notable work was that of Ghiringhelli [33,34], where the thermal problem of general composite beams is solved using a finite element semi-discretization approach. The thermal field within a beam cross-section subject to prescribed boundary conditions was attained first taking into account any kind of thermal anisotropy or inhomogeneity. Then thermoelastic problem in a beam having arbitrary nonhomogeneous, anisotropic material properties over the cross-section was solved under the thermal loads obtained in the previous step. This model can capture the thermal and mechanical fields on the cross-sections.

1.2.2 Multiphysics Beam Modeling

The most popular model for multiphysics analysis is piezoelectric beam model. Pierre and Jacques Curie discovered direct effect in the 1880s which exhibits mechanical-to-electrical coupling of piezoelectrics. Soon after this, Gabriel Lippman predicted the converse effect, that is, these materials undergo deformation when an electric field is applied. It is also known that piezoelectric materials exhibit a thermomechanical coupling called the pyroelectric effect. The recent review of literature associated with piezoelectric beam modeling can be found in Refs. [4, 35–38]. Most of these models can be termed as ad hoc models, where the special variation of the field variables (displacements and potentials) are assumed a priori at the very beginning of the analysis. Depending on the assumptions the models rely on, these models can be classified into two groups: Euler-Bernoulli model (classical model) and Timoshenko model (refined model).

Crawley and Anderson [39] proposed a beam model based on Euler-Bernoulli assumptions. Both surface-bonded and embedded strain actuators are considered and they compared this model with a previously formulated uniform strain model [40], a finite element model and experiment results. This model was found to be accurate in predicting bending and extensional response for low actuator-to-host structure thickness ratios. Park and Chopra [41] developed a 1D model to predict the coupled extension, bending, and torsion response for piezoceramic actuated beams. The numerical results were compared with those
of experiment. The refined models, including Timoshenko model and higher order shear deformation model, are needed if the shear effect is not negligible. Shen [42] proposed a Timoshenko beam model to predict the actuation mechanisms of integrated active beams. Since this model does not require integrating the piezoelectric devices into the governing equation derivation process for the main structures, it can be easily implemented to any general-purpose finite element code. Abramovich [43] derived a closed form solution for sandwich beam structure containing piezoelectric layers. Ghiringhelli et al. [44] extended the previously developed beam model to analyze beams with embedded piezoelectric elements, the results match well with those obtained by 3D model. Banks and Zhang [45] developed a curved beam analysis for a pair of surface-attached piezoelectric patches based on the Donnell-Mushtari theory for shell models and B-spline basis elements. Two layer-wise models for piezoelectric beam analysis can be found in Refs. [46,47]. Some researchers further studied the static response of piezoelectric beams subject to combined electric and thermal load [48–50]. Blandford et al. [51] formulated two models: an uncoupled model that ignores the direct effect while a coupled model included this effect. The beam displacement is based on the first-order shear-deformation theory, and the electromagnetic potential is assumed to vary piecewise linearly through each piezoelectric layer. Lee and Saravanos [52] extended the previously developed discrete layer model [47] to incorporate thermal effects to account for the complete coupled mechanical, electrical, and thermal response of piezoelectric composite beams. Song et al. [53] conducted both finite element analysis and experiments to study the shape control of composite beams using piezoelectric actuators.

Recent advances of composites are the electromagnetoelastic materials consist of piezoelectric and piezomagnetic phases and the structures including layers made of these materials. The mechanical, electrical, and magnetic fields interact one with another, for example, a strain is produced when a magnetic or electric potential applied to the structure [54–57]. Various forms of constitutive equations and variational principles for magneto-electro-elastic solids were derived in Refs. [58–62]. The open literature relevant to the analysis of electro-
magnetoelastic beam structure is rather scanty. Jiang and Ding [63] presented an analytical solution to magneto-electro-elastic beams with different boundary conditions. Kumaravel et al. [64] investigated a three-layered electro-magneto-elastic strip under a plane stress condition. The thermal loading conditions, including uniform temperature rise and non-uniform temperature distribution, were considered. Two types of stacking sequence of the laminate beam under different boundary conditions were studied. Davi et al. [65] analyzed magneto-electro-elastic bimorph beams using a boundary element approach.

1.2.3 Beam Modeling Based on Variational-Asymptotic Method

Most of the previous described beam models can be classified into ad hoc models, which are based on a priori kinematic assumptions, and asymptotic models, which are derived by asymptotic expansions of the displacement field. The advantage of ad hoc models is that the reduced governing equations can be derived in a straightforward manner using variational statements; and this procedure is simple and straightforward for engineers to understand. While the disadvantages are (1) it is a common source of error that the kinematic assumptions contradict each other in 1D and 2D analysis, and (2) it is difficult to determine the shear correction factors needed in refined theories for composite laminated structures. Comparing with ad hoc models, the asymptotic method could develop elegant and rigorous models; however, it is very cumbersome and restricted from both geometric and material points of view.

Variational-asymptotic method (VAM) is a powerful mathematical approach which is first proposed by Berdichevsky about three decades ago [66]. It is applicable to any problem that can be posed in terms of seeking the stationary points of a functional involving some inherent small parameters. This method combines both merits of variational methods and asymptotic methods thus it does not rely on any ad hoc assumptions while is systematic and easy to be implemented numerically. Since the behavior of an elastic body is governed by a variational statement, and there are some small geometrical parameters characterizing the beam structure, say the dimension of the cross-section, VAM is especially the right tool for construction of accurate beam models.
Hodges et al. [67] first applied VAM to yield the cross-sectional properties for prismatic beam. Cesnik and Hodges extended the model taking into consideration the influence of initially curvatures [68–70]. Researchers have proposed the refined theories based on VAM [71, 72]. Since introducing two transverse shear strains in the beam model, these Timoshenko-like refined theories can handle the short-wavelength modes associated with transverse shear effects. However, the refined strain energy obtained by VAM is not convenient for practical use, even if it is possible [73], due to the existence of derivatives of the classical strain measures [74]. To derive an user-friendly form, or generalized Timoshenko strain energy, Berdichevsky and Starosel’skii [75] used changes of variables to tackle this problem, while Hodges et al. [71, 72] used the 1D beam equilibrium equations to build a relationship between the strains and derivatives of strains. Recently, two inconsistencies in the transformation are identified and resolved [76]. VAM has also been applied to construct piezoelectric beam models. Cesnik et al. [77, 78] carried out dimensional reduction of piezoelectric beams. Roy et al. [79–81] developed an 1D model for piezoelectric beam which allows the electric load varies along the axial direction of the beam.

1.3 Present Work and Outline

The ultimate goal of this study is to develop beam models for thermoelastic and multiphysics analysis. As reviewed in the above section, most beam models for thermoelastic analysis are built upon ad hoc assumptions which cannot handle complex geometry and significant heterogeneity over the cross-section of the beam. Moreover, there is no reliable and general tool for multiphysics analysis of beam structures.

The methodology of the present work is demonstrated in Fig. 1.2. Theoretical derivations begin with 3D formulations of energy that governs the behavior of structure in terms of intrinsic 1D variables and 3D warping functions. The VAM is then applied to rigorously split the 3D thermoelastic and multiphysics problems into a 1D global beam analysis and a 2D cross-sectional analysis. The 2D cross-sectional analysis will provide necessary constitutive models, including classical beam model and Timoshenko model, for the 1D global analysis. One cross-sectional analysis is sufficient for beams with uniform cross-sections.
because constitutive models are intrinsic to the structures and will not change with respect to loading, boundary conditions, and operating time. Recovery relations for the 3D displacement, stress, and strain fields are found from expressions that are consistent with the described procedure to attain the energy. This thesis proposal is organized in the following way

- Chapter 2 discusses the theoretical foundations of the present work.

- Chapter 3 is devoted to beam modeling for thermoelastic analysis. Examples are given and numerical results are compared with those available in the literature and 3D FEA results obtained by commercial software ANSYS.

- Chapter 4 presents a beam model for multiphysics analysis taking into account piezoelectric, piezomagnetic, pyroelectric, and pyromagnetic effects. Examples are given to validate this model.

- Chapter 5 summarizes the conclusions and offers recommendations for future works.
Chapter 2
Theoretical Foundations

This chapter discusses theoretical foundations needed for the current work including beam kinematics, Variational Asymptotic Method, and 3D Hamilton’s principle. Please note the introduction here only reflects the author’s understanding of these theories and all credit should go to the original author.

The following conventions apply to this chapter and throughout this thesis. Greek indices assume values 2 and 3 while Latin indices assume values 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated. The operator \( (\cdot)' \) implies taking the derivative with respect to \( x_1 \) so that \( (\cdot)' = \partial(\cdot)/\partial x_1 \), and \( \partial_\alpha(\cdot) = \partial(\cdot)/\partial x_\alpha \). The operator \( (\cdot) \) is the partial derivative with respect to time. The notation \( (\cdot) \) forms an antisymmetric matrix from a vector according to \( (\cdot)_{ij} = -e_{ijk} (\cdot)_k \) using the permutation symbol \( e_{ijk} \). The notations \( \langle \cdot \rangle = \int_S \cdot dx_2 dx_3 \) and \( \langle\langle \cdot \rangle \rangle = \int_S \cdot \sqrt{g} \ dx_2 dx_3 \) are also used.

2.1 Beam Kinematics

2.1.1 Undeformed and Deformed Configurations

As sketched in Fig. 2.1, a beam can be represented by a reference line \( r \), described by its arc-length \( x_1 \), and a typical reference cross-section normal to the reference line, described by local Cartesian coordinates \( x_\alpha \). Note that the undeformed cross section is restricted normal to the reference line for simplicity, which can be relaxed as shown in Ref. [82].

At each point along the reference line, an orthonormal triad \( b_i \) is introduced such that \( b_i \) is tangent to \( x_i \). Any point of the undeformed beam structure is then located by the position vector \( \hat{r} \) as

\[
\hat{r}(x_1, x_2, x_3) = r(x_1) + x_\alpha b_\alpha
\] (2.1)
where $\mathbf{r}$ is the position vector of a point on the reference line, $\mathbf{r}' = \mathbf{b}_1$. It is noted that $\mathbf{b}_i$ could be functions of $x_1$ due to existence of initial curvatures or twist.

When the beam deforms, the triad $\mathbf{b}_i$ rotates to coincide with a new triad $\mathbf{B}_i$. Here $\mathbf{B}_1$ is not tangent to the deformed beam reference line if the transverse shear deformation is considered. $\mathbf{B}_i$ coincides with $\mathbf{b}_i$ before deformation and during deformation they can be related as

$$
\mathbf{B}_i = C^{Bb} \cdot \mathbf{b}_i = C^{Bb}_{ij} \mathbf{b}_j
$$

where $C^{Bb} = C^{Bb}_{ij} \mathbf{b}_j \mathbf{b}_i$ denotes the rotation tensor, and $C^{Bb}_{ij}$ are the components of the corresponding direction cosine matrix. The deformed position, $\hat{\mathbf{R}}$, of the point which had $\hat{\mathbf{r}}$ in the undeformed state can be express as

$$
\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + \mathbf{u}(x_1) + x_\alpha \mathbf{B}_\alpha + \bar{w}_i(x_1, x_2, x_3) \mathbf{B}_i
$$

where $\mathbf{u} = u_i \mathbf{b}_i$ is the displacement vector of the reference line from the reference configuration and $\bar{w}_i(x_1, x_2, x_3)$ are the warping functions.

Although the expression in Eq. (2.3) is mathematically correct, it is not convenient for
carrying out the dimensional reduction from the original 3D model to a 1D beam model using the variational-asymptotic method. Instead, we introduce another triad $T_i$ associated with the deformed beam (see Fig. 2.2), with $T_1$ tangent to the deformed beam reference line and $T_\alpha$ determined by a rotation about $T_1$. The difference in the orientations of $T_i$ and $B_i$ is due to small rotations associated with transverse shear deformation. The relationship between these two basis vectors can be expressed as

$$
\begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix} =
\begin{bmatrix}
1 & -2\gamma_{12} & -2\gamma_{13} \\
2\gamma_{12} & 1 & 0 \\
2\gamma_{13} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_3
\end{bmatrix}
$$

where $2\gamma_{12}$ and $2\gamma_{13}$ are the small angles characterizing the transverse shear deformation, and we know $2\gamma_{1\alpha} \ll 1$ due to the small strain assumption. Note that this relationship can be derived by considering $B_i$ as formed by two consecutive rotations from $T_i$: first around positive $T_2$ with angle $2\gamma_{13}$ and then around negative $T_3$ with angle $2\gamma_{12}$. The distinction between these two frames is important for the development of different levels of approximation.

Fig. 2.2: Coordinate systems used for transverse shear formulation.
The material point having position vector \( \hat{\mathbf{r}} \) in the undeformed beam can also be expressed as

\[
\hat{\mathbf{R}}(x_1, x_2, x_3) = \mathbf{r}(x_1) + \mathbf{u}(x_1) + x_\alpha T_\alpha(x_1) + w_i(x_1, x_2, x_3) T_i(x_1)
\]

(2.5)

where \( w_i \) are the components of warping expressed in \( T_i \) base system. Note that in this formulation we choose \( T_1 \) to be tangent to the deformed beam reference line, which means we classify the transverse shear deformation as part of the warping field. Within the framework of small strains this neither introduces any additional approximations nor results in any loss of information.

For clarification, it is also pointed out that the warping functions \( \bar{w}_i \) in Eq. (2.3) are not the same as the warping functions \( w_i \) in Eq. (2.5). The difference between them is due to the difference between \( B_i \) and \( T_i \) as exemplified in Eq. (2.4). In view of Eqs. (2.3), (2.4), and (2.5), we can obtain the following relationship between these two sets of warping functions as

\[
\bar{w}_i = w_1 - 2\gamma_{1\alpha}(w_\alpha + x_\alpha) \quad \bar{w}_\alpha = w_\alpha + 2\gamma_{1\alpha}w_1
\]

(2.6)

### 2.1.2 Constraints on Warping Functions

It is noted that Eq. (2.5) is four times redundant because of the way the warping functions were introduced. Therefore, four appropriate constraints must be imposed on the displacement field to make it determined. These constraints are not unique and any constraints will work as long as they make the formulation determined. To this end, one can choose to define that the 1D displacement is equal to the difference between the position vector for the deformed and undeformed beam reference line, \( \text{i.e., } \mathbf{u}(x_1) = \hat{\mathbf{R}}(x_1, 0, 0) - \mathbf{r}(x_1) \), which implies the following three constraints

\[
w_i(x_1, 0, 0) = 0
\]

(2.7)

However, sometimes the undeformed reference line is not formed by material points such
as the centroid of the box beam, applying constraints at the reference line will not be meaningful. Thus if one define the 1D displacement vector \( \mathbf{u} \) in terms of the 3D displacement field as

\[
(1) \quad \mathbf{u} = \langle \mathbf{R} - \mathbf{r} \rangle - \langle x_\alpha \rangle (\mathbf{T}_\alpha - \mathbf{b}_\alpha)
\]

(2.8)

In view of Eqs. (2.1) and (2.5), the warping functions in Eq. (2.7) can be expressed as

\[
\langle w_i(x_1, x_2, x_3) \rangle = 0
\]

(2.9)

The fourth constraint is chosen related to twisting. The local rotation about \( x_1 \) can be obtained from the elasticity theory as

\[
\theta_1(x_1, x_2, x_3) = \frac{1}{2}(\hat{u}_{3,2} - \hat{u}_{2,3})
\]

\[
= \frac{1}{2} (\mathbf{b}_3 \cdot \mathbf{T}_2 - \mathbf{b}_2 \cdot \mathbf{T}_3) + \frac{1}{2} (w_{3,2} - w_{2,3})
\]

(2.10)

where \( \hat{u}_\alpha \) are the 3D displacement components in \( \mathbf{T}_i \). If we define \( \frac{1}{2} (\mathbf{b}_3 \cdot \mathbf{T}_2 - \mathbf{b}_2 \cdot \mathbf{T}_3) \) the same as average of the local rotation about \( x_1 \) over the cross-section, \( \langle \theta_1(x_1, x_2, x_3) \rangle \), then the fourth constraint on warping functions can be deduced as

\[
\langle w_{3,2}(x_1, x_2, x_3) - w_{2,3}(x_1, x_2, x_3) \rangle = 0
\]

(2.11)

The four constraints in Eqs. (2.9) and (2.11) can be written in a matrix form as

\[
\langle \Gamma_c \mathbf{w} \rangle = 0
\]

(2.12)

with \( \mathbf{w} = [w_1 \ w_2 \ w_3]^T \) and

\[
\Gamma_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \partial_3 & \partial_2 \end{bmatrix}
\]

(2.13)
2.1.3 3D Strain Field

Wempner [83] defines the covariant base vectors in the undeformed state as

\[ \mathbf{g}_i(x_1, x_2, x_3) = \frac{\partial \mathbf{r}}{\partial x_i} \]  

(2.14)

while the contravariant base vectors can be obtained by standard means

\[ \mathbf{g}^i(x_1, x_2, x_3) = \frac{1}{2\sqrt{g}} \epsilon_{ijk} \mathbf{g}_j \times \mathbf{g}_k \]  

(2.15)

where \( g \) is the determinant of the metric tensor, \( \text{i.e.,} \)

\[ g = \det(\mathbf{g}_i \cdot \mathbf{g}_j) \]  

(2.16)

For the beam structure, it is helpful to note that

\[ \sqrt{g} = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) = 1 - x_2 k_3 + x_3 k_2 \]  

(2.17)

where \( k_1 \) is the initial twist and \( k_\alpha \) are the initial curvatures. The formulas for differential volume element and differential lateral surface element can be derived here. Denoting the differential volume element occupied by the undeformed body by \( dV \), we have

\[ dV = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) \, dx_1 dx_2 dx_3 \]  

(2.18)

and the differential lateral surface element occupied by the undeformed body \( dS \) can be calculated as

\[ dS = |\mathbf{\tau} \times \mathbf{g}_1| \, ds \, dx_1 = \sqrt{g + \left( x_2 \frac{dx_2}{ds} + x_3 \frac{dx_3}{ds} \right)^2 k_1^2} \, ds \, dx_1 \]

\[ \equiv \sqrt{c} \, ds \, dx_1 \]  

(2.19)
where $\boldsymbol{r} = \frac{dr^1}{ds} \mathbf{g}_2 + \frac{dr^3}{ds} \mathbf{g}_3$ is the unit vector tangent to the boundary curve, and $ds$ is the differential arc length along the boundary curve.

Based on the concept of decomposition of the rotation tensor [84], the Jauman-Biot-Cauchy strain components for small local rotation are given by

$$
\Gamma_{ij} = \frac{1}{2} (\chi_{ij} + \chi_{ji}) - \delta_{ij}
$$

(2.20)

where $\delta_{ij}$ is the Kronecker symbol and $\chi_{ij}$ is the mixed-basis component of the deformation gradient tensor which can be calculated as

$$
\chi_{ij} = \mathbf{T}_i \cdot \mathbf{G}_k g^k \cdot \mathbf{b}_j
$$

(2.21)

where $\mathbf{G}_k$ is the covariant base vector in the deformed configuration and can be calculated as

$$
\mathbf{G}_i(x_1, x_2, x_3) = \frac{\partial \mathbf{r}}{\partial x_i}
$$

(2.22)

In order to construct a reduced beam model, 1D generalized strains are needed to express the original 3D strain field. First, we define the generalized strains of classical theory as

$$
\bar{\epsilon} = \begin{bmatrix} \bar{\gamma}_{11} & \bar{\kappa}_1 & \bar{\kappa}_2 & \bar{\kappa}_3 \end{bmatrix}
$$

(2.23)

The 1D force strain measures [74, 85] are defined as

$$
\bar{\gamma} = \mathbf{C}_b^T \cdot (\mathbf{r} + \mathbf{u})' - \mathbf{r}'
$$

(2.24)

where $\mathbf{C}_b^T = \mathbf{b}_iT_i$. The moment strain measures $\bar{\kappa}_i$ are defined based on the rate of change along $x_1$ of the triad $\mathbf{T}_i$, viz.

$$
\mathbf{T}_i' = (k_j + \bar{\kappa}_j) \mathbf{T}_j \times \mathbf{T}_i \equiv \bar{\mathbf{K}} \times \mathbf{T}_i
$$

(2.25)
It is noted that the initial twist and curvatures $k_i$ are measured in $b_i$ such that

$$b'_i = k_j b_j \times b_i \equiv k \times b_i \quad (2.26)$$

To illustrate the physical meaning of the moment strain clearly, we can rewrite Eq. (2.25) in the following equivalent form

$$\tilde{\kappa} = C^{bT} \cdot \bar{K} - k \quad (2.27)$$

Another set of 1D strains is needed for the Timoshenko-like beam model which considers the transverse shear effect. The generalized Timoshenko strains associated with $B_i$ basis are defined as

$$\gamma = C^{bB} \cdot (r + u)' - r' \quad (2.28)$$
$$\kappa = C^{bB} \cdot K - k \quad (2.29)$$

where

$$\gamma = [\gamma_{11} \ 2\gamma_{12} \ 2\gamma_{13}]^T \quad (2.30)$$
$$\kappa = [\kappa_1 \ \kappa_2 \ \kappa_3]^T \quad (2.31)$$

and $C^{bB} = b_i \cdot B_i$ is the direction cosine matrix, $K$ is defined in the similar manner as $\bar{K}$ in Eq. (2.25).

Denoting $\epsilon = [\gamma_{11} \ \kappa_1 \ \kappa_2 \ \kappa_3]^T$ and $\gamma_s = [2\gamma_{12} \ 2\gamma_{13}]^T$, it is clear that the generalized classical strains ($\bar{\epsilon}$) must be related with the generalized Timoshenko strains ($\epsilon$ and $\gamma_s$) in some fashion because both sets are used to describe the deformation of the same structure. As shown in Refs. [74, 82], we can attain the following kinematic relations between these two sets of 1D strains

$$\bar{\epsilon} = \epsilon + Q \gamma'_s + P \gamma_s \quad (2.32)$$
The matrices $Q$ and $P$ are given by

\[
Q = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -1 \\
1 & 0
\end{bmatrix}, \quad
P = \begin{bmatrix}
0 & 0 \\
k_2 & k_3 \\
-k_1 & 0 \\
0 & -k_1
\end{bmatrix}
\] (2.33)

It is clear that

\[
\bar{\gamma}_{11} = \gamma_{11}\big|_{\gamma_{1\alpha} = 0} \quad \bar{\kappa}_i = \kappa_i\big|_{\gamma_{1\alpha} = 0}
\] (2.34)

Substituting Eqs. (2.15), (2.22), and (2.21) into Eq. (2.20) along with the 1D strains defined in Eq. (2.23), one can obtain the 3D strain field in the following matrix form as

\[
\Gamma = \Gamma_a w + \Gamma_\epsilon \bar{\epsilon} + \Gamma_R w + \Gamma_t w'
\] (2.35)

where

\[
\Gamma_a = \begin{bmatrix}
0 & 0 & 0 \\
\partial_2 & 0 & 0 \\
\partial_3 & 0 & 0 \\
0 & \partial_2 & 0 \\
0 & \partial_3 & \partial_2 \\
0 & 0 & \partial_3
\end{bmatrix}
\] (2.36)

\[
\Gamma_\epsilon = \frac{1}{\sqrt{\eta}} \begin{bmatrix}
1 & 0 & x_3 & -x_2 \\
0 & -x_3 & 0 & 0 \\
0 & x_2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (2.37)
\[ \Gamma_R = \frac{1}{\sqrt{g}} \begin{bmatrix} k^* & -k_3 & k_2 \\ k_3 & k^* & -k_1 \\ -k_2 & k_1 & k^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (2.38)

where \( k^* = k_1 (x_3 \partial_2 - x_2 \partial_3) \), and

\[ \Gamma_i = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (2.39)

This form of the 3D strain field is of great importance because it is linear in \( \bar{\epsilon} \) and the warping functions \( w \) and their derivatives \( w' \).

There is an important property of the operator matrix \( \Gamma_a \) which is needed in the following chapters. This operator has a kernel, which implies

\[ \Gamma_a w = 0 \] (2.40)

for nontrivial warping functions. It is a set of six differential equations. One can solve these equations and find that the following warping functions satisfying this requirement

\[ w_1 = c_1 \]
\[ w_2 = c_2 - c_4 x_3 \] (2.41)
\[ w_3 = c_3 + c_4 x_2 \] (2.42)
where $c_1$, $c_2$, $c_3$, and $c_4$ are arbitrary constants. If we choose one of these constants to be 1 and the others to be 0, the “kernel matrix” for $\Gamma_a$ can be obtained as

$$
\psi = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -x_3 \\
0 & 0 & 1 & x_2 \\
\end{bmatrix}
$$

(2.43)

2.2 Variational-Asymptotic Method

Variational-asymptotic method is meant as a method of asymptotic analysis of functionals. This method allows one to consider the minimization problems for functions of a finite number of variables and the problems for differential equations possessing the variational structure from a common point of view. A recent monograph [86] covers all aspects of this theory, so more details may be found there for interested readers.

2.2.1 Basics of Asymptotic Analysis

The variational-asymptotic method is based on the idea of neglecting the small terms in energy. To apply it in a systematic way, one has to learn how to recognize small terms. First we need to learn some terminology frequently used in asymptotic analysis: $O$, $o$, and $\sim$. Suppose $f(x)$ and $g(x)$ are continuous functions defined on some domain and possessing limits as $x \to x_0$ in the domain. We can define the following shorthand notation for the relative properties of these functions in the limit $x \to x_0$.

- $f(x) = O(g(x))$ as $x \to x_0$ if $|f(x)| \leq K|g(x)|$ in the neighborhood of $x_0$ with $K$ denoting a constant. We say that $f(x)$ is asymptotically bounded by $g(x)$ in magnitude as $x \to x_0$ or $f(x)$ is of the order of $g(x)$.

- $f(x) = o(g(x))$ as $x \to x_0$ if $|f(x)| \leq \epsilon|g(x)|$ in the neighborhood of $x_0$ for all positive value $\epsilon$. We say that $f(x)$ is asymptotically smaller than $g(x)$.

- $f(x) \sim g(x)$ as $x \to x_0$ if $f(x) = g(x) + o(g(x))$ in the neighborhood of $x_0$. We say that $f(x)$ is asymptotically equal to $g(x)$. 

To correctly recognize small terms in a functional, we not only need to know the asymptotic order of the functions, but also often need to know the asymptotic order of their derivatives. To this end, we need to introduce the notation of the characteristic length. Consider a function \( f(x) \) defined for \( x \in [a, b] \) and sufficiently smooth in this domain. We denote the amplitude of change of \( f(x) \) on \( [a, b] \) as the maximum difference of the function evaluated at any two points in the domain, \( i.e. \)

\[
\bar{f} = \max_{x_1, x_2 \in [a, b]} |f(x_1) - f(x_2)|
\] (2.44)

Then for a sufficiently small number \( l \), the following inequality holds

\[
\left| \frac{df}{dx} \right| \leq \frac{\bar{f}}{l}
\] (2.45)

The largest constant \( l \) satisfying the above inequality is termed the characteristic length of function \( f(x) \) in its own definition domain. If we need to estimate higher derivatives, then the corresponding terms are included in the definition of \( l \), and the characteristic length is the largest constant satisfying the following inequalities

\[
\left| \frac{df}{dx} \right| \leq \frac{\bar{f}}{l}, \quad \left| \frac{d^2 f}{dx^2} \right| \leq \frac{\bar{f}}{l^2}, \quad \cdots, \quad \left| \frac{d^k f}{dx^k} \right| \leq \frac{\bar{f}}{l^k}
\] (2.46)

where \( k \) is the highest derivative we want to estimate the asymptotic order. This definition of characteristic length can be easily generalized to functions of multiple variables.

### 2.2.2 Variational-Asymptotic Method - an Example

The variational-asymptotic method is of heuristic character. It does not have a strict mathematical foundation, while is formulated as a set of rules, along with their applications illustrated using examples. Here we use an example to show the basic idea and procedure of this method. Let a functional \( I(u, \eta) \) depending on a small parameter \( \eta \) be given at some set \( M \) of elements \( u \). For a beam-like structure, the variable \( u \) represents the 3D displacement field and \( \eta \) is the aspect ratio of the cross-section with respect to the span.
Assuming that the functional $I(u, \eta)$ has a stationary point, denoted by $\bar{u}$, it is clear that $\bar{u}$ is a function of the small parameter $\eta$. This will be emphasized by placing $\eta$ in the index, $\bar{u}_\eta$. We further assume that $\bar{u}$ approaches its asymptotic limit $u_0$ as $\eta \to 0$, which is often called the zeroth-order approximation of $\bar{u}$. It is natural to start from investigating the functional where all small terms were dropped, i.e., the functional $I_0(u) = I(u, 0)$.

**Example** Using variational-asymptotic method to investigate the stationary points of the function of one variable $u$

$$f(u, \eta) = u^2 + u^3 + 2\eta u + \eta u^2 + \eta^2 u$$  \hspace{1cm} (2.47)

with $\eta$ as a small parameter.

The stationary points of the function $f(x, \eta)$ can be analytically solved as

$$u = \frac{1}{3} \left( -1 - \eta \pm \sqrt{1 - 4\eta - 2\eta^2} \right)$$  \hspace{1cm} (2.48)

This exact solution can be expanded asymptotically in terms of $\eta$ as

$$u = \begin{cases} 
-\frac{2}{3} + \frac{\eta}{3} + \eta^2 + o(\eta^2) \\
0 - \eta - \eta^2 + o(\eta^2) 
\end{cases}$$  \hspace{1cm} (2.49)

According to VAM, the zeroth-order approximation of the function is obtained as $f_0(u) = f(u, 0) = u^2 + u^3$, which has two stationary points $u = 0$ and $u = -\frac{2}{3}$. It is clear that the stationary points found from $f_0(u)$ are the zeroth-order approximation of the stationary points of $f(u, \eta)$.

To find the next approximation, we need to start with the two stationary points in the zeroth-order approximation. First consider the stationary point of the function Eq. (2.47) in the neighborhood of $-\frac{2}{3}$, which is one of $\bar{u}_0$. Setting $u = -\frac{2}{3} + u'$ (note that $u'$ here is not the derivative with respect to $x_1$ but the first asymptotic expansion of $u$), we obtain
the following function

\[
f(-\frac{2}{3} + u', \eta) = -u'^2 + \frac{2u'\eta}{3} + u'^3 + u^2\eta + u'\eta^2 + \frac{4}{27} - \frac{8\eta}{9}
\]  \quad (2.50)

The double underlined terms are additive constants that will not affect the stationary points and can be simply dropped. The underlined terms are much smaller than those non-underlined terms. To be specific,

\[
|u'^3| \ll |u'^2| \quad |u'^2\eta| \ll |u'^2| \quad |u'\eta^2| \ll \left|\frac{2u'\eta}{3}\right|
\]  \quad (2.51)

in view of the fact that both \( u' \) and \( \eta \) are small. Keeping the leading terms with respect to \( u' \) in the function \( f(-\frac{2}{3} + u', \eta) \), we arrived at the following function

\[
f_1(u', \eta) = -u'^2 + \frac{2u'\eta}{3}
\]  \quad (2.52)

It is stationary when \( u' = \frac{1}{3}\eta \). Note that the asymptotic order of \( u' \) is not assumed \textit{a priori}, but is determined as the stationary point of the function \( f_1(u', \eta) \). Hence, we have obtained the first-order approximation of the stationary point in the neighborhood of \(-\frac{2}{3}\) as

\[
\bar{u}_\eta = -\frac{2}{3} + \frac{1}{3}\eta + o(\eta)
\]  \quad (2.53)

The first-order approximation in the neighborhood of 0, which is the other solution of \( \bar{u}_0 \), can be obtained analogously. Setting \( u = 0 + u' \), we obtain the following function

\[
f(u', \eta) = u'^2 + 2\eta u' + u'^3 + \eta u'^2 + \eta^2 u'
\]  \quad (2.54)

The underlined terms are much smaller than those non-underlined terms. That is

\[
|u'^3| \ll |u'^2| \quad |u'^2\eta| \ll |u'^2| \quad |u'\eta^2| \ll |2u'\eta|
\]  \quad (2.55)
in view of the fact that both $u'$ and $\eta$ are small. Keeping the leading terms with respect to
$u'$ in the function $f(u', \eta)$, we arrive at the following function

$$f_1(u', \eta) = u'^2 + 2u'\eta$$

(2.56)

It is stationary when $u' = -\eta$. Hence, we have obtained the first-order approximation of
the stationary point in the neighborhood of 0 such that

$$\hat{u}_\eta = 0 - \eta + o(\eta)$$

(2.57)

Till now, we have reproduced that the first two terms of the asymptotic expansion of the
exact solution. We can continue this process to find higher-order approximations.

This example demonstrates that the main issue in the asymptotic analysis is to recog-
nize the leading terms and the negligible terms. Usually, this is the most important and
most difficult point of the asymptotic analysis. To determine which terms are negligible,
we need to consider the following two conditions.

- For two terms $A(u, \eta)$ and $B(u, \eta)$ which are summed in the functional $I(u, \eta)$, if

$$\lim_{\eta \to 0} \max_{u \in M} \frac{|B(u, \eta)|}{|A(u, \eta)|} = 0$$

(2.58)

then $B(u, \eta)$ is negligible in comparison to $A(u, \eta)$ for all stationary points. Such
terms are called globally secondary.

- Let $\hat{u} \to 0$ for $\eta \to 0$, and for any sequence $\{u_n\}$ converging to $u = 0$. If

$$\lim_{n \to \infty, \eta \to 0} \frac{|B(u, \eta)|}{|A(u, \eta)|} = 0$$

(2.59)

then $B(u, \eta)$ is negligible in comparison to $A(u, \eta)$ for the stationary point $\hat{u}_\eta$. Such
terms are called locally secondary.
In the illustrative example, the term $\eta u^2$ is globally secondary with respect to $u^2$, the term $\eta^2 u$ is globally secondary with respect to $2\eta u$ while $u^3$ is locally secondary with respect to $u^2$ in the neighborhood of the point $u = 0$.

### 2.3 Hamilton’s Principle

The elastodynamic behavior of a structure is governed by the extended Hamilton’s principle [87]

$$\int_{t_1}^{t_2} \left[ \delta(K - U) + \delta W \right] dt = 0$$

(2.60)

where $t_1$ and $t_2$ are arbitrary fixed times, $K$ and $U$ are the kinetic and internal energy, respectively, $\delta W$ is the virtual work of applied loads. The overbar indicates that the virtual work needs not be the variation of a functional. Only for a very limited number of cases, this 3D variational statement can be solved exactly using analytical methods. To simplify the original 3D problem to a 1D variational statement, we need to express these quantities, $K$, $U$, and $\delta W$ in terms of 1D beam variables.

#### 2.3.1 Kinetic Energy

To calculate the kinetic energy, we need to know the absolute velocity of a generic point in the structure by taking a time derivative of Eq. (2.5), such that

$$v = V + \Omega(\xi + \bar{w}) + \dot{\bar{w}}$$

(2.61)

where $V$ is the absolute velocity of a point in the deformed reference line, $\Omega$ is the inertial angular velocity of $B_i$ bases. In Eq. (2.61), the symbols $v, V, \Omega, w$ denote column matrices containing the components of corresponding vectors in $B_i$ bases, and $\xi = [0 \ x_2 \ x_3]^T$. The kinetic energy of a beam can be obtained by

$$K = \frac{1}{2} \int_{\mathcal{V}} \rho v^T vd\mathcal{V} = K_{1D} + K^*$$

(2.62)

where $K_{1D}$ is the portion of the kinetic energy which is not related with the 3D unknown
warping functions $\tilde{w}_i(x_1, x_2, x_3)$, $V$ is the volume occupied by the structure, $\rho$ is the mass density, and

$$K_{1D} = \int_0^l K dx_1$$

$$K^* = \frac{1}{2} \int_V \rho \left[ (\tilde{\Omega} \tilde{w} + \dot{\tilde{w}})^T (\tilde{\Omega} \tilde{w} + \dot{\tilde{w}}) + 2(V + \tilde{\Omega} \xi)^T (\tilde{\Omega} \tilde{w} + \dot{\tilde{w}}) \right] dV$$

where $l$ is the length of the beam; $K$ is the 1D kinetic energy per unit span and can be calculated as

$$K = \frac{1}{2} (\mu V^T V + 2\Omega^T \mu \tilde{\xi} V + \Omega^T i \Omega)$$

with $\mu$, $\mu \tilde{\xi}$, and $i$ defined as mass per unit length, the first and second distributed mass moments of inertia respectively, which can be trivially obtained through simple integrals over the cross-section as

$$\mu = \langle \langle \rho \rangle \rangle \quad \mu \tilde{\xi} = \langle \langle \rho \xi \rangle \rangle \quad i = \langle \langle \rho (\xi^T \xi \Delta - \xi \xi^T) \rangle \rangle$$

where $\Delta$ is a unit matrix.

### 2.3.2 Strain Energy

If the structure is made of linear elastic material, the 3D Biot stress tensor $\sigma_{ij}$, which is conjugate to the Jaumann-Biot-Cauchy strain tensor, is related with the 3D strain $\Gamma$ using the generalized Hooke’s law

$$\sigma = \mathcal{D} \Gamma$$

where the 3D stress components are elements of the matrix $\sigma = [\sigma_{11} \sigma_{12} \sigma_{13} \sigma_{22} \sigma_{23} \sigma_{33}]^T$. It is noted that $\mathcal{D}$ could be fully populated if the beam is made of general anisotropic material.

The strain energy of the beam structure can be written as

$$U = \int_V \frac{1}{2} \Gamma^T \Gamma dV = \int_0^l U dx_1$$
with
\[ U = \frac{1}{2} \left\langle \left\langle f^T \mathcal{D} f \right\rangle \right\rangle \quad (2.69) \]
where \( U \) is the strain energy per unit span.

### 2.3.3 Virtual Work

The virtual work done by applied loads can be calculated as
\[
\delta W = \int_0^L \left( \left\langle \left\langle p \cdot \delta R \right\rangle \right\rangle + \int_{\partial \Omega} Q \cdot \delta \hat{R} \sqrt{c} \, ds \right) \, dx_1 + \left( Q \cdot \delta \hat{R} \right)_{|x_1=0} + \left( Q \cdot \delta \hat{R} \right)_{|x_1=l} \quad (2.70)
\]
where \( \partial \Omega \) denotes the lateral surface of the undeformed beam, \( p = p_i \mathbf{B}_i \) is the applied body force per unit undeformed volume, \( Q = Q_i \mathbf{B}_i \) is the applied surface tractions of the undeformed beam. Note if the displacements on the end surfaces are prescribed, the last two terms of Eq. (2.70) will vanish. \( \delta \hat{R} \) is the Lagrangian variation of the displacement field in Eq. (2.3), such that
\[
\delta \hat{R} = \overline{\delta q_i \mathbf{B}_i} + x_\alpha \delta B_\alpha + \delta \bar{w}_i \mathbf{B}_i + \bar{w}_j \delta B_j \quad (2.71)
\]
The virtual displacements and rotations are defined as
\[
\overline{\delta q_i} = \delta u \cdot \mathbf{B}_i \quad \delta \mathbf{B}_i = \overline{\delta \psi_j B_j} \times \mathbf{B}_i \quad (2.72)
\]
where \( \overline{\delta q_i} \) and \( \overline{\delta \psi_i} \) contain the components of the virtual displacement and rotation in the \( \mathbf{B}_i \) system, respectively. Since the warping functions are small, one may safely ignore products of the warping and virtual rotation in \( \delta \hat{R} \) and obtain the virtual work due to applied loads as
\[
\overline{\delta W} = \overline{\delta W}_{1D} + \overline{\delta W}' \quad (2.73)
\]
where \( \overline{\delta W}_{1D} \) is the virtual work not related with the warping functions \( \bar{w}_i(x_1, x_2, x_3) \) and...
\( \delta W^* \) is the virtual work related with the warping functions. The expression of \( \delta W_{1D} \) and \( \delta W^* \) can be written as

\[
\delta W_{1D} = \int_0^l \delta \bar{W} dx_1 + \bar{F}_i \delta \bar{q}_i |_{x_1=0} + \bar{M}_i \delta \bar{\psi}_i |_{x_1=l}
\]

\[
\delta W^* = \int_0^l \left( \langle p_i \delta \bar{w}_i \rangle + \oint Q_i \delta \bar{w}_i \sqrt{c} \, ds \right) dx_1 + \langle Q_i \delta \bar{w}_i \rangle |_{x_1=0} + \langle Q_i \delta \bar{w}_i \rangle |_{x_1=l}
\]

where \( \delta \bar{W} \) is the virtual work per unit span and is expressed as

\[
\delta \bar{W} = f_i \delta \bar{q}_i + m_i \delta \bar{\psi}_i
\]

with the generalized forces \( f_i \) and moments \( m_i \) defined as

\[
f_i = \langle p_i \rangle + \oint Q_i \sqrt{c} \, ds \quad m_i = e_{i\alpha j} \left( \langle x_\alpha p_j \rangle + \oint x_\alpha Q_j \sqrt{c} \, ds \right)
\]

\( \bar{F}_i \) and \( \bar{M}_i \) are generalized forces applied at the ends of the beam and they are defined as

\[
\bar{F}_i = \langle Q_i \rangle \quad \bar{M}_i = e_{i\alpha j} \langle x_\alpha Q_j \rangle
\]

2.3.4 Reformulation of Hamilton’s Principle

The Hamilton’s principle in Eq. (2.60) becomes

\[
\int_{t_1}^{t_2} \left[ \delta \left( K_{1D} + K^* - U \right) + \delta \bar{W}_{1D} + \delta \bar{W}^* \right] dt = 0
\]

So far, we have presented a 3D formulation for the beam structure in terms of 1D displacements (represented by \( u \)) and rotations (represented by \( b_i B_i \)) and 3D warping functions (\( \bar{w}_i \)). Note that these quantities also can be expressed in \( T_i \) basis using Eqs. (2.4) and (2.6).

It is noted that only two assumptions are invoked in the derivation of this 3D variational statement including the small local rotation assumption which is used in the decomposition of rotation tensor and the small strain assumption which is essential for obtaining a geometrically nonlinear and material linear theory.
The difficulty of solving this problem directly comes from the unknown 3D warping functions $\tilde{w}_i$ or $w_i$. Fortunately, VAM provides a powerful technique to attain the warping functions through an asymptotical analysis of the variational statement in Eq. (2.79) in terms of small parameters inherent to the structure to construct asymptotically correct 1D beam models represented by a variational statement as

$\int_{t_1}^{t_2} \int_0^l \left[ \delta (K - U) + \delta W \right] dx_1 dt = \delta \mathcal{A}$

(2.80)

here $K$, $U$, and $\delta W$ are functions of 1D beam variables which are functions of the reference line $x_1$. Note that $\delta \mathcal{A}$ consists of the last terms due to applied loads at the ends in Eq. (2.74)

$\delta \mathcal{A} = - \int_{t_1}^{t_2} \left( \delta \bar{q}^T \bar{F} + \delta \bar{\psi}^T \bar{M} \right) |^l_0 dt$

(2.81)

where the quantities with a hat ($\hat{\cdot}$) are the force and moment evaluated at the ends of space interval ($x_1 = 0$ and $x_2 = l$).

### 2.4 Order Assessment

The dimensional reduction from the original 3D formulation, Eq. (2.79), to a 1D formulation, Eq. (2.80), can only be done approximately. According to VAM, we can take advantage of the small parameters inherent to the structure to construct the 1D formulation. In order to apply this method, first it is necessary to assess the order of quantities in terms of small parameters.

For a structure to be modeled as a beam, it should be slender, which means $h/l \ll 1$ and $h/R \ll 1$, with $h$ as the characteristic size of the cross-section, $l$ the characteristic wavelength of the deformation along the axial direction, and $R$ the characteristic radius of initial curvatures and twist of the beam. For simplicity, we assume $R$ and $l$ are of the same order, which means $\tilde{h} \sim h/l \sim h/R \ll 1$.

We have assumed both the 3D strain field and the 1D strain field to be small as we are only interested in a geometrically nonlinear but physically linear theory, i.e., $\hat{\epsilon} = O(\bar{\epsilon}) =$
\( O(\varepsilon) = O(\Gamma) \ll 1 \) with \( \varepsilon \) denoting the characteristic magnitude of both 3D and 1D strain field. It is noted that \( \varepsilon = O(\varepsilon) = O(\varepsilon) \) actually means \( \varepsilon = O(\gamma_{11}) = O(h\kappa_i) = O(\gamma_{11}) = O(h\kappa_i) \) as one can easily conclude from Eq. (2.35). From the same equation, we can deduce that \( w_i = O(h\varepsilon) \) as the first term cannot be asymptotically larger than \( \varepsilon \). It is also obvious that the order of the last two terms in this equation are one order of \( \tilde{h} \) smaller than the first two terms. From Eq. (2.6) we can conclude that \( O(\bar{w}_i) = O(w_i) \) because the transverse shear strain \( 2\gamma_{1\alpha} \) cannot be asymptotically larger than \( \varepsilon \).

The 1D strain energy density, \( \mathcal{U} \) in Eq. (2.69), will be of the order of \( \nu h^2\varepsilon^2 \) with \( \nu \) denoting the order of the elastic constants. The condition of the boundedness of deformations for \( \tilde{h} \to 0 \) puts some constraints on the external forces of how fast the structure can vibrates. It is clear that the virtual work and the kinetic energy must be of the same order as the strain energy, i.e., \( \Delta W \sim K \sim O(\nu h^2\varepsilon^2) \). Here we assume the asymptotic order of the variation of a quantity is inherited from the order of the corresponding quantity. Clearly, we have \( \Delta W \sim K \sim O(\nu h^2\varepsilon^2) \). This constraint on the order of virtual work can help us estimate the order of applied forces. If we estimate the asymptotic order of the virtual displacements and rotations as

\[
\begin{align*}
\delta q_1 & \sim O(\varepsilon\ell) & \delta q_\alpha & \sim O(\varepsilon\ell^2/h) & \delta \psi_i & \sim O(\varepsilon\ell/h)
\end{align*}
\]

then we can estimate the orders of the generalized forces \( f_i \) and moments \( m_i \)

\[
\begin{align*}
f_1 & \sim O(\nu h\varepsilon) & f_\alpha & \sim O(\nu h^2\varepsilon) & m_i & \sim O(\nu h^2\varepsilon) \\
(2.82)
\end{align*}
\]

Then we can estimate the orders of the body forces and tractions can be estimated as

\[
\begin{align*}
p_1 & \sim O(\mu h\varepsilon/h) & Q_1 & \sim O(\mu h\varepsilon) & p_\alpha & \sim O(\mu h^2\varepsilon/h) & Q_\alpha & \sim O(\mu h^2\varepsilon) \\
(2.84)
\end{align*}
\]
The present work is restricted to those dynamical process for which the following equation holds

\[
\frac{h}{c_s \tau} \sim O(\dot{h}) \ll 1 \quad (2.85)
\]

where \( \tau \) is the characteristic scale of change of the displacement and warping functions in time and \( c_s = \sqrt{\mu/\rho} \) is the characteristic velocity of the shear waves.

In view of the orders estimated in Eq. (2.85) and Eq. (2.61), we can conclude that \( K^* \) is much smaller than \( K_{1D} \) so that it can be dropped in the present work. Physically, it indicates that we are dealing with low frequency vibration problems.

The virtual work term \( \delta W^* \) is much smaller than \( \delta W_{1D} \) as we know that \( \delta \bar{w}_i \ll \delta q_i \).

### 2.5 Composite Beam Analysis Based on VAM

To illustrate the procedure of beam analysis based on VAM, two examples are presented here and the results are compared with those obtained by 3D finite element analysis and other beam theory. The origins of the coordinates used in these two examples are located at the geometric centers of the cross-section at the fixed ends; \( x_1, x_2, \) and \( x_3 \) represents the axis along the reference line, the width of the beam cross-section, and thickness of the beam cross-section, respectively.

#### 2.5.1 Example 1: Four-layer Cross-ply Composite Beam

The first illustrative example is analysis of a four-layer composite beam with the stacking sequence \([0^\circ/90^\circ/90^\circ/0^\circ]\), as shown in Fig. 2.3. The beam is cantilevered at one end and a shear force in the negative \( x_3 \) direction is applied at the free end. The material properties, geometric properties and loading are given in Table 2.1.

A 2D cross-section analysis is carried out first. The width is divided by five eight-noded quadrilateral elements and along the thickness each layer is divided into two elements, see Fig. 2.4. The non-zero cross-sectional stiffness constants are listed in Table 2.2.

The second step is to carry out a 1D beam analysis. Here we turn to Hodges beam theory [74]. Using the stiffness constants obtained in the previous step, the deflection can
Table 2.1: Material Properties, Geometric Properties and Loading for Example 1

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Geometric Properties</th>
<th>Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1 = E_3 = 206.8$ GPa</td>
<td>$L = 254$ mm</td>
<td>$P = 44.4528$ N</td>
</tr>
<tr>
<td>$E_2 = 83.74$ GPa</td>
<td>$b = 25.4$ mm</td>
<td></td>
</tr>
<tr>
<td>$\nu_{12} = \nu_{13} = \nu_{23} = 0.12$</td>
<td>$t = 12.7$ mm</td>
<td></td>
</tr>
<tr>
<td>$G_{12} = G_{13} = G_{23} = 48.27$ GPa</td>
<td>$t_1 = t_4 = 2.54$ mm</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$t_2 = t_3 = 3.81$ mm</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Cross-sectional Constants of Four-layer Cross-ply Composite Beam

<table>
<thead>
<tr>
<th>Stiffness</th>
<th>VABS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{11}(N)$</td>
<td>$4.292 \times 10^7$</td>
</tr>
<tr>
<td>$s_{22}(N)$</td>
<td>$1.286 \times 10^7$</td>
</tr>
<tr>
<td>$s_{33}(N)$</td>
<td>$1.347 \times 10^7$</td>
</tr>
<tr>
<td>$s_{44}(N \text{ m}^2)$</td>
<td>$5.748 \times 10^2$</td>
</tr>
<tr>
<td>$s_{55}(N \text{ m}^2)$</td>
<td>$7.819 \times 10^2$</td>
</tr>
<tr>
<td>$s_{66}(N \text{ m}^2)$</td>
<td>$2.307 \times 10^4$</td>
</tr>
</tbody>
</table>
be calculated by hand as

\[ u_3(x_1) = x_1 \frac{P}{s_{33}} + \left( \frac{x_1^3}{6} - \frac{x_1^2 L}{2} \right) \frac{P}{s_{55}} \]  

(2.86)

The maximum displacement in \( x_3 \) direction obtained from the current model is compared with those obtained by 3D ANSYS analysis and Lin’s [88] model, see Table 2.3. Excellent agreement between current result and those obtained by other method can be found.

Finally a convergence study is presented. The cross-section is divided into five elements along the width while the element used in discretizing each layer (along the thickness) changes from one to four; and these four meshing schemata are termed as Mesh 1, Mesh 2, Mesh 3, and Mesh 4, respectively. The discretized cross-section shown in Fig. 2.4 is Mesh 2 according to this definition. Fig. 2.5 shows the convergence of the cross-sectional constants obtained by VABS; and these values are scaled as the ratio between the current results and the results obtained by Mesh 4 calculation. Here the results obtained by Mesh 4 is considered as converged values. From Fig. 2.5, one can find that the VABS results converged very fast and Mesh 2 schema is sufficient for this case.

### 2.5.2 Example 2: Eight-layer Composite Beam Composed of Two Materials

The second illustrative example is analysis of a cantilevered laminated beam consisting of eight layers with two materials. Two load cases are considered here, i.e., a concentrated tip load of \( Q = 200 \text{ N} \) (Case A) and a uniformly distributed load of \( q = 100 \text{ N/mm} \) (Case B), as shown in Fig. 2.6. The material properties, geometric properties and loading are given in Table 2.4.

In the 2D cross-sectional analysis, the width is divided into six quadrilateral elements and along the thickness each layer is divided into five elements, see Fig. 2.7. The 1D

<table>
<thead>
<tr>
<th>Max. deflection (10^-3 m)</th>
<th>VABS</th>
<th>ANSYS 3D</th>
<th>Lin and Zhang</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.31139</td>
<td>0.31136</td>
<td>0.31015</td>
</tr>
</tbody>
</table>
Fig. 2.5: Convergence study of Example 1.

Fig. 2.6: Configuration and cross-section of a laminated cantilever beam.
Table 2.4: Material Properties, Geometric Properties and Loading for Example 2

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Geometric Properties</th>
<th>Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1: $E_1 = E_2 = 30.0 \text{ MPa}$</td>
<td>$L_2 = 90 \text{ mm}$</td>
<td>$Q = 200 \text{ N}$</td>
</tr>
<tr>
<td>$E_3 = 1.0 \text{ MPa}$</td>
<td>$t_2 = 10 \text{ mm}$</td>
<td>$q = 100 \text{ N/mm}$</td>
</tr>
<tr>
<td>$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$</td>
<td>$b_2 = 1 \text{ mm}$</td>
<td></td>
</tr>
<tr>
<td>$G_{12} = G_{13} = G_{23} = 0.5 \text{ MPa}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Material 2: $E_1 = E_2 = 5.0 \text{ MPa}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_3 = 1.0 \text{ MPa}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_{12} = G_{13} = G_{23} = 0.5 \text{ MPa}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The constitutive law for a composite beam is

$$
\begin{align*}
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} &= \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{bmatrix} 
\end{align*}
$$

(2.87)

where the non-zero cross-sectional stiffness constants are listed in Table 2.5.

To recover the stress field at the mid-span, the 1D stress resultants, $F_i$ and $M_i$, are needed which can be calculated by a 1D beam analysis. For Case A, the stress resultants are $F_1(x_1) = F_2(x_1) = M_1(x_1) = M_3(x_1) = 0$, $F_3(x_1) = Q$, and $M_2(x_1) = -Q(x_1 - L_2)$. For Case B, the solutions are $F_1(x_1) = F_2(x_1) = M_1(x_1) = M_3(x_1) = 0$, $F_3(x_1) = -qx_1$, and

Table 2.5: Cross-sectional Constants of Eight-layer Composite Beam Composed of Two Materials

<table>
<thead>
<tr>
<th>Stiffness</th>
<th>VABS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{11}(\text{N})$</td>
<td>$1.75 \times 10^8$</td>
</tr>
<tr>
<td>$s_{22}(\text{N})$</td>
<td>$3.122 \times 10^9$</td>
</tr>
<tr>
<td>$s_{33}(\text{N})$</td>
<td>$4.386 \times 10^9$</td>
</tr>
<tr>
<td>$s_{44}(\text{N m}^2)$</td>
<td>$1.562 \times 10^9$</td>
</tr>
<tr>
<td>$s_{55}(\text{N m}^2)$</td>
<td>$1.849 \times 10^9$</td>
</tr>
<tr>
<td>$s_{66}(\text{N m}^2)$</td>
<td>$1.458 \times 10^9$</td>
</tr>
</tbody>
</table>
Fig. 2.7: Discretization of cross-section of an eight-layer composite beam.

\[ M_2(x_1) = \frac{1}{2} qx_1^2. \]

The recovered stress distribution at mid-span along thickness of the beam are plotted in Fig. 2.8 and Fig. 2.9 for Case A and Case B, respectively. Good correlation between the results obtained by different models can be observed. It is pointed out that these two illustrative examples are very simple since the 1D stiffness matrices are diagonal which mean coupling effects between extension, torsion, and bendings do not exist in these two beams. Therefore all other beam theories, including the one presented and conventional Timoshenko beam theory, can have a good answer. However, for beams composed of highly heterogeneous materials and complicate layer-up sequences, VABS can accurately predict all stress components \( \sigma_{ij} \) while some other theories cannot, see Refs. [89,90].
Fig. 2.8: Distribution of $\sigma_{11}$ along thickness at mid-span for Case A.

Fig. 2.9: Distribution of $\sigma_{11}$ along thickness at mid-span for Case B.
Chapter 3

Asymptotic Construction of Thermoelastic Model

This chapter is devoted to the thermoelastic beam modeling. Based on the quasisteady theory of linear thermoelasticity, the thermal problem separates into two problems to be solved consecutively: the heat conduction problem and the one-way coupled thermoelastic problem. Firstly, the 3D formulation of heat conduction problem and thermoelastic problem are presented. Then the dimensional reduction for these two problems are carried out up to different orders. Finally the recovery relations based on the cross-sectional analysis of beams are given in terms of sectional resultants and applied loads.

3.1 3D Formulations

3.1.1 Heat Conduction Problem

The 3D steady heat conduction problem of a composite beam is governed by the variation of the following functional

\[ \Pi = U_T - I_T \]  

where we term \( U_T \) as the thermal potential and \( I_T \) as the power input with expressions as

\[ U_T = \int_0^l \left( \frac{1}{2} (\nabla T)^T K \nabla T \right) \, dx_1 \equiv \int_0^l U_T \, dx_1 \]  

and

\[ I_T = \int_0^l \left[ \langle \langle Q_T \rangle \rangle + \int \bar{q} T \sqrt{c} \, ds \right] \, dx_1 + \langle \bar{q}_e T \rangle \bigg|_{x_1=L} + \langle \bar{q}_e T \rangle \bigg|_{x_1=0} \]  

\( T \) is the 3D temperature field, \( K \) is the conductivity matrix representing the second-order conductivity tensor expressed in the triad \( b_i \), \( Q \) is the density of internal heat source, \( \bar{q} \) is the given heat flux on the lateral boundary surfaces \( \partial \Omega \), and \( \bar{q}_e \) is the given heat flux on the end
surfaces. It is pointed out that if the convection heat transfer is taken into consideration, one more term should be included in the power input $I_T$ as:

$$\int_0^l f \left(-\frac{1}{2} h_c T (T - 2T_\infty) \right) \sqrt{c} ds dx_1$$

where $h_c$ is the convective heat transfer coefficient and $T_\infty$ is the temperature of adjacent fluid outside the boundary layer. The temperature gradient $\nabla T$ in a curvilinear coordinate system can be expressed as:

$$\nabla T = \frac{\partial T}{\partial x_i} g^i$$ (3.4)

There are two types of thermal load for heat conduction analysis.

- **Thermal load case 1**: temperature field is not prescribed at any point over the cross-section except the end surfaces at $x_1 = 0$ and $x_1 = l$ (see Fig. 3.1). For this case we are free to use the following change of variables for the 3D temperature field

$$T(x_1, x_2, x_3) = \mathcal{T}(x_1) + w_T(x_1, x_2, x_3)$$ (3.5)

with the 1D temperature variable $\mathcal{T}(x_1)$ defined as the average of $T$ over the cross-section and $w_T$ is the thermal warping functions that describe the difference between the 3D temperature field and its cross-sectional average. According to the definition, we have the following constraint on thermal warping functions

$$\langle w_T(x_1, x_2, x_3) \rangle = 0$$ (3.6)

and this should be valid for all the surfaces along the beam span including the end surfaces. Note for the thermal load case 1, it is possible that there is no temperature prescribed at one of the end surfaces or both end surfaces.

- **Thermal load case 2**: temperature is prescribed at least one point of the cross-section along the beam span (see Fig. 3.2). For this case, we lose the freedom of introducing the 1D temperature variable $\mathcal{T}(x_1)$ as what we did in Eq. (3.5), and all the 3D
Fig. 3.1: Thermal load case 1: temperature prescribed at the end surfaces of the beam.

Fig. 3.2: Thermal load case 2: temperature prescribed at the lateral surfaces of the beam.

3.1.2 Thermoelasticity Problem

The variational statement for a beam structure has been discussed in Chapter 2, that is, the temperature field must be represented by the thermal warping function as

$$ T(x_1, x_2, x_3) = w_T(x_1, x_2, x_3) $$

And we cannot constrain the thermal warping functions as we did in Eq. (3.6) either.

$$ (3.7) $$
Hamilton’s principle. For the thermoelastic problem, the internal strain energy is replaced with Helmholtz free energy that considers the thermal effect. The expression of Helmholtz free energy per unit span can be written as

$$U_A = \left\langle \frac{1}{2} \Gamma^T D \Gamma - \Gamma^T D \alpha \Delta T \right\rangle$$

(3.8)

where $\Delta T$ is the difference between the temperature in the structure and the reference temperature when the beam is stress free, $D$ is the $6 \times 6$ material matrix, which contains elements of the fourth-order elasticity tensor expressed in the triad $b_i$, and $\alpha$ is a $6 \times 1$ column matrix containing the components of the second-order thermal expansion tensor expressed in the triad $b_i$. These matrices are in general fully populated.

The Hamilton’s principle of thermal problem can be rewritten as

$$\int_{t_1}^{t_2} \left[ \delta (K_{1D} + K^* - U_A) + \delta \bar{W}_{1D} + \delta \bar{W}^* \right] dt = 0$$

(3.9)

Here the definition of $U_A$ is

$$U_A = \int_0^l U_A dx_1$$

(3.10)

while the other terms are the same as those in Chapter 2. It needs to be pointed out that Eq. (3.8) is based on small strain assumption and small temperature change assumption and the material properties are independent of temperature change. However, it can be directly generalized to handle finite temperature change and account for the dependency of material properties which will be shown later.
3.2 Dimensional Reduction of Heat Conduction Problem

3.2.1 Dimensional Reduction of Case 1

In view of Eq. (3.5), the temperature gradient components in Eq. (3.4) can be expressed in the following column matrix

\[
\nabla T = e_1 T' + \Gamma_T w_T + \Gamma_{RT} w_T + e_1 w_T
\]

(3.11)

with \( e_1 = \frac{1}{\sqrt{g}} [1 \ 0 \ 0]^T \), \( T' \) as 1D temperature gradient, and

\[
\Gamma_T = \begin{bmatrix}
0 \\
\partial_2 \\
\partial_3
\end{bmatrix}
\]

(3.12)

\[
\Gamma_{RT} = \frac{1}{\sqrt{g}} \begin{bmatrix}
k_1(x_3 \partial_2 - x_2 \partial_3) \\
0 \\
0
\end{bmatrix}
\]

(3.13)

Zeroth-order Approximation

Substituting Eq. (3.11) into Eq. (3.1), we can obtain the first approximation of the functional as

\[
\Pi_0 = \int_0^l \left[ U_{T0} - \left( \langle \mathcal{Q} \rangle + \oint_{\partial \Omega} \bar{q} \, ds \right) T \right] dx_1 - \langle \bar{q}_e \rangle T|_{x_1=l} - \langle \bar{q}_e \rangle T|_{x_1=0}
\]

(3.14)

where

\[
U_{T0} = \left\langle \frac{1}{2} \left( e_1 T' + \Gamma_T w_T \right)^T K \left( e_1 T' + \Gamma_T w_T \right) \right\rangle
\]

(3.15)

Here terms higher than \( O(\nu T^2) \) are neglected, and \( \nu \) denotes the order of heat conduction coefficients. It is clear that the thermal warping functions can be solved by minimizing the zeroth-order thermal potential \( U_{T0} \) subject to the constraint in Eq. (3.6) as other terms do not contain \( w_T(x_1, x_2, x_3) \).
To deal with the arbitrary cross-sectional geometry and anisotropic materials, one may turn to a numerical approach to find the stationary value of the functional. The thermal warping field can be discretized as

$$w_T(x_1, x_2, x_3) = S(x_2, x_3)V_T(x_1) \quad (3.16)$$

with $S(x_2, x_3)$ representing the matrix of finite element shape functions, and $V_T$ as a column matrix of the nodal values of the thermal warping over the cross-section.

Substituting Eq. (3.16) back into Eq. (3.15) one obtains

$$2\mu_{T0} = V_T^T E_T V_T + 2V_T^T D_T T' + \bar{K} T'^2 \quad (3.17)$$

where the newly introduced matrices are defined as

$$E_T = \left\langle [\Gamma_T S]^{T} K [\Gamma_T S] \right\rangle$$

$$D_T = \left\langle [\Gamma_T S]^{T} K e_1 \right\rangle$$

$$\bar{K} = \langle K_{11} \rangle \quad (3.18)$$

It is pointed out that since we do not consider the initial curved or twisted beam in the zeroth-order approximation, the operator $\langle \bullet \rangle$ is used in the formulation. However, when we take the initial curvature or twist into account in the first-order approximation, the matrices in Eq. (3.18) should be calculated by operator $\langle\langle \bullet \rangle \rangle$. The difference between these two operators has been discussed at the beginning of Chapter 2.

Minimizing Eq. (3.17) subject to constraints in Eq. (3.6), we can obtain the thermal warping function in the following form

$$V_T = \tilde{V}_{T0} T' = V_{T0} \quad (3.19)$$
Having solved $\bar{V}_T$, we can approximate the original functional in Eq. (3.14) using the following 1D functional

$$\Pi_0 = \int_0^l \left[ U_{T0} - \left( \left\langle Q \right\rangle + \int_{\partial \Omega} \bar{q} \, ds \right) T \right] \, dx_1 - \left. \left\langle \bar{q}_e \right\rangle T \right|_{x_1=l} - \left. \left\langle \bar{q}_e \right\rangle T \right|_{x_1=0} \quad (3.20)$$

and

$$U_{T0} = \frac{1}{2} \left( \bar{V}_{T0}^T D_T + \bar{K} \right) T'^2 \equiv \frac{1}{2} \bar{K}_0 T'^2 \quad (3.21)$$

The scalar $\bar{K}_0$ can be viewed as a generalized heat conduction coefficient of a classical model for composite beams. Now we have constructed a reduced beam model under thermal load case 1 for heat conduction analysis. It is clear that the 1D constitutive model in Eq. (3.21) is asymptotically correct through the order of $O(\nu T'^2)$.

**Discussion on Convection Heat Transfer**

It is known that there are three modes of heat transfer: conduction, convection, and radiation. In this chapter, a conduction model for beam analysis is constructed. Now this model is extended to incorporate convection heat transfer for thermal load case 1. An important application of model is analysis of a cooling fin.

The convection heat transfer is governed by Newton’s law of cooling [87], which states that at a solid-fluid interface the heat flux is related to the difference between the temperature at the interface and that in the fluid

$$q_n = h_c(T - T_\infty) \quad (3.22)$$

where $h_c$ is the heat transfer coefficient or film conductance. The functional that governs convection can be written as

$$\Pi_h = \int_0^l \left( U_T - I_{Th} \right) \, dx_1 \quad (3.23)$$
where $\mathcal{I}_{Th} = - \oint \frac{1}{2} h_c T^2 - h_c T T_\infty \sqrt{c} ds$. Substitute Eq. (3.11) and Eq. (3.5) into Eq. (3.23), the first approximation can be written as

$$\Pi_{h0} = \int_0^l \left[ \mathcal{U}_{T0} + \oint \frac{1}{2} h_c (T^2 - 2 T T_\infty) \sqrt{c} ds \right] dx_1 \quad (3.24)$$

Here terms are kept up to the order of $O(\nu T')$. From this equation we can see that the convection does not influence the zeroth-order solution of warping function. Thus, the governing functional for the first approximation can be written as

$$\Pi_{h0} = \int_0^l \left[ \frac{1}{2} \bar{K}_0 T'^2 + \oint \frac{1}{2} h_c (T^2 - 2 T T_\infty) \sqrt{c} ds \right] dx_1 \quad (3.25)$$

where $\bar{K}_0$ can be found in Eq. (3.21). The governing equation and boundary conditions can be obtained by carrying out calculus of variations to the above 1D functional

$$\delta \Pi_{h0} = \int_0^l \left[ \bar{K}_0 T'' \delta T' + \oint (h_c T - h_c T_\infty) \delta T \sqrt{c} ds \right] dx_1$$

$$= \bar{K}_0 T'(l) \delta T(l) - \bar{K}_0 T'(0) \delta T(0) - \int_0^l \left[ \bar{K}_0 T'' + \oint h_c (T - T_\infty) \sqrt{c} ds \right] dx_1 \quad (3.26)$$

which indicates the following boundary value problem

Euler-Lagrange Equation

$$\bar{K}_0 T'' + \oint h_c (T - T_\infty) \sqrt{c} ds = 0 \quad (3.27)$$

with boundary conditions

$$T(0) = T_0 \quad (3.28)$$

$$T'(l) = 0 \quad (3.29)$$

where $T_0$ is the temperature at the end of beam. The 1D temperature $T$ can be obtained by solving this boundary value problem.
First-order Approximation

To obtain the first-order approximation with respect to initial twist and curvatures, we simply perturb the thermal warping function as

\[ V_T = V_{T0} + V_{TR} \]  \hspace{1cm} (3.30)

where \( V_{TR} \sim O(\hat{h}T') \). As introduced in Chapter 2, \( \hat{h} \) denotes the order of \( h/R \), i.e., \( \hat{h} \sim h/R \) with \( R \) as the characteristic radius of initial curvatures and twist.

Now we proceed to solve for the first-order approximation of the thermal warping function, \( V_{TR} \). Substituting Eq. (3.30) along with Eq. (3.19) into Eq. (3.2), and neglecting all the terms higher than \( O(\nu T'^2 \hat{h}^2) \), we obtain

\[ 2\mathcal{U}_{T1} = 2\mathcal{U}_{T0} + 2V_{T0}^T D_{Re} T' + V_{T0}^T (D_{RT} + D_{RT}^T) V_{T0} \]

\[ + V_{TR}^T E_T V_{TR} + 2V_{TR}^T D_{Re} T' + 2V_{TR}^T (D_{RT} + D_{RT}^T) V_{T0} + V_{T0}^T D_{RR} V_{T0} \]  \hspace{1cm} (3.31)

where the newly defined matrices are

\[ D_{Re} = \left\langle \left\langle [\Gamma_{RT} S]^T K e_1 \right\rangle \right\rangle \]

\[ D_{RT} = \left\langle \left\langle [\Gamma_{RT} S]^T K [\Gamma_T S] \right\rangle \right\rangle \]  \hspace{1cm} (3.32)

\[ D_{RR} = \left\langle \left\langle [\Gamma_{RT} S]^T K [\Gamma_{RT} S] \right\rangle \right\rangle \]

The leading terms with respect to the unknown \( V_{TR} \) from Eq. (3.31) are

\[ 2\mathcal{U}_{T1}^* = V_{TR}^T E_T V_{TR} + 2V_{TR}^T D_{Re} T' + 2V_{TR}^T (D_{RT} + D_{RT}^T) V_{T0} \]

\[ \equiv V_{TR}^T E_T V_{TR} + 2V_{TR}^T D_R T' \]  \hspace{1cm} (3.33)
with \( D_R = D_{Re} + (D_{RT} + D_{RT}^T) \bar{V}_T \). It is noted that \( \sqrt{g} \) contains \( k_\alpha \), and it is should be expanded in the asymptotic analysis so that

\[
E_T = \left\langle [\Gamma_T S]^T K [\Gamma_T S] \right\rangle + \left\langle [\Gamma_T S]^T K [\Gamma_T S] (x_3 k_2 - x_2 k_3) \right\rangle \tag{3.34}
\]

For simplicity of notation, we continue to use \( E_T \) in derivation with the understanding that such expansion are actually carried out in the numerical implementation. The details can be found in Appendix B.

Minizing the leading terms in Eq. (3.33) subject to the constraint in Eq. (3.6), we can obtain the thermal warping function in the following form

\[
V_{TR} = \bar{V}_{TR} T' \tag{3.35}
\]

Having obtained \( \bar{V}_{T0} \) and \( \bar{V}_{TR} \), we can approximate the original functional in Eq. (3.1) using the following 1D functional

\[
\Pi_1 = \int_0^l \left[ U_{T1} - \left( \langle \langle Q \rangle \rangle + \oint_{\partial \Omega} \bar{q} \ ds \right) T \right] dx_1 - \langle \bar{q}_e \rangle T |_{x_1=l} - \langle \bar{q}_e \rangle T |_{x_1=0} \tag{3.36}
\]

where

\[
U_{T1} = \frac{1}{2} \left( V_{T0}^T D_T + K + \bar{V}_{T0}^T D_R + \bar{V}_{T0}^T D_{Re} + \bar{V}_{T0}^T D_{RR} \bar{V}_{T0} + \bar{V}_{TR}^T D_R \right) T'^2 \]

\[
\equiv \frac{1}{2} K_R T'^2 \tag{3.37}
\]

### 3.2.2 Dimensional Reduction of Case 2

**Zeroth-order Approximation**

The temperature field of this case is provided in Eq. (3.7) and can be discretized as

\[
T(x_1, x_2, x_3) = w_T(x_1, x_2, x_3) = S(x_2, x_3) V_T(x_1) \tag{3.38}
\]
For simplicity of illustration, let us assume $Q$, $\bar{q}$ and $\bar{q}_e$ vanish, then the zeroth-order approximation of the thermal warping function should be solved from minimizing the following functional

$$2\mathcal{U}_T = V_T^T E_T V_T$$

with some values of $V_{T0}$ prescribed at certain nodes. As long as there is at least one point having prescribed temperature, the thermal warping function can be solved uniquely.

**First-order Approximation**

For the refined model with respect to initial twist and curvatures based on the first-order approximation, we expand the unknown thermal warping function $V_T$ asymptotically as we did for thermal load case 1 in Eq. (3.30) where $V_{TR} \sim O(\bar{h}T')$ and it should be zero at the points where the temperature is prescribed as the prescribed condition has already been satisfied by $V_{T0}$. Following what we did before, we have the Euler-Lagrange equation for the first-order thermal warping $V_{TR}$

$$E_T V_{TR} = - (D_{RT}^T + D_{RT} + E_T) V_{T0}$$

Here it is noted that the zero constraints for $V_{TR}$ at the prescribed points should be introduced to solve the linear system.

The prescribed temperature at certain nodes can be considered as single-point constraint that sets a single degree of freedom to a known value. The solution procedure of this kind of problem can be found in a typical textbook of finite element method such as Ref. [91].

**3.3 Recovery of 3D Thermal Field**

Thus far, we have obtained a generalized beam model for heat conduction analysis of thermal load case 1. The generalized heat conduction coefficients $\bar{K}_0$ and $\bar{K}_R$ can be used as an input for a 1D beam analysis to calculate the global thermal behavior. In other words, $T(x_1)$ can be solved using Eq. (3.20) or Eq. (3.36). However, as mentioned in Chapter 1,
only predicting the global behavior is not sufficient, and the original 3D results should be recovered for detailed analysis. The recovery process of heat conduction analysis can be summarized as

1. Using Eq. (3.20) or Eq. (3.36) to find \( \mathcal{T}(x_1) \);
2. Using Eq. (3.16) along with Eq. (3.30) to calculate the thermal warping function;
3. Using Eq. (3.5) to obtain the 3D temperature field;
4. Having the 1D temperature \( \mathcal{T}(x_1) \) and warping functions \( \bar{V}_{T0} \) and \( \bar{V}_{TR} \), it is straightforward to obtain the 3D temperature gradient using Eq. (3.11) as

\[
\nabla T = \left[ e_1 + (\Gamma_T S + \Gamma_{RT} S) (\bar{V}_{T0} + \bar{V}_{TR}) \right] \mathcal{T}' \quad (3.41)
\]

and 3D heat flux within the beam using

\[
\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix} = -K \nabla T \quad (3.42)
\]

The recovery of temperature field for thermal load case 2 is easier than that for case 1. Since there is no 1D variables like \( \mathcal{T}(x_1) \), the temperature can be obtained directly from Eq. (3.38) along with the solution of thermal warping functions in Eqs. (3.39) and (3.40).

### 3.4 Dimensional Reduction of Thermoelasticity Problem

#### 3.4.1 Classical Theory

First we are applying VAM to construct a classical beam model for thermoelastic analysis. In view of order assessment in Chapter 2, we can neglect the asymptotically smaller terms and keep the leading terms of the variational statement in Eq. (2.80) as

\[
\int_{t_1}^{t_2} \int_0^l \left[ \delta (K - U_{A0}) + \delta W \right] dx_1 dt = \delta A \quad (3.43)
\]
where $U_{A0}$ denotes the zeroth-order Helmholtz free energy per unit span such that

$$U_{A0} = \left\langle \frac{1}{2} \Gamma_0^T \mathcal{D} \Gamma_0 - \Gamma_0^T \mathcal{D} \alpha \Delta T \right\rangle$$  \hspace{1cm} (3.44)

with $\Gamma_0$ obtained from Eq. (2.35) by dropping the last two higher order terms as

$$\Gamma_0 = \Gamma_a w + \Gamma_e \bar{\epsilon}$$  \hspace{1cm} (3.45)

As mentioned in Chapter 2, the unknown 3D warping functions $w_i(x_1, x_2, x_3)$ can be solved from the following much simpler variation statement

$$\delta U_{A0} = 0$$  \hspace{1cm} (3.46)

along with the constraints in Eq. (2.12). Similarly, to deal with arbitrary cross-sectional geometry and anisotropic materials, we need to discretize the mechanical warping field as

$$w(x_1, x_2, x_3) = S(x_2, x_3)V(x_1)$$  \hspace{1cm} (3.47)

with $S(x_2, x_3)$ representing the matrix of finite element shape functions, and $V$ as a column matrix of the nodal values of the warping functions over the cross-section.

Substituting Eq. (3.45) into Eq. (3.44), one can express the zeroth-order Helmholtz free energy in discretized form as

$$2U_{A0} = V^T \mathcal{E} V + 2V^T D_a \bar{\epsilon} + \bar{\epsilon}^T D_e \bar{\epsilon} - 2 \left( V^T \alpha_a + \bar{\epsilon}^T \alpha_e \right)$$  \hspace{1cm} (3.48)
where the newly introduced matrices carry information on the properties of both the geometry of the cross-section and material, defined as

\[ E = \langle \langle [\Gamma_a S]^T D [\Gamma_a S] \rangle \rangle \]
\[ D_{ae} = \langle \langle [\Gamma_a S]^T D \Gamma_e \rangle \rangle \]
\[ D_{\epsilon \epsilon} = \langle \langle [\Gamma_e]^T D [\Gamma_e] \rangle \rangle \]
\[ \alpha_a = \langle \langle [\Gamma_a S]^T D \alpha \Delta T \rangle \rangle \]
\[ \alpha_\epsilon = \langle \langle [\Gamma_e]^T D \alpha \Delta T \rangle \rangle \]

Substituting Eq. (3.47) into Eq. (2.12), we can express the constraints in a discretized form as

\[ V^T D_c = 0 \] (3.50)

with \( D^T_c = (\Gamma_c S) \). We also use shape functions to discretize the kernel matrix in Eq. (2.43) in terms of its nodal value \( \Psi \) as

\[ \psi = S \Psi \] (3.51)

where we recall that \( \Gamma_a \psi = \Gamma_a S \Psi = 0 \), so that \( E \Psi = 0 \), which implies that \( \Psi \) is the kernel matrix of \( E \).

Now the problem has been transformed to numerical minimization of Eq. (3.48) subject to constraints Eq. (3.50). The Euler-Lagrange equation for this problem can be obtained by usual procedure of calculus of variations with the aid of Lagrange multipliers \( \Lambda \) as follows

\[ EV + D_{ae} \bar{\epsilon} - \alpha_a = D_c \Lambda \] (3.52)

Multiplying both sides by \( \Psi^T \) and considering the properties of the kernel matrix \( \Psi \), one calculates the Lagrange multiplier \( \Lambda \) as

\[ \Lambda = (\Psi^T D_c)^{-1} \Psi^T (D_{ae} \bar{\epsilon} - \alpha_a) \] (3.53)
It is clear that $\Lambda$ vanishes because $\Psi^T D_{ae} = \langle (\Gamma_a S \Psi)^T D \Gamma_\epsilon \rangle = 0$, similarly $\Psi^T \alpha_a = 0$, which implies that the constraints will not affect the minimum value of $U_{A0}$. Then the linear system in Eq. (3.52) becomes

$$EV = -D_{ae} \bar{\epsilon} + \alpha_a$$  \hspace{1cm} (3.54)

There exists a unique solution linearly independent of $\Psi$, the null space of $E$, for $V$ because the right-hand-side of Eq. (3.54) is orthogonal to the null space. Because of the uniqueness of the solution, the linear system in Eq. (3.54) can be solved by letting the numerical algorithm to determine where the singularities are and properly remove the singularities of the coefficient matrix. Let us denote the solution of Eq. (3.54) obtained this way as $V^*$, the complete solution can be written as

$$V = V^* + \Psi \lambda$$  \hspace{1cm} (3.55)

where $\lambda$ can be determined by Eq. (3.50) as

$$\lambda = - (\Psi^T D_e)^{-T} D_e^T V^*$$  \hspace{1cm} (3.56)

Hence the final solution minimizing the functional in Eq. (3.48) subject to constraints in Eq. (3.50) is

$$V = \left[ \Delta - \Psi \left( \Psi^T D_e \right)^{-T} D_e^T \right] V^* = \hat{V}_0 \bar{\epsilon} + V_{t0} \equiv V_0$$  \hspace{1cm} (3.57)

where $V_{t0}$ is the mechanical warping caused by the applied temperature field.

Substituting Eq. (3.57) back into Eq. (3.48), one can obtain the total energy asymptotically correct up to the $O(\nu \bar{\epsilon})$ as

$$2U_{A0} = \bar{\epsilon}^T \left( \hat{V}_0^T D_{ae} + D_{ee} \right) \bar{\epsilon} - 2\bar{\epsilon}^T \left[ \alpha_a + \frac{1}{2} \left( \hat{V}_0^T \alpha_a - D_{ae}^T V_{t0} \right) \right]$$  \hspace{1cm} (3.58)

Note the quadratic terms associated with temperature $V_{t0}^T \alpha_a$ and $V_{t0}^T EV_{t0}$ are dropped.
because they will not contribute to the corresponding 1D thermoelastic beam model. This
is the asymptotically correct energy for a beam without correction for initial curvature and
twist. This energy can be written in an explicit matrix form as

$$2\mathcal{U}_{A0} = \begin{bmatrix} \ddot{\gamma}_{11} \\
\ddot{\kappa}_1 \\
\ddot{\kappa}_2 \\
\ddot{\kappa}_3 \end{bmatrix}^T \begin{bmatrix} \ddot{S}_{11} & \ddot{S}_{12} & \ddot{S}_{13} & \ddot{S}_{14} \\
\ddot{S}_{12} & \ddot{S}_{22} & \ddot{S}_{23} & \ddot{S}_{24} \\
\ddot{S}_{13} & \ddot{S}_{23} & \ddot{S}_{33} & \ddot{S}_{34} \\
\ddot{S}_{14} & \ddot{S}_{24} & \ddot{S}_{34} & \ddot{S}_{44} \end{bmatrix} \begin{bmatrix} \ddot{\gamma}_{11} \\
\ddot{\kappa}_1 \\
\ddot{\kappa}_2 \\
\ddot{\kappa}_3 \end{bmatrix} - 2 \begin{bmatrix} \ddot{\gamma}_{11} \\
\ddot{\kappa}_1 \\
\ddot{\kappa}_2 \\
\ddot{\kappa}_3 \end{bmatrix}^T \begin{bmatrix} f_1^t \\
m_1^t \\
m_2^t \\
m_3^t \end{bmatrix} \tag{3.59}$$

which implies a 1D constitutive model of the form

$$\begin{bmatrix} F_1 \\
M_1 \\
M_2 \\
M_3 \end{bmatrix} = \begin{bmatrix} \ddot{S}_{11} & \ddot{S}_{12} & \ddot{S}_{13} & \ddot{S}_{14} \\
\ddot{S}_{12} & \ddot{S}_{22} & \ddot{S}_{23} & \ddot{S}_{24} \\
\ddot{S}_{13} & \ddot{S}_{23} & \ddot{S}_{33} & \ddot{S}_{34} \\
\ddot{S}_{14} & \ddot{S}_{24} & \ddot{S}_{34} & \ddot{S}_{44} \end{bmatrix} \begin{bmatrix} \ddot{\gamma}_{11} \\
\ddot{\kappa}_1 \\
\ddot{\kappa}_2 \\
\ddot{\kappa}_3 \end{bmatrix} - \begin{bmatrix} f_1^t \\
m_1^t \\
m_2^t \\
m_3^t \end{bmatrix} \tag{3.60}$$

where $F_1$ is the axial stress resultant conjugate to the extensional strain $\ddot{\gamma}_{11}$ and $M_i$ are the
moment resultants conjugate to the twists and curvatures $\ddot{\kappa}_i$, i.e.,

$$F_1 = \frac{\partial \mathcal{U}_{A0}}{\partial \ddot{\gamma}_{11}} \quad M_i = \frac{\partial \mathcal{U}_{A0}}{\partial \ddot{\kappa}_i} \tag{3.61}$$

This model can be considered as a generalized Euler-Bernoulli beam model while we have
not used any ad hoc kinematic assumptions. Next we will construct a refined thermoelastic
beam model to capture the transverse shear effects as well as the effects due to initial twist
and curvatures.

### 3.4.2 Refined Theory

For the refined modeling, we keep terms up to $O(\nu \hat{\epsilon}^2 \hat{h}^2)$ in the expression of the
Helmholtz free energy. Perturbing the warping functions to be

$$V = V_0 + V_1 = \hat{V}_0 \hat{\epsilon} + V_{00} + V_1 \tag{3.62}$$
Substituting Eq. (3.62) into Eq. (3.47), then into Eq. (2.35), and finally into Eq. (3.8), we obtain the following functional after neglecting terms higher than $O(ν^2 h^2)$

\[
2\mathcal{U}_{A1} = \bar{\epsilon}^T (V_0^T D_t \bar{\epsilon} + D_t \bar{\epsilon}) - 2\bar{\epsilon}^T \left[ \alpha_t + \frac{1}{2} \left( \dot{V}_0^T \alpha_0 - D_t \bar{\epsilon} V_0 \right) \right] \\
+ 2V_0^T D_{tR} V_0 + 2V_0^T D_{tS} V_0' + 2V_0^T D_{tR} \bar{\epsilon} + 2V_0^T D_{tS} \bar{\epsilon} - 2V_0^T \alpha_t - 2V_0^T \alpha_R \\
+ V_1^T EV_1 + 2V_1^T (D_{tR} V_0 + D_{tS} V_0' + D_{tR} \bar{\epsilon}) + 2V_1^T D_{tS} V_0' + 2V_0^T D_{td} V_0' + \\
2V_{1t}^T \alpha_t - 2V_0^T \alpha_t - 2V_1^T \alpha_R \\
(3.63)
\]

where

\[
D_{tR} = \langle [\Gamma_t S]^T \mathcal{D} [\Gamma_{tR} S] \rangle \\
D_{tS} = \langle [\Gamma_t S]^T \mathcal{D} [\Gamma_{tS}] \rangle \\
D_{tL} = \langle [\Gamma_t L]^T \mathcal{D} [\Gamma_{tL}] \rangle \\
D_{tR} = \langle [\Gamma_t S]^T \mathcal{D} [\Gamma_{tR}] \rangle \\
\alpha_t = \langle [\Gamma_t S]^T \mathcal{D} \alpha \Delta T \rangle \\
\alpha_R = \langle [\Gamma_{tR} S]^T \mathcal{D} \alpha \Delta T \rangle \\
(3.64)
\]

As we are interested in the interior solution for the beam without consideration of edge effects, we can integrate by parts to get rid of the derivatives of the warping $V_1'$ and neglect the boundary terms. The leading terms (without the constant terms) of Eq. (3.63) are

\[
2\mathcal{U}_{A1}^* = V_1^T EV_1 + 2V_1^T D_{tR} \bar{\epsilon} + 2V_1^T D_{tS} \bar{\epsilon}' + 2V_1^T (\mathcal{D}_{tR} + D_{tS}) \\
(3.65)
\]
where

\[
D_R = D_{aR} V_0 + D_{aR}^T \dot{V}_0 + D_{Re} 
\]  
(3.66)

\[
D_S = D_{al} \dot{V}_0 - D_{al}^T \dot{V}_0 - D_{le} 
\]  
(3.67)

\[
D_{RT} = (D_{aR}^T + D_{aR}) V_{t0} - \alpha_R 
\]  
(3.68)

\[
D_{ST} = (D_{al} - D_{al}^T) V_{t0}' + \alpha_l' 
\]  
(3.69)

Similar to the zeroth-order warping, the first-order warping could be solved as

\[
V_1 = V_{1R} \epsilon + V_{1S} \epsilon' + V_{1T} 
\]  
(3.70)

Using Eq. (3.70), the second-order asymptotically correct Helmholtz free energy can now be obtained from Eq. (3.8) as

\[
2\mathcal{U}_{A1} = \epsilon^T A \epsilon + 2 \epsilon^T B \epsilon' + \epsilon'^T C \epsilon' + 2 \epsilon^T D \epsilon'' - 2 \epsilon^T F_{t1} - 2 \epsilon'^T F_{t2} - 2 \epsilon''^T F_{t3} 
\]  
(3.71)

where

\[
A = \dot{V}_0^T D_{al} + D_{le} + \dot{V}_0^T (D_{aR} + D_{aR}^T + D_{RR}) \dot{V}_0 + 2 \dot{V}_0^T D_{Re} + V_{1R}^T D_R 
\]

\[
B = \dot{V}_0^T (D_{al} + D_{RL}) \dot{V}_0 + D_{le}^T \dot{V}_0 + (\dot{V}_0^T D_{al} + D_{le}^T) V_{1R} + \frac{1}{2} (D_{R}^T V_{1S} + V_{1T}^T \dot{D}_S) 
\]

\[
C = V_{1S}^T \dot{D}_S + \ddot{V}_0^T D_{il} \dot{V}_0 
\]

\[
D = (D_{le}^T + \ddot{V}_0^T D_{al}) V_{1S} 
\]

\[
F_{t1} = N_T - (D_{RT}^T + \dot{V}_0^T D_{RR}) V_{t0} - (\dot{V}_0^T D_{al} + D_{le}^T + \dot{V}_0^T D_{RL}) V_{t0}' - \frac{1}{2} D_{R}^T V_{1T} 
\]  
(3.72)

\[
F_{t2} = (\dot{V}_0^T + \dot{V}_{1R}) \alpha_l - (\dot{V}_0^T D_{al} + V_{1R}^T D_{al}^T + \dot{V}_0^T D_{RL}^T) V_{t0} - \frac{1}{2} \dot{D}_S^T V_{1T} 
\]

\[
F_{t3} = -V_{1S}^T D_{al} V_{t0} + V_{1S}^T \alpha_l 
\]
with

\[ N_T = \alpha_{\epsilon} + \frac{1}{2} \hat{V}_0^T \alpha_a - D_{aT}^T V_0 \]

\[ \bar{D}_S = (D_{al} + D_{al}^T) \hat{V}_0 + D_{te} \]

\[ \bar{D}_{ST} = (D_{al} + D_{al}^T) V'_{t0} - \alpha'_{l} \] (3.73)

### 3.4.3 Transformation to Generalized Timoshenko Model

The energy of the form in Eq. (3.71) is not convenient for engineering applications because it involves derivatives of the 1D generalized strains. To get rid of these derivatives, we can transform this asymptotically correct energy expression to a generalized Timoshenko model following the equilibrium-equation approach.

The key to the energy transformation is to find expressions for \( \bar{\epsilon}, \bar{\epsilon}' \) and \( \bar{\epsilon}'' \) in terms of \( \epsilon \) and \( \gamma_s \). Following the procedure in Ho et al. [76], we can express the energy up to the second order as

\[ 2U_{AT} = \epsilon^T X \epsilon + 2 \epsilon^T Y \gamma_s + \gamma_s^T G \gamma_s - 2 \epsilon^T F_1^T - 2 \gamma_s^T F_2^T \] (3.74)

where \( F_1 = [F_1^T \quad M_1^T \quad M_2^T \quad M_3^T]^T \) and \( F_2 = [F_2^T \quad F_3^T]^T \). These terms can be solved asymptotically up to different orders. The details of transformation can be found in Appendix A.

We can rewrite this model in an explicit matrix form as

\[
2U_A = \left[
\begin{array}{c}
\gamma_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{array}
\right]^T 
\begin{pmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}
\end{pmatrix}
\left[
\begin{array}{c}
\gamma_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{array}
\right] 
- 2
\left[
\begin{array}{c}
\gamma_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{array}
\right]^T 
\left[
\begin{array}{c}
F_1^T \\
F_2^T \\
F_3^T \\
M_1^T \\
M_2^T \\
M_3^T
\end{array}
\right]
\] (3.75)
which implies the following 1D constitutive model

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}
\end{bmatrix}
\begin{bmatrix}
\gamma_{11} \\
2\gamma_{12} \\
2\gamma_{13} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3
\end{bmatrix} -
\begin{bmatrix}
\mathcal{F}'_1 \\
\mathcal{F}'_2 \\
\mathcal{F}'_3 \\
\mathcal{M}'_1 \\
\mathcal{M}'_2 \\
\mathcal{M}'_3
\end{bmatrix}
\]

(3.76)

where \([F_1 \ F_2 \ F_3]^T\) are the stress resultants conjugate to the force strain \([\gamma_{11} \ 2\gamma_{12} \ 2\gamma_{13}]^T\)
and \([M_1 \ M_2 \ M_3]^T\) are the moment resultants conjugate to the moment strains \([\kappa_1 \ \kappa_2 \ \kappa_3]^T\);
\([\mathcal{F}'_1 \ \mathcal{F}'_2 \ \mathcal{F}'_3]^T\) and \([\mathcal{M}'_1 \ \mathcal{M}'_2 \ \mathcal{M}'_3]^T\) are thermal load related terms corresponding to the
force strains and moment strains, respectively.

### 3.5 Recovery of 3D Mechanical Field

In this section, we are going to recover the original 3D results based on the developed
1D constitutive models.

For the generalized thermoelastic Timoshenko model of an initially curved and twisted
beam, the warping function that is asymptotically correct up to the order of \(h \sim h/R \sim h/l\)
can be expressed as

\[
w(x_1, x_2, x_3) = S \left( \hat{V}_0 + V_{1R} \right) \hat{\epsilon} + SV_1 S' \epsilon' + S (V_{00} + V_{1T})
\]

(3.77)

where \(w(x_1, x_2, x_3)\) is a column matrix containing the 3D warping functions, and \(V_0, V_{1R},
V_{1S}\) are the nodal values of the asymptotically correct warping functions for classical mod-
eling, the correction for initial curvatures and twist, the refined warping of the order of \(h/l\),
respectively. \(V_{00}\) and \(V_{1T}\) are the warping functions caused by thermal loads.

From Eq. (2.5) and Eq. (2.1), we can calculate the 3D displacement field as

\[
\bar{u}_i(x_1, x_2, x_3) = u_i(x_1) + x_\alpha \left[ C_{\alpha i}^T (x_1) - \delta_{\alpha i} \right] + C_{ji}^{Th} w_j(x_1, x_2, x_3)
\]

(3.78)
where $\bar{u}_i$ are the 3D displacements, $u_i$ the 1D beam displacements, and $C_{ij}^{Tb}$ the components of the direction cosine matrix representing the finite rotation from triad $b_i$ to triad $T_i$.

The 3D strain field can be recovered by substituting 1D strain measures, cross-sectional warping and their derivatives into Eq. (2.35). Substituting Eq. (3.77) into Eq. (2.35), we obtain

$$
\Gamma = \left[ (\Gamma_a + \Gamma_R) S \left( \tilde{V}_0 + V_{1R} \right) + \Gamma_t \right] \bar{\varepsilon} + \left[ (\Gamma_a + \Gamma_R) SV_1S + \Gamma_t S \left( \tilde{V}_0 + V_{1R} \right) \right] \bar{\varepsilon}' + \Gamma_t SV_1S \bar{\varepsilon}' + \left( \Gamma_a + \Gamma_R \right) S \left( V_{00} + V_{11}T \right) + \Gamma_t S \left( V_{0}' + V_{1}'T \right) 
$$

(3.79)

Finally, the stress can be obtained from the 3D constitutive relations based on the Helmholtz free energy in Eq. (3.8) so that

$$
\sigma = D\Gamma - D\alpha \Delta T
$$

(3.80)

where $\sigma$ is a column matrix containing $\sigma_{ij}$ as

$$
\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{22} & \sigma_{23} & \sigma_{33} \end{bmatrix}^T
$$

(3.81)

### 3.6 Thermoelastic Beam Modeling Under Large Temperature Changes

The constitutive framework of thermoelasticity in the previous sections is based on temperature-independent condition, small temperature assumption, and small strain assumption. For some cases, it is reasonable to assume the strains are small while the temperature change cannot be considered as “small.” And also, if the temperature changes are large enough, the material properties including elastic constants, coefficients of thermal expansion become temperature dependent [92,93]. In this section, we will discuss how to extend current VABS thermoelastic model to incorporate temperature-dependent materials experiencing large temperature changes.
To relax the assumption of small temperature changes, we need to derive a Helmholtz free energy suitable for materials with temperature dependent properties and experiencing finite temperature changes. This implies that we need first to define the material properties of interest as temperature dependent, such as the coefficient of thermal expansion \( \alpha(\sigma_{ij}, T) \), the elastic constants \( C_{ijkl}(T) \), the thermal strain tensor \( m(T) \), and the thermal stress tensor \( l(T) \) and etc. The symbol outside the parenthesis denotes the physical quantity while the symbols inside parenthesis are regarded as the independent variables used to describe the state of function. Note that for a defined function \( F(\sigma_{ij}, T) \) or \( F(\epsilon_{ij}, T) \), the quantity \( F(0, T) \) means \( F(\sigma_{ij} = 0, T) \)(constant stress state) or \( F(\epsilon_{ij} = 0, T) \)(constant strain state) depending on how the function is defined.

The Helmholtz free energy density \( f(\epsilon_{ij}, T) \) is a function of strain field \( \epsilon_{ij} \) and the absolute temperature \( T \). Let us not put any restriction on \( T \) but assuming \( \epsilon_{ij} \) to be small, then we can carry out a Taylor expansion of \( f(\epsilon_{ij}, T) \) in terms of the small strain field, \( \epsilon_{ij} \), as

\[
f(\epsilon_{ij}, T) = f(0, T) + \epsilon_{ij} \frac{\partial f(\epsilon_{ij}, T)}{\partial \epsilon_{ij}} |_{\epsilon_{ij}=0} + \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \frac{\partial^2 f(\epsilon_{ij}, T)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} |_{\epsilon_{ij}=0} \tag{3.82}
\]

Here only up to the quadratic terms of the strain field are kept due the assumption of small strains. As the constant term \( f(0, T) \) will not affect our thermoelastic analysis [12], the constant term \( f(0, T) \) is dropped. We know \( \sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}} \), that is

\[
\sigma_{ij} = C_{ijkl}(T)\epsilon_{kl} + l_{ij}(T) \tag{3.83}
\]

with \( C_{ijkl}(T) = \frac{\partial^2 f(\epsilon_{ij}, T)}{\partial \epsilon_{ij} \partial \epsilon_{kl}} |_{\epsilon_{ij}=0} \) as the fourth-order elasticity tensor and \( l_{ij}(T) = \frac{\partial f(\epsilon_{ij}, T)}{\partial \epsilon_{ij}} |_{\epsilon_{ij}=0} \) as the second-order thermal stress tensor. We can also rewrite the stress-strain relations as

\[
\epsilon_{ij} = S_{ijkl}(T)\sigma_{kl} + m_{ij}(T) \tag{3.84}
\]

with \( S_{ijkl} \) as the fourth-order compliance tensor and \( m_{ij} \) as the second-order thermal strain tensor and we have \( m_{ij} = -S_{ijkl}l_{kl} \). The coefficients of thermal expansion, \( \alpha_{ij} \), as a function
of stress field and temperature, is defined as

\[ \alpha_{ij} = \frac{\partial \epsilon_{ij}}{\partial T} \Big|_{\sigma_{ij}=\text{constant}} \quad (3.85) \]

Then from Eq. (3.84) and (3.85), we have

\[ \alpha_{ij} = S'_{ijkl} \sigma_{kl} + m'_{ij} \quad (3.86) \]

where prime is used to denote derivative with respect to \( T \), i.e., \( m'_{ij} = \frac{dm_{ij}}{dT} \). From Eq. (3.86), we have

\[ \alpha_{ij}(0, T) = m'_{ij} \quad (3.87) \]

where we can obtain

\[ m_{ij} = \int_{T_0}^{T} \alpha_{ij}(0, \zeta) d\zeta + m_{ij}(T_0) \quad (3.88) \]

Note here \( \alpha_{kl}(0, T) \) are the stress-free coefficients of thermal expansion which can be easily measured at a specific temperature \( T \). We normally choose our reference state to be at \( T = T_0 \) with stress and strain free, which implies \( m_{ij}(T_0) = 0 \) in view of Eq. (3.84). Then we can express our thermal strain tensor in a form similar as that we used for small temperature variations

\[ m_{ij} = \tilde{\alpha}_{ij}(T) \Delta T \quad (3.89) \]

with

\[ \tilde{\alpha}_{ij}(T) = \frac{1}{\Delta T} \int_{T_0}^{T_0+\Delta T} \alpha_{ij}(0, \zeta) d\zeta \quad (3.90) \]

Normally, \( \tilde{\alpha}_{ij}(T) \) is termed as the secant free thermal expansion coefficients. We can also express the thermal stress tensor as

\[ l_{ij}(T) = -C_{ijkl}(T)m_{kl}(T) = -C_{ijkl}(T)\tilde{\alpha}_{ij}(T) \Delta T \equiv \tilde{\beta}_{ij}(T) \Delta T \quad (3.91) \]

Here, \( \tilde{\beta}_{ij}(T) \) can be similarly called secant free thermal stress coefficients.

Substituting Eq. (3.91) into Eq. (3.82), we have the Helmholtz energy for thermoelastic
analysis considering the temperature-dependent material properties without assuming small
temperature changes as

\[ f(\epsilon_{ij}, T) = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl} + \tilde{\beta}_{ij}(T) \epsilon_{ij} \Delta T \quad (3.92) \]

Given the derivation, current VABS thermoelastic model can be easily extended to
incorporate the temperature dependent properties of materials. If temperature change
is small, \( \alpha \) is the conventional CTE, also know as the tangent or instantaneous CTE.
Otherwise, for finite temperature change, one just needs to use the secant CTE, which can
be obtained from Eq. (3.90), in Eq. (3.8).

3.7 Validation of Thermoelastic Model

The theory developed in the previous chapter has been implemented into the computer
program Variational Asymptotic Beam Sectional Analysis (VABS). To validate the present
model, we have used VABS to analyze several examples and the results are compared with
those available in the literature and 3D finite element analysis in commercial software
ANSYS. It is pointed out that for thermal load case 1 problems, one needs to solve a 1D
heat conduction problem to obtain the 1D variable \( T \), which is used to recover the thermal
field.

3.7.1 Heat Conduction Analysis

Example 1: Two-layer Beam Under Thermal Load Case 1 and Convection Anal-
ysis

The first example is a two-layer angle-ply composite beam with the lay-up angle as
\([30^\circ/ - 30^\circ]\). The length of the beam is 0.2 m \( (x_1 \) direction), the thickness of each layer
is 0.01 m \( (x_3 \) direction), and the width equals to 0.04 m \( (x_2 \) direction). The beam is
made of an orthotropic material with thermal conductivities given by \( k_{11} = 0.3 \text{ W/(m.}\-period\text{C)} \),
\( k_{22} = k_{33} = 0.16 \text{ W/(m.}\period\text{C)} \). Three thermal load cases are considered
• Case A: constrained temperature on the end surfaces of the beam such as

\[ T_0 = 0^\circ \text{C} \quad T_L = 100^\circ \text{C} \quad (3.93) \]

• Case B: the constrained temperature in the previous case together with input heat flux of 5 W/m\(^2\) in \( x_3 \) direction on the top surface of the beam.

• Case C: temperature is prescribed at the left end as \( T(x_1 = 0) = 50^\circ \text{C} \) and the tip is insulated. The ambient temperature is 20\(^\circ\)C and the heat transfer coefficient \( h_c \) is 2 W/(m\(^2\)K).

The geometry and loads of this beam refer to Fig. 3.3. For VABS analysis, this cross-section is meshed with 32 four-noded quadrilateral elements (eight elements along the width, two elements along the thickness of each layer). SOLID70 thermal elements are used to carry out a thermal analysis in ANSYS with the same cross-sectional mesh and the length is discretized into eight divisions. For comparison, we plot the temperature distribution over the cross-section at \( x_1 = 0.1 \) m along the width and the thickness in Fig. 3.4 and Fig. 3.5, respectively. It can be observed that VABS agrees with ANSYS very well for both cases along the width and the thickness of the cross-section, within differences less than 0.02\% and 0.04\% for the first and second load case, respectively. In addition to accurately predicting the temperature field, the developed model is also able to predict the heat flux within the structure. As shown in Figs. 3.6, 3.7, and 3.8, this model also accurately predicts the heat flux. For the first load case, the error compared with 3D analysis is less than 0.1\% in predicting the heat flux. For the second load case, the distribution of \( q_3 \) has been sharply changed due to the heat flux input on the top surface of the beam. The maximum error between VABS calculation and 3D analysis for this case is around 3\%.

For Case C, the temperature distribution at the centroid along the beam axis is plotted in Fig. 3.9. Good agreement between results from VABS calculation and those from 3D analysis can be observed.
Fig. 3.3: Sketch of a composite beam used in heat conduction Example 1.

Fig. 3.4: Temperature distribution along the width at $x_3 = -0.005$ m for heat conduction Example 1.
Fig. 3.5: Temperature distribution along the thickness at $x_2 = 0.01$ m for heat conduction Example 1.

Fig. 3.6: Distribution of heat flux $q_1$ along width at $x_3 = -0.005$ m for heat conduction Example 1.
Fig. 3.7: Distribution of heat flux $q_2$ along width at $x_3 = -0.005$ m for heat conduction Example 1.

Fig. 3.8: Distribution of heat flux $q_3$ along width at $x_3 = -0.005$ m for heat conduction Example 1.
Example 2: Square Cross-section Beam Under Thermal Load Case 2

This example is to check the accuracy of the present model for the problem that prescribes the temperature at some specific points over the cross-section. Here we use a square cross-section beam with constant temperature on three sides and a sinusoidal temperature distribution on the fourth one: \( T = T_0 \sin \left( \frac{\pi x_2}{a} \right) \), where \( a \) is the width of the cross-section. For numerical calculation in VABS, we set \( a = 0.08 \text{ m} \) and \( T_0 = 100^\circ \text{C} \) and the constant temperature along the other three sides is zero. The origin of the cross-sectional coordinates are located at the bottom left corner, that is, \( 0 \leq x_2 \leq a \) and \( 0 \leq x_3 \leq a \). For a constant temperature distribution along the beam axis, the temperature has the following exact solution available in Refs. [34,94]

\[
T(x_2, x_3) = T_0 \frac{\sinh \left( \frac{\pi x_3}{a} \right)}{\sinh (\pi)} \sin \left( \frac{\pi x_2}{a} \right) 
\] (3.94)
Fig. 3.10: Temperature distribution along $x_3$ at $x_2 = -0.02$ m for heat conduction Example 2.

We mesh the cross-section by 64 four-noded quadrilateral elements. As shown in Figs. 3.10-3.11, the temperature distribution over the cross-section predicted by VABS has excellent agreement with the exact solution.

**Example 3: A Composite Box-beam**

The third example is a composite box-beam constructed from a two-layer beam by cutting a hole at the center. The sketch of the cross-section is shown in Fig. 3.12.

The upper layer angle is $45^\circ$ and the lower layer angle is $-45^\circ$. The length of the beam is 0.8 m, the outside both has the size as 0.08 m $\times$ 0.08 m and the hole at the center is 0.04 m $\times$ 0.04 m. The material properties are the same as the Example 1. The cross-section is meshed with four-noded quadrilateral elements in such a way that all elements have the same size of 0.01 m $\times$ 0.01 m. At the two ends of the beam, the temperature is prescribed to be $0^\circ$C and $100^\circ$C at $x_1 = 0$ m and $x_1 = 0.8$ m, respectively (thermal load case 1, Case A).

We use SOLID5 elements to carry out a 3D heat conduction analysis in ANSYS. In ANSYS 3D model, the cross-section is meshed the same and the length along $x_1$ direction is divided
Fig. 3.11: Temperature distribution along $x_2$ for heat conduction Example 2.

Fig. 3.12: Sketch of the cross-section of a composite box-beam used in heat conduction Example 3.
Fig. 3.13: Temperature distribution along $x_2$ at $x_3 = 0.03$ m of Case A for heat conduction Example 3.

with 80 segments. The temperature distributions at the center of the beam length ($x_1 = 0.4$ m) predicted by VABS and ANSYS are plotted in Figs. 3.13-3.14. One can observe that VABS accurately predicts the temperature distribution with much less computational effort.

Next we are going to check the accuracy of the our developed model and the VABS code for the same composite box-beam with temperature prescribed at some points (thermal load case 2, Case B). To this end, we prescribe the temperature along the outside surfaces to be 50°C and the temperature along the inside surfaces to be 100°C. The same mesh as the previous composite box-beam example is used for both VABS 2D cross-sectional heat conduction analysis and ANSYS 3D heat conduction analysis. The results obtained by both VABS and ANSYS are plotted in Figs. 3.15-3.16 for comparison. Again, VABS computes accurate temperature distribution based on a 2D cross-sectional analysis, in comparison to the 3D finite element analysis using ANSYS.
Fig. 3.14: Temperature distribution along $x_3$ at $x_2 = -0.03$ m of Case A for heat conduction Example 3.

Fig. 3.15: Temperature distribution along $x_2$ at $x_3 = 0.03$ m of Case B for heat conduction Example 3.
Fig. 3.16: Temperature distribution along $x_3$ at $x_2 = -0.03$ m of Case B for heat conduction Example 3.

**Example 4: Initially Curved Composite Beam with $k_2 = 1.25$ rad/m**

The fourth example is a two-layer angle-ply composite beam with the lay-up angle as $[45^\circ/ -45^\circ]$ and initial curvature $k_2 = 1.25$ rad/m. The beam is made of the same orthotropic material as we used in Example 1. The length of the beam is 0.8 m and the thickness of each layer is 0.04 m. The thermal loads are the same as in Example 2, that is, we apply a sinusoidal temperature distribution on one side: $T = T_0 \sin\left(\frac{\pi x}{a}\right)$, where $a = 0.08$ m is the width of the cross-section and $T_0 = 100^\circ C$, and constant temperature ($0^\circ C$) on the other three sides (thermal load case 2).

For VABS 2D cross-sectional discretization, we divide the width by 20 eight-noded quadrilateral elements and along the thickness each layer is divided into 10 elements. The total number of 2D elements in the cross-section is $20 \times 20$. In the ANSYS model, a prismatic
beam with the same length has been constructed. The cross-sections are divided by the same mesh and we divide the length into 10 elements. Thus, the ANSYS model uses a total of $10 \times 20 \times 20$ SOLID 90, twenty-noded thermal brick elements.

Figs. 3.17 and 3.18 show the temperature distribution along the width and thickness at the mid-span, respectively. The curves and data points labeled “Prismatic” are calculated from ANSYS 3D analysis and VABS prismatic model, respectively. Excellent agreement can be observed for this case. The dashed lines labeled “Curved” show the results obtained by VABS curved model. Fig. 3.19 shows the difference between results obtained from VABS prismatic and curved calculation, which indicates the influence of initially curvature on the final results. The difference is calculated as $(VABS(Curved) - VABS(Prismatic)) \times 100/VABS(Prismatic)$.

From all the previous examples, we can confidently state that the newly developed model can accurately solve the heat conduction problem of composite beams no matter whether the temperature is prescribed at two ends of the beam or the temperature is pre-
Fig. 3.18: Temperature distribution along $x_3$ at $x_2 = 0$ for heat conduction Example 4.

Fig. 3.19: Difference of temperatures between VABS prismatic and curved results for heat conduction Example 4.
scribed at some specific points of the cross-section of the beam. In comparison to the much more computationally intensive 3D heat conduction analysis, the new model implemented in VABS provides a very efficient alternative while maintaining a similar accuracy for the original 3D analysis. This computational saving becomes more significant for composite beams with sophisticated configurations such as modern composite rotor blades made of hundreds of composite layers.

3.7.2 Thermoelastic Analysis

Example 1: Composite Beam Under Uniform Temperature Change

To validate the thermoelastic capability of the present theory and the companion code VABS, we first use a beam of length $l$ and width $w$, made up of two layers of different materials, is subjected to a uniform rise in temperature from $T_{ref}$ to $T_0$ and a bending moment $M_2$ at the free-end, see Fig. 3.20. This example is also used by ANSYS to validate its thermoelastic capability. The material properties taken from [95], geometric properties and loading are given in Table 3.1. The beam is idealized to match the theoretical assumptions by taking $\nu = \alpha_2 = \alpha_3 = 0$. The free-end displacement (in the $x_3$ direction) and the stress component $\sigma_{11}$ at the top and the bottom surfaces of the layered beam are compared with analytical solution and 3D solution. Here we use 16 solid brick elements, SOLID186 in ANSYS, along $x_1$ direction where 1 element for a layer on the cross-section to obtain the 3D results. In the VABS 2D cross-sectional model, we have the same mesh over the cross-section. These results are tabulated in Table 3.2. The numerical prediction of VABS achieves an excellent agreement with the analytical solution and 3D solution.

Example 2: Composite Beam Under Thermal Load Case 1

Next, we use VABS to analyze a cantilever composite beam under thermal load case 1. The length of the beam is 0.8 m and the material properties are given in Table 3.3. This beam consists of four layers of equal thickness 0.02 m with a symmetric $[30^\circ/-60^\circ/-60^\circ/30^\circ]$ layer-up. These four layers are of materials 2/1/1/2. We prescribe the temperature to be
0°C and 100°C at \( x_1 = 0 \) m and \( x_1 = 0.8 \) m, respectively. A 3D finite element model of the beam is constructed in ANSYS using solid elements. For VABS 2D cross-sectional discretization, we divide the width by 40 eight-noded quadrilateral elements and along the thickness each layer is divided into 10 elements. The total number of 2D elements in the cross-section is \( 40 \times 40 \). In the ANSYS model, we use a total of 47,200 twenty-noded SOLID186 brick elements. To demonstrate the predictive capability of VABS for the detailed distributions of 3D variables, we recovered the 3D field using VABS based on the global beam behavior at the mid span (\( x_1 = 0.4 \) m). Figs. 3.21-3.23 show the stress components obtained by VABS and ANSYS. Excellent agreement between these two approaches has been observed for these quantities.

**Table 3.1: Material Properties, Geometric Properties and Loading for Thermoelastic Example 1**

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Geometric Properties</th>
<th>Loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_1 = 1.2 \times 10^6 ) psi</td>
<td>( l = 8 ) in</td>
<td>( T_0 = 100^\circ C )</td>
</tr>
<tr>
<td>( \alpha_1 = 1.8 \times 10^{-4} ) in / in / ( ^\circ C )</td>
<td>( w = 0.5 ) in</td>
<td>( T_{ref} = 0^\circ F )</td>
</tr>
<tr>
<td>Material 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E_1 = 0.4 \times 10^6 ) psi</td>
<td>( t_1 = 0.2 ) in</td>
<td>( M_2 = 10.0 ) in-lb</td>
</tr>
<tr>
<td>( \alpha_1 = 0.6 \times 10^{-4} ) in / in / ( ^\circ F )</td>
<td>( t_2 = 0.1 ) in</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.2: Comparison of Analytical, 3D FEA and VABS Results

<table>
<thead>
<tr>
<th>Displacement ($u_3$, in)</th>
<th>Analytical Solution</th>
<th>ANSYS 3D</th>
<th>VABS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top Surface</td>
<td>0.832</td>
<td>0.83221</td>
<td>0.83446</td>
</tr>
<tr>
<td>Bottom Surface</td>
<td>0.83446</td>
<td>0.83158</td>
<td>0.82989</td>
</tr>
<tr>
<td>Average</td>
<td>0.832</td>
<td>0.832</td>
<td>0.832</td>
</tr>
</tbody>
</table>

| $\sigma_{11}$ (psi)     | Top Surface         | 2258       | 2266.6     | 2257.6     |
|                         | Bottom Surface      | 1731       | 1737.3     | 1730.6     |

Table 3.3: Thermoelastic Properties of Composites in Thermoelastic Example 2

<table>
<thead>
<tr>
<th>Properties</th>
<th>Material 1: SiC/polyimide</th>
<th>Material 2: T300/epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{11}$ (MPa)</td>
<td>259.4</td>
<td>133.4</td>
</tr>
<tr>
<td>$E_{22} = E_{33}$ (MPa)</td>
<td>14.90</td>
<td>14.90</td>
</tr>
<tr>
<td>$\nu_{12} = \nu_{23} = \nu_{13}$</td>
<td>0.25</td>
<td>0.26</td>
</tr>
<tr>
<td>$G_{12} = G_{13} = G_{23}$ (MPa)</td>
<td>5.53</td>
<td>3.81</td>
</tr>
<tr>
<td>$\alpha_{11}$ ($/^{\circ}$C)</td>
<td>$4.56 \times 10^{-6}$</td>
<td>$2.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\alpha_{22} = \alpha_{33}$ ($/^{\circ}$C)</td>
<td>$14.21 \times 10^{-6}$</td>
<td>$27.34 \times 10^{-6}$</td>
</tr>
<tr>
<td>$k_{11}$ (W/(m(^2)·C))</td>
<td>602.7</td>
<td>601.9</td>
</tr>
<tr>
<td>$k_{22} = k_{33}$ (W/(m·°C))</td>
<td>5.61</td>
<td>0.72</td>
</tr>
</tbody>
</table>

Fig. 3.21: Distribution of $\sigma_{11}$ along thickness at $x_2 = 0.01$ m for thermoelastic Example 2.
Fig. 3.22: Distribution of $\sigma_{12}$ along thickness at $x_2 = 0.01$ m for thermoelastic Example 2.

Fig. 3.23: Distribution of $\sigma_{22}$ along thickness at $x_2 = 0.01$ m for thermoelastic Example 2.
Example 3: Composite Beam Under Thermal Load Case 2

This example is to check the accuracy of the stress results of the present model under thermal load case 2. We use a simple two-layer angle-ply composite beam with lay-up as $[45^\circ/\ -45^\circ]$. The width of the beam is 0.04 m and the thickness is 0.04 m for each layer. The total thickness of the beam is 0.08 m. The length of the beam is 0.8 m. The beam is made of T300/epoxy with the same material properties as the previous Example 2. We prescribe the temperature as we did in Example 2 of heat conduction analysis, that is, we apply constant temperature on three sides and a sinusoidal temperature distribution on the fourth one. Figs. 3.24-3.26 presented here show the comparison of the stress components obtained by VABS and ANSYS. The element size is the same as the previous example. Quadratic elements are used here to carry out the analysis, that is, we use twenty-noded brick elements in ANSYS analysis and eight-noded elements in VABS.

Example 4: Thermomechanical Analysis of A Multilayer Beam Under Combined Thermal and Mechanical Loading

This example is to demonstrate that VABS can handle thermomechanical coupling and
Fig. 3.25: Distribution of $\sigma_{13}$ along width at $x_3 = -0.02$ m for thermoelastic Example 3.

Fig. 3.26: Distribution of $\sigma_{33}$ along width at $x_3 = -0.02$ m for thermoelastic Example 3.
that Timoshenko refined model makes an improvement comparing to the classical model for certain cases. A cantilever beam which is same as the one in Example 2 is considered. The thermal load is also same with Example 2 and we apply a 10 N force in negative $x_3$ direction at the center of the free end of the beam. Figs. 3.27-3.29 show the results obtained by VABS and ANSYS. One can observe from the figures that VABS does a fairly good job of predicting the stress components. As shown in Fig. 3.29, one can find that for certain cases, like with in-plane or out-plane shear stress, the results obtained by Timoshenko model is much better than that obtained by the classical model.

Example 5: Thermomechanical Analysis of a Curved Composite Beam

This example is a curved beam spans a 90° arc as shown in Fig. 3.30. This beam made up of two layers of different materials and the material properties and geometric properties are given in Table 3.4. A uniform temperature rise $T = 100^\circ F$ is applied to the beam and the top end is built-in. The cross-section is divided into 400 eight-noded quadrilateral...
Fig. 3.28: Distribution of $\sigma_{12}$ along $x_3$ at $x_2 = 0.01$ m for thermoelastic Example 4.

Fig. 3.29: Distribution of $\sigma_{13}$ along $x_3$ at $x_2 = 0.01$ m for thermoelastic Example 4.
Table 3.4: Material Properties and Geometric Properties for Thermoelastic Example 5

<table>
<thead>
<tr>
<th>Material Properties</th>
<th>Geometric Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1:</td>
<td></td>
</tr>
<tr>
<td>$E_1 = 10.3 \times 10^6$ psi</td>
<td>$r_{inner} = 4.12$ in.</td>
</tr>
<tr>
<td>$\nu_1 = 0.35$</td>
<td>$r_{outer} = 4.32$ in.</td>
</tr>
<tr>
<td>$\alpha_1 = 1.42 \times 10^{-5}$ in. / in. /°F</td>
<td>$w = 0.2$ in.</td>
</tr>
<tr>
<td>Material 2:</td>
<td>$t = 0.1$ in.</td>
</tr>
<tr>
<td>$E_2 = 20.6 \times 10^6$ psi</td>
<td></td>
</tr>
<tr>
<td>$\nu_2 = 0.3$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2 = 8.11 \times 10^{-6}$ in. / in. /°F</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3.30: Sketch of curved composite beam used in thermoelastic Example 5.

elements in VABS and 16,000 twenty-noded brick elements are used in ANSYS 3D analysis.
Figs. 3.31-3.32 show the stress distributions along the thickness ($x_3$) of the mid-surface ($\theta = 45^\circ$) at $x_2 = 0$. The stress components predicted by VABS are almost on top of the 3D solutions.

Example 6: Thermoelastic Analysis of a Realistic Rotor Blade

The author is not aware of any previous studies on thermoelastic analysis of realistic blade with prediction of the stresses over the whole cross-section, so this example shows
Fig. 3.31: Distribution of $\sigma_{\theta\theta}$ along $x_3$ at $x_2 = 0$ in for thermoelastic Example 5.

Fig. 3.32: Distribution of $\sigma_{rr}$ along $x_3$ at $x_2 = 0$ in for thermoelastic Example 5.
that the current VABS thermal model has the ability of analyzing a realistic blade structure at an affordable computational cost.

A NACA2412 airfoil is used in this case. A schematic of this blade as well as coordinate system is depicted in Fig. 3.33 where $x_1$ direction is coming out of the page. The chord length $l$ is 0.1524m while the length of the realistic blade $L$ is 1.524m. This realistic blade is made of Aluminum as the skin and a typical aerospace foam as the core. The Aluminum has the properties $E = 72.4\text{GPa}$, $\nu = 0.3$, and $\alpha = 22.5 \times 10^{-6}/\circ\text{C}$, and the properties for the aerospace foam are $E = 2.76\text{GPa}$, $\nu = 0.22$, and $\alpha = 2.2 \times 10^{-6}/\circ\text{C}$. This blade is cantilevered since most applications like helicopter rotor blade and wind turbine blade can be analyzed as cantilevered beam. A uniform temperature of 100$\circ\text{C}$ is applied to this blade.

The contour plots of the non-zero stress components $\sigma_{11}$, $\sigma_{22}$, $\sigma_{33}$, and $\sigma_{23}$ are shown in Figs. 3.34, 3.35, 3.36, and 3.37, respectively. For quantitative comparison, we plot these non-zero stress components at mid-span of the blade along two comparison paths shown in Fig. 3.33 in the following four figures. From Figs. 3.38, 3.39, 3.40, and 3.41, it is observed that the predictions of VABS have excellent agreement with those of ANSYS 3D along both chord-line-direction (Comparison Path 1) and through-the-thickness (Comparison Path 2) direction.

The computational efficiency of the two models is now shown. Figs. 3.42 and 3.43 show the mesh used in 3D and VABS calculations, respectively. Type of elements used, the total number of elements and nodes in calculations, and the running time of each model are tabulated in Table 3.5. It needs to be pointed out that the two running times for VABS
Fig. 3.34: Contour plot of $\sigma_{11}$ within the cross-section at mid-span for thermoelastic Example 6.

Fig. 3.35: Contour plot of $\sigma_{22}$ within the cross-section at mid-span for thermoelastic Example 6.

Fig. 3.36: Contour plot of $\sigma_{33}$ within the cross-section at mid-span for thermoelastic Example 6.

Fig. 3.37: Contour plot of $\sigma_{23}$ within the cross-section at mid-span for thermoelastic Example 6.
Fig. 3.38: Distributions of normal stresses along comparison path 1 at mid-span.

Fig. 3.39: Distribution of $\sigma_{23}$ along comparison path 1 at mid-span.
Fig. 3.40: Distributions of normal stresses along comparison path 2 at mid-span.

Fig. 3.41: Distribution of $\sigma_{23}$ along comparison path 2 at mid-span.
Fig. 3.42: 3D mesh of a realistic blade for thermoelastic Example 6.

Fig. 3.43: 2D mesh of a realistic blade for VABS calculation.
Table 3.5: Summary of Model Statistics

<table>
<thead>
<tr>
<th></th>
<th>ANSYS 3D</th>
<th>VABS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element Type</td>
<td>SOLID186</td>
<td>8-noded quadrilateral</td>
</tr>
<tr>
<td>Number of Elements</td>
<td>362,408</td>
<td>2,459</td>
</tr>
<tr>
<td>Number of Nodes</td>
<td>1,638,866</td>
<td>7,965</td>
</tr>
<tr>
<td>Running Time</td>
<td>3h 5min 23s</td>
<td>11s + 26s</td>
</tr>
</tbody>
</table>

analysis are for constitutive modeling and recovery, respectively. Both programs are running on a computing server with AMD Opteron(tm) Processor 6174 2.20 GHz (2 processors) and 128 GB RAM. The operating system is 64-bit Windows 7 Professional. It can be observed that the computational cost of VABS calculation is several orders lower than that of 3D analysis.

3.7.3 Analysis of a Sandwich Beam

Sandwich beam structure refers to a special class of composite beams that is fabricated by attaching two thin but stiff skins, often not identical, to a lightweight but thick core [96, 97]. Comparing to the skins, the rigidity of the core is about several orders lower. But its higher thickness provides the sandwich structure with high bending stiffness with overall low density. A typical application of sandwich structure is thermal protection system. Here we analyze a sandwich beam with different configurations under thermal environment, and the results are compared with those obtained by 3D ANSYS analysis.

Fig. 3.44 shows the configuration of the cross-section of the sandwich beam used in the current example. It is a beam of infinity long along $x_1$ direction and the core can be Material 1 and Material 2, so two cases are analyzed. The first case where Core Material 1 is used is called Case 1, and the second case is called Case 2 where Core Material 2 is used in the beam. The geometric and material properties are listed in Table 3.6. A temperature of 2000°F is applied at the top surface, and the bottom surface is constrained as 600°F.

The temperature distributions are plotted in Fig. 3.45, and the non-zero stress components are plotted in Figs. 3.46 and 3.47. Although the ratio of rigidity between skin and core materials changing from 1,000 to 10,000, the current model does a pretty good job in
Fig. 3.44: Sketch of the cross-section of a sandwich beam.

Table 3.6: Material Properties and Geometric Properties of a Sandwich Beam

<table>
<thead>
<tr>
<th>Layer</th>
<th>Material Properties</th>
<th>Geometric Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top Skin</td>
<td>$E = 59$ MPsi, $\nu = 0.14$</td>
<td>$t_1 = 0.125$ in.</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 4 \times 10^{-6}$/K, $k = 80$ W/(m $^\circ$ K)</td>
<td>$w = 30$ in.</td>
</tr>
<tr>
<td>Core Material 1</td>
<td>$E = 0.048$ MPsi, $\nu = 0.12$</td>
<td>$t_2 = 1.5$ in.</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 1 \times 10^{-6}$/K, $k = 0.3$ W/(m $^\circ$ K)</td>
<td>$w = 30$ in.</td>
</tr>
<tr>
<td>Core Material 2</td>
<td>$E = 0.004$ MPsi, $\nu = 0.49$</td>
<td>$t_2 = 1.5$ in.</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 39.9 \times 10^{-6}$/K, $k = 0.13$ W/(m $^\circ$ K)</td>
<td>$w = 30$ in.</td>
</tr>
<tr>
<td>Bottom Skin</td>
<td>$E = 7.64$ MPsi, $\nu = 0.32$</td>
<td>$t_3 = 0.25$ in.</td>
</tr>
<tr>
<td></td>
<td>$\alpha_x = 1.6 \times 10^{-6}$/K, $k = 80$ W/(m $^\circ$ K)</td>
<td>$w = 30$ in.</td>
</tr>
<tr>
<td></td>
<td>$\alpha_y = 28.1 \times 10^{-6}$/K</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 3.45: Distribution of temperature along thickness at $x_2 = \frac{1}{2}w$ for sandwich example.

Fig. 3.46: Distribution of $\sigma_{11}$ along thickness at $x_2 = \frac{1}{2}w$ for sandwich example.
Fig. 3.47: Distribution of $\sigma_{22}$ along thickness at $x_2 = \frac{1}{2}w$ for sandwich example.

predicting thermal and mechanical behavior of the sandwich beam.

3.7.4 Thermoelastic Analysis of a Composite Beam Under Finite Temperature Change

In this section, a cantilever two-layer composite beam is used to examine the temperature-dependent properties and the framework of thermoelasticity based on finite temperature change. Two load cases are studied here, one is the beam under small temperature change and the other is the beam experiencing finite temperature change. The geometry is given by Fig. 3.48 with the dimensions $L = 1$ m, $b = 0.1$ m, and $t = 0.05$ m. The material properties are listed in Table 3.7. Firstly, the beam is experiencing a small temperature change, from 480°C to 500°C. The stress distributions of mid-span along thickness are plotted in Figs. 3.49 to 3.51. The curves and data points labeled “Inst” are calculated based on the traditional framework of thermoelasticity where instantaneous CTEs are used. The curves and data points labeled “Sect” are from the current framework of thermoelasticity
where secant CTEs are used in the analysis. Excellent agreement exists between predictions from VABS and ANSYS 3D analysis. Moreover, due to the load chosen, the newly developed framework of finite temperature change thermoelasticity does not have a huge impact on the results. In other words, the predictions from traditional constitutive framework of thermoelasticity may be adequate for this case.

For the second case, the composite beam is experiencing a large temperature change from 0°C to 500°C. Figs. 3.52, 3.53, and 3.54 show the plots of non-zero stress components $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{33}$, respectively. Again, excellent agreements exist between results from 3D analysis and VABS based on different theories. A striking observation from these three figures is that two different frameworks of thermoelasticity result in huge different stress

![Fig. 3.48: Schematic of a two-layer composite beam for finite temperature change analysis.](image)

Table 3.7: Material Properties of Two-layer Composite Beam in Fig. 3.48

<table>
<thead>
<tr>
<th>Material</th>
<th>0°C</th>
<th>200°C</th>
<th>500°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material 1:</td>
<td>$E = 83\text{GPa}$</td>
<td>$E = 82.47\text{GPa}$</td>
<td>$E = 81.67\text{GPa}$</td>
</tr>
<tr>
<td>CFCCs</td>
<td>$\nu = 0.27$</td>
<td>$\nu = 0.27$</td>
<td>$\nu = 0.27$</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 4.28 \times 10^{-6}/\circ\text{C}$</td>
<td>$\alpha = 4.278 \times 10^{-6}/\circ\text{C}$</td>
<td>$\alpha = 4.275 \times 10^{-6}/\circ\text{C}$</td>
</tr>
<tr>
<td>Material 2:</td>
<td>$E = 2.76\text{GPa}$</td>
<td>$E = 2.76\text{GPa}$</td>
<td>$E = 2.76\text{GPa}$</td>
</tr>
<tr>
<td>DCF</td>
<td>$\nu = 0.22$</td>
<td>$\nu = 0.22$</td>
<td>$\nu = 0.22$</td>
</tr>
<tr>
<td></td>
<td>$\alpha = 1.22 \times 10^{-6}/\circ\text{C}$</td>
<td>$\alpha = 2.06 \times 10^{-6}/\circ\text{C}$</td>
<td>$\alpha = 2.56 \times 10^{-6}/\circ\text{C}$</td>
</tr>
</tbody>
</table>
Fig. 3.49: Distributions of $\sigma_{11}$ along thickness at $x_2 = 0$ for small temperature change.

Fig. 3.50: Distributions of $\sigma_{22}$ along thickness at $x_2 = 0$ for small temperature change.
Fig. 3.51: Distributions of $\sigma_{33}$ along thickness at $x_2 = 0$ for small temperature change.

distributions for this case. It demonstrates that the influence of temperature-dependent material properties and framework of thermoelasticity on thermal stresses is quite significant for finite temperature change cases.
Fig. 3.52: Distributions of $\sigma_{11}$ along thickness at $x_2 = 0$ for large temperature change.

![Graph showing distributions of $\sigma_{11}$](image1)

Fig. 3.53: Distributions of $\sigma_{22}$ along thickness at $x_2 = 0$ for large temperature change.

![Graph showing distributions of $\sigma_{22}$](image2)
Fig. 3.54: Distributions of $\sigma_{33}$ along thickness at $x_2 = 0$ for large temperature change.
Chapter 4

Asymptotic Construction of Multiphysics Model

In this chapter, we proposed an efficient high-fidelity beam model for predicting multiphysics behavior of smart slender structures using the variational-asymptotic method. This model can handle piezoelectric, piezomagnetic, pyroelectric and pyromagnetic effects in a beam structure in addition to the traditional mechanical and thermoelastic behavior. Two multiphysics load types are considered in the current work.

4.1 Theoretical Formulation

4.1.1 3D Formulation

As discussed in Chapter 2, the dynamic behavior of solids is governed by the extended Hamilton’s principle. To capture the multiphysics behavior of a structure, we rewrite the variational statement in Eq. (2.79) as

\[
\int_{t_1}^{t_2} \left[ \delta \left( K_{1D} + K^* - U_M \right) + \delta W_{1D} + \delta W^* \right] dt = 0 \tag{4.1}
\]

Here the kinetic energy and virtual work are expressed in terms of 1D variables. For structures active to electromagnetic fields, the internal energy is the electromagnetic enthalpy containing contributions from mechanical, electric, magnetic, and thermal fields and the coupling effects among them

\[
U_M = \frac{1}{2} \int_V \left[ \Gamma : C^{E,H} : \Gamma - E : k^\Gamma,H : E - H : \mu^\Gamma,E \cdot H \\
- 2E : e^H : \Gamma - 2H : q^E : \Gamma - 2E : a^\Gamma \cdot H - 2(\Gamma : \Lambda + E : p + H \cdot m) \cdot \Delta T \right] dV \tag{4.2}
\]
where $\mathbf{\Gamma}$, $\mathbf{E}$ and $\mathbf{H}$ are the strain, electric field and magnetic field tensors, respectively; $\mathbf{e}^H$, $\mathbf{q}^E$, and $\mathbf{a}^\Gamma$ are piezoelectric tensor (measured at constant magnetic field), piezomagnetic tensor (measured at constant electric field), and magnetoelectric tensor (measured at constant strain), respectively; $\mathbf{C}^{E,H}$, $\mathbf{k}^{\Gamma,H}$, and $\mathbf{\mu}^{\Gamma,E}$ are elastic tensor (measured at constant electric and magnetic field), dielectric tensor (measured at constant strain and magnetic field), and magnetic permeability tensor (measured at constant strain and electric field), respectively; $\mathbf{\Lambda}$, $\mathbf{p}$, and $\mathbf{m}$ are thermal stress tensor, pyroelectric vector, and pyromagnetic vector, respectively; $\Delta T$ is the temperature change from reference temperature; $\mathcal{V}$ is the space occupied by the structure. For beam structures, we can also express the internal energy in Eq. (4.2) as

$$U_M = \int_0^l \mathcal{U}_M dx_1$$

with $\mathcal{U}_M$ defining the internal energy per unit span.

The kinematics of beam structures has been formulated in Chapter 2. However, to deal with the multiphysics problems of beams, a complete description requires electric field and magnetic field in addition to the mechanical field. These two fields are characterized by the electric potential $\phi(x_i)$ and magnetic potential $\theta(x_i)$ as

$$\mathbf{E} = -\nabla \phi = -\frac{\partial \phi}{\partial x_i} \mathbf{g}^i$$
$$\mathbf{H} = -\nabla \theta = -\frac{\partial \theta}{\partial x_i} \mathbf{g}^i$$

The internal energy density in Eq. (4.3) can be written in matrix form as

$$\mathcal{U}_M = \left\langle \mathbf{\hat{\Gamma}}^T \tilde{\mathbf{D}} \mathbf{\hat{\Gamma}} - 2\mathbf{\hat{\Gamma}}^T \eta \Delta T \right\rangle$$
where $\hat{D}$ is a $12 \times 12$ matrix containing all the necessary material constants for characterizing the fully coupled electromagnetoelastic materials

$$
\hat{D} = \begin{bmatrix}
C_{E,H} & -e^H & -q^E \\
-e^{H^T} & -k_{\Gamma,H} & -a_{\Gamma} \\
-q^{E^T} & -a_{\Gamma^T} & -\mu_{\Gamma,E}
\end{bmatrix}
$$

(4.6)

$\eta$ is a $12 \times 1$ matrix containing the second-order thermal stress tensor $\Lambda$, pyroelectric vector $p$ and pyromagnetic vector $m$

$$
\eta = [\Lambda_{11} \, \Lambda_{12} \, \Lambda_{13} \, \Lambda_{22} \, \Lambda_{23} \, \Lambda_{33} \, p_1 \, p_2 \, p_3 \, m_1 \, m_2 \, m_3]^T
$$

(4.7)

For convenience of derivation, we define the generalized 3D strain vector as

$$
\hat{\Gamma} = [\Gamma_{11} \, 2\Gamma_{12} \, 2\Gamma_{13} \, \Gamma_{22} \, 2\Gamma_{23} \, \Gamma_{33} \, E_1 \, E_2 \, E_3 \, H_1 \, H_2 \, H_3]^T
$$

(4.8)

There are two types of boundary conditions for the applied electric or magnetic field.

- Multiphysics load case 1: electric or magnetic field is not prescribed at any point over the cross-section, for example, if only the end surfaces at $x_1 = 0$ and $x_1 = l$ have prescribed potential (see Fig. 4.1), we are free to use the following change of variables for the 3D electric and magnetic fields

$$
\phi(x_1, x_2, x_3) = \Phi(x_1) + w_\phi(x_1, x_2, x_3)
$$

$$
\theta(x_1, x_2, x_3) = \Theta(x_1) + w_\theta(x_1, x_2, x_3)
$$

(4.9)

with the 1D variables $\Phi(x_1)$ and $\Theta(x_1)$ defined as the average of $\phi$ and $\theta$ over the cross-section, which implies

$$
\langle w_\phi(x_1, x_2, x_3) \rangle = 0
$$

$$
\langle w_\theta(x_1, x_2, x_3) \rangle = 0
$$

(4.10)
Multiphysics load case 2: electric or magnetic field is prescribed at some points over the cross-section (see Fig. 4.2), we lose the freedom to introduce the 1D variables $\Phi$ and $\Theta$. The 3D electric and magnetic fields can be expressed as

\[
\begin{align*}
\phi(x_1, x_2, x_3) &= w_\phi(x_1, x_2, x_3) \\
\theta(x_1, x_2, x_3) &= w_\theta(x_1, x_2, x_3)
\end{align*}
\] (4.11)

and we cannot constrain the electric and magnetic warping functions as we did in Case 1 either; see Eq. (4.10).

Most of the current applications of smart beams belong to either one of these two cases.
or a combination of these two cases. In the following sections, we will carry out dimensional reduction for these two cases and then recover the multiphysics fields.

### 4.1.2 Units and Scaling Method

In this section, we will present a detailed description of the units because it is a constant confusion regarding the units used in the multiphysics modeling. In order to avoid ill-conditioned matrix, a scaling method is also presented.

To avoid confusion, firstly, we express the $\mathbf{D}$ matrix in the explicit form of the $12 \times 12$ matrix as follows:

$$
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & -e_{11} & -e_{21} & -e_{31} & -q_{11} & -q_{21} & -q_{31} \\
c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & -e_{12} & -e_{22} & -e_{32} & -q_{12} & -q_{22} & -q_{32} \\
c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} & -e_{13} & -e_{23} & -e_{33} & -q_{13} & -q_{23} & -q_{33} \\
c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} & -e_{14} & -e_{24} & -e_{34} & -q_{14} & -q_{24} & -q_{34} \\
c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} & -e_{15} & -e_{25} & -e_{35} & -q_{15} & -q_{25} & -q_{35} \\
c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} & -e_{16} & -e_{26} & -e_{36} & -q_{16} & -q_{26} & -q_{36} \\
-e_{11} & -e_{12} & -e_{13} & -e_{14} & -e_{15} & -e_{16} & -k_{11} & -k_{12} & -k_{13} & -a_{11} & -a_{12} & -a_{13} \\
-e_{21} & -e_{22} & -e_{23} & -e_{24} & -e_{25} & -e_{26} & -k_{21} & -k_{22} & -k_{23} & -a_{21} & -a_{22} & -a_{23} \\
-e_{31} & -e_{32} & -e_{33} & -e_{34} & -e_{35} & -e_{36} & -k_{31} & -k_{32} & -k_{33} & -a_{31} & -a_{32} & -a_{33} \\
-q_{11} & -q_{12} & -q_{13} & -q_{14} & -q_{15} & -q_{16} & -a_{11} & -a_{21} & -a_{31} & -\mu_{11} & -\mu_{12} & -\mu_{13} \\
-q_{21} & -q_{22} & -q_{23} & -q_{24} & -q_{25} & -q_{26} & -a_{12} & -a_{22} & -a_{32} & -\mu_{12} & -\mu_{22} & -\mu_{23} \\
-q_{31} & -q_{32} & -q_{33} & -q_{34} & -q_{35} & -q_{36} & -a_{13} & -a_{23} & -a_{33} & -\mu_{13} & -\mu_{23} & -\mu_{33}
\end{pmatrix}
$$

(4.12)

According to the International Standard unit system, we use $\text{Pa}$ (i.e., $\text{N/m}^2$) for the elastic constants $C_{ijkl}$ and the stress field $\sigma_{ij}$ (note the strain field $\Gamma_{ij}$ is unitless), $\text{C/m}^2$ for piezoelectric constants $e_{ijk}$ and electric displacement $D_i$, $\text{N/(A\cdot m)}$ for piezomagnetic constants $q_{ijk}$ and magnetic induction $B_i$, $\text{C/(V\cdot m)}$ for dielectric constants $k_{ij}$, $\text{N/A}^2$ (or $\text{N\cdot s}^2/\text{C}^2$) for magnetic permeability $\mu_{ij}$, $\text{C/(A\cdot m)}$ for electromagnetic coefficients $a_{ij}$, $\text{V/m}$ for electric field $E_i$, $\text{A/m}$ for magnetic field $H_i$, $\text{K}$ for the temperature field $\Delta T$ (note $\degree\text{C}$ has
the same unit dimension as $\mathbf{K}$, $1/K$ for thermal expansion coefficients $\alpha_{ij}$ (correspondingly $\text{Pa/K}$ for thermal stress coefficients $\Lambda_{ij}$), $C/\mathbf{m}^2 \cdot \mathbf{K}$ for pyroelectric constants $p_i$, and $\mathbf{N}/(\mathbf{A} \cdot \mathbf{m} \cdot \mathbf{K})$ for pyromagnetic $m_i$. With all these units, the energy density $U$ will be in the unit of $\mathbf{N}/\mathbf{m}^2$, which is the same as $\mathbf{J}/\mathbf{m}^3$.

Although the units aforementioned are consistent with each other, direct use of these units will introduce an extremely ill-conditioned generalized stiffness matrix $\hat{D}$ as for regular materials, we will have $C_{ijkl}$ in the order of $10^{11}$, while $k_{ij}$ in the order of $10^{-9}$. Proper scaling is needed even if double precision is used in computing. To this end, we define $E_i^* = 10^{-9}E_i, H_i^* = 10^{-9}H_i$, then the energy density $U_M$ in Eq. (4.5) can be rewritten as

$$\frac{U_M}{10^9} = \frac{1}{2} \begin{bmatrix} \Gamma \\ E^* \\ H^* \end{bmatrix}^T \begin{bmatrix} C^* & -e & -q \\ -e^T & -k^* & -a^* \\ -q^T & -a^{*T} & -\mu^* \end{bmatrix} \begin{bmatrix} \Gamma \\ E^* \\ H^* \end{bmatrix} + \begin{bmatrix} \Gamma \\ E^* \\ H^* \end{bmatrix}^T \begin{bmatrix} -C^* \alpha \\ -p \\ -m \end{bmatrix} \Delta T \quad (4.13)$$

with

$$C^* = \frac{C}{10^9}, \quad a^* = a \times 10^9, \quad k^* = k \times 10^9, \quad \mu^* = \mu \times 10^9 \quad (4.14)$$

According to the generalized Hooke’s law, the constitutive equations for multiphysics modeling can be rewritten in the following matrix form

$$\sigma^* = C^* \Gamma - e E^* - q H^* - \Lambda^* \Delta T$$

$$D = e^T \Gamma + k^* E^* + a^* H^* + p_i \Delta T \quad (4.15)$$

$$B = q^T \Gamma + a^{*T} E^* + \mu^* H^* + m_i \Delta T$$

with $\sigma^* = \frac{\mathbf{D}}{10^9}$. For VABS multiphysics constitutive modeling, we input $C^*, e, q, k^*, a^*, \mu^*, \alpha, p, m$ as material properties, and for the recovery, we input $\Gamma, E^*, H^*$ as the field vectors. In other words, if the quantities are given in IS units, we need to divide $C, E, H$ by $10^9$, and multiply $k, a, \mu$ by $10^9$, and all the other quantities remain the same. The output stiffness constants are also scaled the same way as the input material properties. As far as the recovered field concerned, the displacements, electromagnetic potentials, strains, electric displacements,
and magnetic induction are the same as SI units, one needs to multiply the stresses, electric and magnetic fields with \(10^9\) to convert these quantities in SI units. It is pointed out that it is just one suggestion to scale the inputs to avoid numerical difficulties. One can certainly devise a different scaling following the same philosophy.

### 4.2 Dimensional Reduction of Multiphysics Load Case 1

The 3D electric and magnetic fields for this case can be obtained by substituting the potentials in Eq. (4.9) into Eq. (4.4), which can be written as

\[
E_1 = \frac{1}{\sqrt{g}} \left[ E_{1D} - w'_\phi + k_1 \left( x_2 \frac{\partial w\phi}{\partial x_3} - x_3 \frac{\partial w\phi}{\partial x_2} \right) \right] \\
E_2 = -\frac{\partial w\phi}{\partial x_2} \\
E_3 = -\frac{\partial w\phi}{\partial x_3}
\]

and

\[
H_1 = \frac{1}{\sqrt{g}} \left[ H_{1D} - w'_\theta + k_1 \left( x_2 \frac{\partial w\theta}{\partial x_3} - x_3 \frac{\partial w\theta}{\partial x_2} \right) \right] \\
H_2 = -\frac{\partial w\theta}{\partial x_2} \\
H_3 = -\frac{\partial w\theta}{\partial x_3}
\]

Here \(E_{1D}\) and \(H_{1D}\) are defined as

\[
E_{1D} = -\Phi' \\
H_{1D} = -\Theta'
\]

Denoting \(\hat{\epsilon} = [\tilde{\gamma}_{11} \quad \tilde{\kappa}_1 \quad \tilde{\kappa}_2 \quad \tilde{\kappa}_3 \quad E_{1D} \quad H_{1D}]^T\) as the generalized 1D strain and \(\hat{w} = [w_1 \quad w_2 \quad w_3 \quad w_\phi \quad w_\theta]^T\) as the generalized warping functions, we can express \(\hat{\Gamma}\) as

\[
\hat{\Gamma} = \hat{\Gamma}_a \hat{w} + \hat{\Gamma}_\epsilon \hat{\epsilon} + \hat{\Gamma}_R \hat{w} + \hat{\Gamma}_l \hat{w}'
\]
The explicit forms of the operator matrices in Eq. (4.19) are given as

\[ \hat{\Gamma}_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 & 0 \\ \partial_3 & 0 & 0 & 0 & 0 \\ 0 & \partial_2 & 0 & 0 & 0 \\ 0 & \partial_3 & \partial_2 & 0 & 0 \\ 0 & 0 & \partial_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_2 & 0 \\ 0 & 0 & 0 & -\partial_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\partial_2 \\ 0 & 0 & 0 & 0 & -\partial_3 \end{bmatrix} \]  \hspace{1cm} (4.20)

\[ \hat{\Gamma}_\epsilon = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & x_3 & -x_2 & 0 & 0 \\ 0 & -x_3 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]  \hspace{1cm} (4.21)
\[ \hat{\Gamma}_R = \frac{1}{\sqrt{g}} \begin{bmatrix} k^* & -k_3 & k_2 & 0 & 0 \\ k_3 & k^* & -k_1 & 0 & 0 \\ -k_2 & k_1 & k^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (4.22)

where \( k^* = k_1(x_3 \partial_2 - x_2 \partial_3) \), and

\[ \hat{\Gamma}_l = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (4.23)
For multiphysics load case 1, because we have two additional constraints in Eq. (4.10), the constraints on the generalized warping function \( \hat{w} \) can be expressed in matrix form as

\[
\langle \hat{\Gamma}_c \hat{w} \rangle = 0 \tag{4.24}
\]

with

\[
\hat{\Gamma}_c = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & \partial_3 & -\partial_2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{4.25}
\]

and the kernel matrix introduced in Eq. (2.43) in Chapter 2 should be revised for this case as

\[
\hat{\psi} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -x_3 & 0 & 0 \\
0 & 0 & 1 & x_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{4.26}
\]

In order to deal with the arbitrary cross-sectional geometry and anisotropic materials, we need to turn to a numerical approach, such as the finite element method (FEM), to find the warping functions. The warping field can be discretized as

\[
\hat{w}(x_1, x_2, x_3) = S(x_2, x_3) V(x_1) \tag{4.27}
\]

with \( S(x_2, x_3) \) representing the element shape functions and \( V \) as a column matrix of the nodal values of the warping functions over the cross-section. Substituting Eq. (4.27) into
Eq. (4.19) and then into Eq. (4.5), we obtain

\[
2\mathcal{M} = V^T EV + 2V^T (D_{ae} \hat{\epsilon} + D_{aR} \epsilon + D_{al} \epsilon') + \hat{\epsilon}^T D_{ae} \hat{\epsilon} + \\
V^T D_{RR} V + V'^T D_{ll} V' + 2V'^T D_{Re} \hat{\epsilon} + 2V'^T D_{Rl} \epsilon + 2V^T D_{Rl} V' \\
- 2 (V^T \alpha_a + \epsilon^T \alpha_a + V^T \alpha_R + V'^T \alpha_l)
\]

(4.28)

The new matrix variables carry the properties of both the geometry and material, defined as

\[
E = \left\langle \left[ \hat{\Gamma}_a S \right]^T \hat{\mathcal{D}} \left[ \hat{\Gamma}_a S \right] \right\rangle \quad D_{ae} = \left\langle \left[ \hat{\Gamma}_a S \right]^T \hat{\mathcal{D}} \hat{\Gamma}_e \right\rangle \\
D_{aR} = \left\langle \left[ \hat{\Gamma}_a S \right]^T \hat{\mathcal{D}} \left[ \hat{\Gamma}_R S \right] \right\rangle \quad D_{RR} = \left\langle \left[ \hat{\Gamma}_R S \right]^T \hat{\mathcal{D}} \left[ \hat{\Gamma}_R S \right] \right\rangle \\
D_{al} = \left\langle \left[ \hat{\Gamma}_a S \right]^T \hat{\mathcal{D}} \left[ \hat{\Gamma}_l S \right] \right\rangle \quad D_{le} = \left\langle \left[ \hat{\Gamma}_l S \right]^T \hat{\mathcal{D}} \hat{\Gamma}_e \right\rangle \\
D_{ll} = \left\langle \left[ \hat{\Gamma}_l S \right]^T \hat{\mathcal{D}} \left[ \hat{\Gamma}_l S \right] \right\rangle \quad D_{Rl} = \left\langle \left[ \hat{\Gamma}_R S \right]^T \hat{\mathcal{D}} \left[ \hat{\Gamma}_l S \right] \right\rangle \\
\alpha_a = \left\langle \left[ \hat{\Gamma}_a S \right]^T \hat{\mathcal{D}} \eta \Delta T \right\rangle \quad \alpha_e = \left\langle \left[ \hat{\Gamma}_e \right]^T \hat{\mathcal{D}} \eta \Delta T \right\rangle \\
\alpha_l = \left\langle \left[ \hat{\Gamma}_l S \right]^T \hat{\mathcal{D}} \eta \Delta T \right\rangle \quad \alpha_R = \left\langle \left[ \hat{\Gamma}_R S \right]^T \hat{\mathcal{D}} \eta \Delta T \right\rangle 
\]

(4.29)

It is noted that \( \sqrt{g} \) contains \( k_\alpha \) and it is should be expanded in the asymptotic analysis. For example, the \( E \) matrix should be expressed as

\[
E = \left\langle \left[ \hat{\Gamma}_a S \right]^T \mathcal{D} \left[ \hat{\Gamma}_a S \right] \right\rangle + \left\langle \left[ \hat{\Gamma}_a S \right]^T D \left[ \hat{\Gamma}_a S \right] (x_3 k_2 - x_2 k_3) \right\rangle
\]

(4.30)

For simplicity of notation, we continue to use \( E \) in derivation with the understanding that such expansions are actually carried out in the numerical implementation. The details can be found in Appendix B.
4.2.1 Classical Theory

Neglecting the terms higher than $O (\nu \dot{\epsilon})$ in Eq. (4.28), we obtain the expression for the zeroth-order energy $U_{M0}$

$$2U_{M0} = V^T E V + 2V^T D_a \dot{\epsilon} + \dot{\epsilon}^T D_\epsilon \dot{\epsilon} - 2(V^T \alpha_a + \dot{\epsilon}^T \alpha_\epsilon)$$ (4.31)

Substituting Eq. (4.27) into Eq. (4.24), the constraints could be expressed in discretized form as

$$V^T D_c = 0$$ (4.32)

with $D_c^T = \langle \hat{\Gamma}_c S \rangle$. We also denote the corresponding discretized kernel matrix of $E$ as $\Psi$ so that $E \Psi = 0$.

Now the problem is transformed to minimize the functional in Eq. (4.31) subject to the constraints in Eq. (4.32). The Euler-Lagrange equation of multiphysics load case 1 can be obtained by the usual procedure of calculus of variation with the aid of a Lagrange multiplier as follows

$$EV + D_a \dot{\epsilon} - \alpha_a = D_c \Lambda$$ (4.33)

Multiplying both sides by $\Psi^T$ and considering the properties of the kernel matrix $\Psi$, one calculates the Lagrange multiplier $\Lambda$ as

$$\Lambda = (\Psi^T D_c)^{-1} \Psi^T (D_a \dot{\epsilon} - \alpha_a)$$ (4.34)

It is clear that $\Lambda$ vanishes because $\Psi^T D_a = \left\langle \left( \hat{\Gamma}_a S \Psi \right)^T \hat{D} \hat{\Gamma}_\epsilon \right\rangle = 0$, similarly $\Psi^T \alpha_a = 0$, which implies that the constraints will not affect the minimum value of $U_{M0}$. Then the linear system in Eq. (4.33) becomes

$$EV = -D_a \dot{\epsilon} + \alpha_a$$ (4.35)

There exists a unique solution linearly independent of the null space of $E$ for $V$ because the right-hand-side of Eq. (4.35) is orthogonal to the null space. Because of the uniqueness
of the solution, the linear system in Eq. (4.35) can be solved by letting the numerical algorithm to determine where the singularities are and properly remove the singularities of the coefficient matrix. Let us denote the solution of Eq. (4.35) obtained this way as $V^*$, the complete solution can be written as

$$V = V^* + \Psi \lambda$$  \hspace{1cm} (4.36)

where $\lambda$ can be determined by Eq. (4.32) as

$$\lambda = - \left( \Psi^T D_c \right)^{-T} D_c^T V^*$$  \hspace{1cm} (4.37)

Hence the final solution minimizing the functional Eq. (4.31) subject to constraints Eq. (4.32) is

$$V = \left[ \Delta - \Psi \left( \Psi^T D_c \right)^{-T} D_c^T \right] V^* = \hat{V}_0 \dot{\epsilon} + \hat{V} = V_0$$  \hspace{1cm} (4.38)

where $V_0$ is the warping caused by the applied temperature field.

Substituting Eq. (4.38) back into Eq. (4.31), one can obtain the total energy of multiphysics load case 1 asymptotically correct up to the zeroth-order as

$$2U_{M0} = \dot{\epsilon}^T \left( \hat{V}_0^T D_{\alpha \epsilon} + D_{\epsilon \epsilon} \right) \dot{\epsilon} - 2\dot{\epsilon}^T \left[ \alpha_{\epsilon} + \frac{1}{2} \left( \hat{V}_0^T \alpha_{\alpha} - D_{\alpha \epsilon} V_{t0} \right) \right]$$  \hspace{1cm} (4.39)

Note the quadratic terms associated with temperature $V_{t0}^T \alpha_{\alpha}$ and $V_{t0}^T E V_{t0}$ are dropped because they will not contribute to the 1D multiphysics beam model. This energy can be
written explicitly as

\[
2\mathcal{U}_{M0} = \begin{bmatrix}
\bar{\gamma}_{11} \\
\bar{\kappa}_1 \\
\bar{\kappa}_2 \\
\bar{\kappa}_3 \\
E_{1D} \\
H_{1D}
\end{bmatrix}^T \begin{bmatrix}
\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{e}_{11} & \bar{q}_{11} \\
\bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} & \bar{e}_{12} & \bar{q}_{12} \\
\bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & \bar{S}_{34} & \bar{e}_{13} & \bar{q}_{13} \\
\bar{S}_{14} & \bar{S}_{24} & \bar{S}_{34} & \bar{S}_{44} & \bar{e}_{14} & \bar{q}_{14} \\
\bar{e}_{11} & \bar{e}_{12} & \bar{e}_{13} & \bar{e}_{14} & \bar{k}_{55} & \bar{a}_{15} \\
\bar{q}_{11} & \bar{q}_{12} & \bar{q}_{13} & \bar{q}_{14} & \bar{a}_{15} & \bar{\mu}_{66}
\end{bmatrix}
\begin{bmatrix}
\bar{\gamma}_{11} \\
\bar{\kappa}_1 \\
\bar{\kappa}_2 \\
\bar{\kappa}_3 \\
E_{1D} \\
H_{1D}
\end{bmatrix}^T
\begin{bmatrix}
f_1^n \\
m_1^n \\
m_2^n \\
m_3^n \\
f_{E}^n \\
f_{H}^n
\end{bmatrix}
\]

(4.40)

and the corresponding 1D constitutive relation for the classical beam model is

\[
\begin{bmatrix}
F_1 \\
M_1 \\
M_2 \\
M_3 \\
F_E \\
F_H
\end{bmatrix} = \begin{bmatrix}
\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} & \bar{e}_{11} & \bar{q}_{11} \\
\bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} & \bar{e}_{12} & \bar{q}_{12} \\
\bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & \bar{S}_{34} & \bar{e}_{13} & \bar{q}_{13} \\
\bar{S}_{14} & \bar{S}_{24} & \bar{S}_{34} & \bar{S}_{44} & \bar{e}_{14} & \bar{q}_{14} \\
\bar{e}_{11} & \bar{e}_{12} & \bar{e}_{13} & \bar{e}_{14} & \bar{k}_{55} & \bar{a}_{15} \\
\bar{q}_{11} & \bar{q}_{12} & \bar{q}_{13} & \bar{q}_{14} & \bar{a}_{15} & \bar{\mu}_{66}
\end{bmatrix}
\begin{bmatrix}
\bar{\gamma}_{11} \\
\bar{\kappa}_1 \\
\bar{\kappa}_2 \\
\bar{\kappa}_3 \\
E_{1D} \\
H_{1D}
\end{bmatrix}^T
\begin{bmatrix}
f_1^n \\
m_1^n \\
m_2^n \\
m_3^n \\
f_{E}^n \\
f_{H}^n
\end{bmatrix}
\]

(4.41)

As an analogy to the mechanical counterpart, \( F_E \) and \( F_H \) can be considered as 1D multiphysics resultants.

4.2.2 Refined Theory

To obtain a Timoshenko-like refined model, terms up to \( O(\nu \varepsilon^2 h^2) \) are kept in total internal energy. As we did in the previous chapters, here we assume that \( \bar{h} \sim h/l \sim h/R \), that is, \( l \) and \( R \) are of the same order. Perturb the warping functions for multiphysics load case 1 as

\[
V = \tilde{V}_0 \varepsilon + V_{i0} + V_1
\]

(4.42)
Substituting Eq. (4.42) into Eq. (4.28), we obtain the following functional

\[
2\mathcal{U}_{M1} = \hat{\epsilon}^T \left( V_0^T D_{aR} + D_{a\ell} \right) \hat{\epsilon} - 2\hat{\epsilon}^T \left[ \alpha_{\ell} + \frac{1}{2} \left( V_0^T \alpha_a - D_{aR}^T V_{10} \right) \right]
+ 2V_0^TD_{aR}V_0 + 2V_0^TD_{a\ell}V_0' + 2V_0^TD_{R\ell} \hat{\epsilon} + 2V_0'^T D_{R\ell} \hat{\epsilon} - 2V_0'^T \alpha_l - 2V_0^T \alpha_R
\]

\[V_1^T EV_1 + 2V_1^T (D_{aR}V_0 + D_{aR}^T V_0 + D_{R\ell} \hat{\epsilon}) + 2V_1^T D_{a\ell}V_0' + 2V_0^T D_{a\ell}V_1' + 2V_1'^T D_{\ell\ell} \hat{\epsilon} + V_0^T D_{RR}V_0 + 2V_0^T D_{R\ell}V_0' + V_0'^T D_{R\ell}V_0' - 2V_1'^T \alpha_l - 2V_1^T \alpha_R \] (4.43)

After integrating by parts, the leading terms (without the constant terms) are

\[
2\mathcal{U}_{M1}^* = V_1^T EV_1 + 2V_1^T D_{R\ell} \hat{\epsilon} + 2V_1^T D_{S\ell} \hat{\epsilon}' + 2V_1^T (D_{RT} + D_{ST}) \] (4.44)

where

\[
D_R = D_{aR} \hat{V}_0 + D_{aR}^T \hat{V}_0 + D_{R\ell} \] (4.45)

\[
D_S = D_{a\ell} \hat{V}_0 - D_{a\ell}^T \hat{V}_0 - D_{\ell\ell} \] (4.46)

\[
D_{RT} = (D_{aR}^T + D_{aR}) V_{00} - \alpha_R \] (4.47)

\[
D_{ST} = (D_{a\ell} - D_{a\ell}^T) V_{00}' + \alpha_1' \] (4.48)

Similar to the zeroth-order warping, the first-order warping functions could be solved as

\[
V_1 = V_{1R} \hat{\epsilon} + V_{1S} \hat{\epsilon}' + V_{1T} \] (4.49)

Using Eq. (4.49), the second-order asymptotically correct energy of multiphysics load case 1 can now be obtained from Eq. (4.28) as

\[
2\mathcal{U}_{M1} = \hat{\epsilon}^T A \hat{\epsilon} + 2\hat{\epsilon}^T B \hat{\epsilon} + \hat{\epsilon}'^T C \hat{\epsilon}' + 2\hat{\epsilon}''^T D \hat{\epsilon}'' - 2\hat{\epsilon}^T F_{a1} - 2\hat{\epsilon}^T F_{a2} - 2\hat{\epsilon}''^T F_{a3} \] (4.50)
where

\[ A = V_0^T D_{ae} + D_{ae} + \hat{V}_0^T (D_{R +} + D_{al}^T + D_{RR}) \hat{V}_0 + 2 \hat{V}_0^T D_{Rc} + V_{1R}^T D_R \]

\[ B = V_0^T (D_{al} + D_{Rl}) \hat{V}_0 + D_{al}^T \hat{V}_0 + \left( \hat{V}_0^T D_{al} + D_{al}^T \right) V_{1R} + \frac{1}{2} (D_R^T V_{1S} + V_{1R}^T D_S) \]

\[ C = V_{1S} \tilde{D}_S + \hat{V}_0^T D_{al} \hat{V}_0 \]

\[ D = \left( D_{al}^T + \hat{V}_0^T D_{al} \right) V_{1S} \]

\[ F_{a1} = N_I - \left( D_R^T + \hat{V}_0^T D_{RR} \right) V_{10} - \left( \hat{V}_0^T D_{al} + D_{al}^T + \hat{V}_0^T D_{Rl} \right) V_{i0} - \frac{1}{2} D_R^T V_{1T} \]  

\[ - \frac{1}{2} V_{1R} (D_{RT} + \tilde{D}_{ST}) + \hat{V}_0^T \alpha_R - \hat{V}_0 D_{al} V_{1T} - D_{al}^T V_{1T} \]

\[ F_{a2} = \left( \hat{V}_0^T + \hat{V}_{1R} \right) \alpha_L - \left( \hat{V}_0^T D_{al} + \hat{V}_{1R} D_{al} + \hat{V}_0^T D_{Rl} \right) V_{10} - \frac{1}{2} \tilde{D}_S^T V_{1T} \]

\[ - \hat{V}_0^T D_{al} V_{10} - \frac{1}{2} V_{1S} (D_{RT} + \tilde{D}_{ST}) \]

\[ F_{a3} = - V_{1S} D_{al} V_{10} + V_{1S} \alpha_L \]

with

\[ N_I = \alpha_L + \frac{1}{2} \left( \hat{V}_0^T \alpha_L - D_{al}^T V_{i0} \right) \]

\[ \tilde{D}_S = (D_{al} + D_{al}^T) \hat{V}_0 + D_{al} \]

\[ \tilde{D}_{ST} = (D_{al} + D_{al}^T) V_{i0} - \alpha_L \]

4.3 Dimensional Reduction of Multiphysics Load Case 2

For this case, we need to change some operators used in multiphysics load case 1 since we do not introduce any 1D variables in this case

1. The last two columns of operator \( \hat{\Gamma}_e \) in Eq. (4.21) should be deleted;

2. The last two rows of operator \( \hat{\Gamma}_c \) in Eq. (4.25) are zeros;

3. The 1D strains are defined as \( \tilde{\epsilon} = [\tilde{\gamma}_{11} \ \tilde{\kappa}_1 \ \tilde{\kappa}_2 \ \tilde{\kappa}_3]^T \) while the other operators used in this case are the same as those used in multiphysics load case 1.
4.3.1 Classical Theory

To deal with applied electric potential or magnetic induction at some specific locations (multiphysics load case 2), we divide the total nodal values of the warping field into two parts such that

\[ V = V_k + V_{u0} \]  \hspace{1cm} (4.53)

where \( V_k \) is a known matrix holding the prescribed electric or magnetic potentials at specific points (nodes), and \( V_{u0} \) is an unknown matrix such that the electric or the magnetic potentials of those prescribed points (nodes) are zeroes. Substituting Eq. (4.53) into Eq. (4.31), we rewrite the zeroth-order energy as

\[ 2\mathcal{U}^*_M = V_{u0}^T EV_{u0} + 2V_{u0}^T EV_k + 2V_{u0}^T D_{ae} \bar{\epsilon} + 2V_k^T D_{ae} \bar{\epsilon} + \bar{\epsilon}^T D_{ee} \bar{\epsilon} - 2V_{u0}^T \alpha_a - 2\bar{\epsilon}^T \alpha \epsilon \]  \hspace{1cm} (4.54)

Here the quadratic terms with known potential \( V_k^T EV_k \) is dropped because it will not affect the solution. For the electric and magnetic field, we can solve the warping function using the standard method introducing prescribed displacements in the conventional displacement-based finite element method. Repeating the solution procedure for multiphysics load case 1, we obtain the solution of the warping function as

\[ V_{u0} = \hat{V}_0 \bar{\epsilon} + V_m + V_{l0} \]  \hspace{1cm} (4.55)

Substituting Eq. (4.55) into Eq. (4.54), we obtain the zeroth-order approximation of the internal energy as

\[ 2\mathcal{U}_{M0} = \bar{\epsilon}^T \left( \hat{V}_0^T D_{ae} + D_{ee} \right) \bar{\epsilon} + \bar{\epsilon}^T \left[ \hat{V}_0 EV_k + D_{ae}^T (V_m + V_{l0}) + 2D_{ae}^T V_k - \hat{V}_0^T \alpha_a - 2\bar{\epsilon} \right] \]  \hspace{1cm} (4.56)
This energy can be written explicitly as

\[ 2\mathcal{U}_{M0} = \begin{bmatrix} \gamma_{11} \\ \tilde{\kappa}_1 \\ \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{bmatrix}^T \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} & \tilde{S}_{13} & \tilde{S}_{14} \\ \tilde{S}_{12} & \tilde{S}_{22} & \tilde{S}_{23} & \tilde{S}_{24} \\ \tilde{S}_{13} & \tilde{S}_{23} & \tilde{S}_{33} & \tilde{S}_{34} \\ \tilde{S}_{14} & \tilde{S}_{24} & \tilde{S}_{34} & \tilde{S}_{44} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \tilde{\kappa}_1 \\ \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{bmatrix} - 2 \begin{bmatrix} \gamma_{11} \\ \tilde{\kappa}_1 \\ \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{bmatrix}^T \begin{bmatrix} f_1^a \\ m_1^q \\ m_2^q \\ m_3^q \end{bmatrix} \] (4.57)

If we define the 1D generalized force resultants conjugate to \( \bar{\epsilon} \), such that

\[ F = \frac{\partial \mathcal{U}_{M0}}{\partial \bar{\epsilon}} \] (4.58)

then we can obtain an 1D constitutive model for the composite beam analysis as

\[ \begin{bmatrix} F_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} & \tilde{S}_{13} & \tilde{S}_{14} \\ \tilde{S}_{12} & \tilde{S}_{22} & \tilde{S}_{23} & \tilde{S}_{24} \\ \tilde{S}_{13} & \tilde{S}_{23} & \tilde{S}_{33} & \tilde{S}_{34} \\ \tilde{S}_{14} & \tilde{S}_{24} & \tilde{S}_{34} & \tilde{S}_{44} \end{bmatrix} \begin{bmatrix} \bar{\gamma}_{11} \\ \tilde{\kappa}_1 \\ \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{bmatrix} - \begin{bmatrix} f_1^a \\ m_1^q \\ m_2^q \\ m_3^q \end{bmatrix} \] (4.59)

Here the stiffness matrix \( S_{II} \) obtained as

\[ S_{II} = \tilde{V}_0^T D_{ae} + D_{e\epsilon} \] (4.60)

Another vector caused by the applied electric or magnetic and thermal loads \( N_{II} \) is:

\[ N_{II} = \alpha_{\epsilon} - D_{ae}^T V_k + \frac{1}{2} \left[ \tilde{V}_0^T \alpha_a - \tilde{V}_0 EV_k - D_{ae}^T (V_m + V_{t0}) \right] \] (4.61)

### 4.3.2 Refined Theory

For multiphysics load case 2, we perturb the warping function as

\[ V = \tilde{V}_0 \bar{\epsilon} + V_m + V_{t0} + V_1 \] (4.62)
Repeating the procedure for multiphysics load case 1, we have the following governing functional up to $O(\nu^2h^2)$

$$2\mathcal{U}_{M1}=\epsilon^T \left( \hat{V}_0^T D_{ae} + D_{e\epsilon} \right) \epsilon - 2\epsilon^T \left[ \alpha_e - D_{ae}^T V_k + \frac{1}{2} \left[ \hat{V}_0^T \alpha_a - \hat{V}_0 E V_k - D_{ae}^T (V_m + V_{t0}) \right] \right]$$

$$+ 2V_{a0}^T (D_{aR} + D_{aR}^T) V_k + 2V_{a0}^T D_{aR} V_{a0} + 2 (V_k + V_{a0})^T D_{al} V_{a0} + 2 (V_k + V_{a0})^T D_{Re} \epsilon$$

$$+ 2V_{a0}^T D_{le} \epsilon + V_1^T E V_1 + 2 (V_k + V_{a0})^T (D_{aR} + D_{aR}^T) V_1 + 2 (V_k + V_{a0}) D_{al} V_1' + 2V_1^T D_{al} V_{a0}$$

$$+ 2V_k^T D_{RR} V_{a0} + V_{a0}^T D_{RR} V_{a0} + V_{a0}^T D_{il} V_{a0} + 2V_1^T D_{Re} \epsilon + 2V_1^T D_{le} \epsilon$$

$$+ 2 (V_k + V_{a0})^T D_{Ri} V_{a0} - 2 \left[ (V_1 + V_{a0})^T \alpha_R + (V_1 + V_{a0})^T \alpha_l \right]$$

(4.63)

The leading terms for the first-order energy of multiphysics load case 2 are

$$2\mathcal{U}_{M1}^* = V_1^T E V_1 + 2V_1^T D_{Ri} \epsilon + 2V_1^T D_{S} \epsilon' + 2V_1^T \left( D_{RT} + D_{ST} + D_{RM} + D_{SM}' \right)$$

(4.64)

where $D_R$, $D_S$, $D_{RT}$ and $D_{ST}$ have the same expression in Eq. (4.45) and the newly introduced matrix $D_{RM}$ is

$$D_{RM} = (D_{aR}^T + D_{aR}) (V_k + V_m)$$

(4.65)

$$D_{SM}' = (D_{al} + D_{al}^T) V_m'$$

(4.66)

It is noted here that since we solve $V_m$ from the equation $E V_m = E V_k$ in classical modeling, the derivative of $V_m$ could be obtained by equation $E V_m' = E V_k'$. As pointed in Ref. [81], the actual representation of $V_m'$ can be only obtained asymptotically.

The first-order warping function of multiphysics load case 2 could be solved follow the same procedure as we did before

$$V_1 = V_{1R} \epsilon + V_{1S} \epsilon' + V_{1T} + V_{1M}$$

(4.67)
Using Eq. (4.67), the internal energy of refined model for multiphysics load case 2 can be expressed as

\[
2\mathcal{U}_{M1} = \tilde{\epsilon}^T A\tilde{\epsilon} + 2\tilde{\epsilon}^T B\tilde{\epsilon}' + \tilde{\epsilon}'^T C\tilde{\epsilon}' + 2\tilde{\epsilon}^T D\tilde{\epsilon}'' - 2\tilde{\epsilon}^T F_{a1} - 2\tilde{\epsilon}'^T F_{a2} - 2\tilde{\epsilon}''^T F_{a3} \quad (4.68)
\]

In the above enthalpy, the expression of matrices \(A, B, C\) and \(D\) are the same as in Eq. (4.51). \(F_{a1}, F_{a2}, \text{and } F_{a3}\) are

\[
F_{a1} = N_{II} - \left( D_{R}^T + V_{0}^T D_{RR} \right) (V_k + V_m + V_{t0}) - \left( \dot{V}_{0}^T D_{al} + D_{\alpha}^T + \dot{V}_{0}^T D_{Rl} \right) V'_{t0}
\]

\[
- \frac{1}{2} D_{R}^T (V_{1T} + V_{1M}) - \frac{1}{2} V_{1R}^T (D_{RT} + \tilde{D}_{ST} + D_{RM}) + \dot{V}_{0}^T \alpha_{R} - \dot{V}_{0} D_{al} V'_{1T} - D_{\alpha}^T V'_{1T}
\]

\[
F_{a2} = \left( \dot{V}_{0}^T + \dot{V}_{1R}^T \right) \alpha_{I} - \left( \dot{V}_{0}^T D_{al} + \dot{V}_{1R}^T D_{al} + \dot{V}_{0}^T D_{Rl} \right) (V_k + V_m + V_{t0}) - \frac{1}{2} \tilde{D}_{S}^T (V_{1T} + V_{1M})
\]

\[
- \dot{V}_{0}^T D_{Rl} V'_{T} - \frac{1}{2} V_{1S}^T (D_{RT} + \tilde{D}_{ST} + D_{RM}) + \left( \dot{V}_{0} D_{al} + D_{\alpha}^T + \dot{V}_{0}^T D_{Rl} \right) V_m
\]

\[
+ \frac{1}{2} (D_{R}^T V_{1M} + V_{1R}^T D_{SM}) + (\dot{V}_{0}^T D_{hl} + D_{\alpha}^T) V_{RM}
\]

\[
F_{a3} = - V_{1S}^T D_{al}^T (V_k + V_m + V_{t0}) + V_{1S}^T \alpha_{I} + \frac{1}{2} \left[ \dot{V}_{0}^T (D_{al}^T - D_{al}) - D_{\alpha}^T \right] V_{1M}
\]

\[
+ \frac{1}{2} V_{1S}^T \tilde{D}_{SM} + \dot{V}_{0}^T D_{hl} V_m
\]

where

\[
\tilde{D}_{SM} = (D_{al} + D_{al}^T) V_m \quad (4.70)
\]

### 4.4 Transformation to Generalized Timoshenko Model

Although Eq. (4.50) and Eq. (4.68) are asymptotically correct through the second order, they are not convenient for engineering applications because they involve derivatives of the 1D generalized strains. To get rid of these, we can transform this asymptotically correct energy expression to a generalized Timoshenko model following the equilibrium-equation approach [74]. The strain energy of the generalized Timoshenko model can be written as

\[
2\mathcal{U}_{MT} = \epsilon^T X \epsilon + 2\epsilon^T Y \gamma_s + \gamma_s^T G \gamma_s - 2\epsilon^T F_{\epsilon} - 2\gamma_s^T F_{\gamma} \quad (4.71)
\]
As shown by Yu et al. in Ref. [72], the kinematic relationships between the strain measures are

\[ \epsilon = \hat{\epsilon} + Q^* \gamma_s' + P^* \gamma_s \] (4.72)

where

\[ Q^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad P^* = \begin{bmatrix} 0 & 0 \\ k_2 & k_3 \\ -k_1 & 0 \\ 0 & -k_1 \end{bmatrix} \] (4.73)

Note that for multiphysics load case 2, the last two rows of \( P \) and \( Q \) need to be deleted.

The key to the energy transformation is to find expressions for \( \hat{\epsilon} \) (strain measures associated with triad \( T_i \)), \( \epsilon' \) and \( \epsilon'' \) in terms of \( \epsilon \) and \( \gamma_s \) (strain measures associated with triad \( B_i \)). Following the procedure in Ho et al. [76], we can finally solve the \( X, Y, G, F_\epsilon \) and \( F_\gamma \), which are expressed in \( A, B, C, D, F_{a1} \) and \( F_{a2} \), up to the second order. The details of this transformation can be found in Appendix A. The 1D constitutive relations for a generalized Timoshenko model can be written as

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
M_1 \\
M_2 \\
M_3 \\
\bar{F}_E \\
\bar{F}_H
\end{bmatrix} =
\begin{bmatrix}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & \bar{e}_{11} & \bar{q}_{11} \\
s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} & \bar{e}_{12} & \bar{q}_{12} \\
s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} & \bar{e}_{13} & \bar{q}_{13} \\
s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} & \bar{e}_{14} & \bar{q}_{14} \\
s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} & \bar{e}_{15} & \bar{q}_{15} \\
s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} & \bar{e}_{16} & \bar{q}_{16} \\
\bar{e}_{11} & \bar{e}_{12} & \bar{e}_{13} & \bar{e}_{14} & \bar{e}_{15} & \bar{e}_{16} & k_{17} & \bar{a}_{17} \\
\bar{q}_{11} & \bar{q}_{12} & \bar{q}_{13} & \bar{q}_{14} & \bar{q}_{15} & \bar{q}_{16} & \bar{a}_{17} & \bar{\mu}_{88}
\end{bmatrix}
\begin{bmatrix}
\gamma_{11} \\
\gamma_{12} \\
\gamma_{13} \\
\gamma_{14} \\
\gamma_{15} \\
\gamma_{16} \\
\kappa_1 \\
\kappa_2 \\
\kappa_3 \\
E_{1D}
\end{bmatrix}
= 
\begin{bmatrix}
f_{1a} \\
f_{2a} \\
f_{3a} \\
m_{1a} \\
m_{2a} \\
m_{3a} \\
E_{aE} \\
H_{1a}
\end{bmatrix}
\] (4.74)
for multiphysics load case 1 and case 2, respectively.

4.5 Validation of Multiphysics Model

The above developed theory for dimensional reduction of the multiphysics problems for composite beams has been implemented into the computer program VABS. In this section, we will use VABS to predict the effective stiffness of different composite beams as well as the recovered 3D fields such as displacement and stress distribution. Some of the results will be compared with those obtained by 3D finite element analysis in the commercial software ANSYS and the analytical solution available in the literature. Materials used in the following examples are magnetostrictive $\text{CoFe}_2\text{O}_4$, piezoelectric PZT-4, PZT5H, and aluminum. Material properties are listed in Table 4.1. It is pointed out that the values of magnetic permeabilities of $\text{CoFe}_2\text{O}_4$ in this table are from Ref. [98] where the authors declared that it should be positive for a stable material in physics although negative values are used in many other references.

4.5.1 Example 1: Two-layer Beam Under Multiphysics Load Case 1

The first example is a 1.0 m long cantilever beam of rectangular cross-section composed of an aluminum layer bounded to a PZT-4 layer. The width of the cross-section is 0.1 m and the thickness of each layer is 0.05 m, see Fig. 4.3. It should be noted that the material properties of PZT-4 given in Table 4.1 are polarized along the thickness direction of the beam ($x_3$). If the piezoelectric material is poled along beam span ($x_1$), the constitutive equations can be obtained from a 90 degree rotation around $x_2$ and then followed by a 180
Table 4.1: Material Properties in Multiphysics Validation Examples

<table>
<thead>
<tr>
<th>Properties</th>
<th>Aluminium</th>
<th>PZT-4</th>
<th>CoFe$_2$O$_4$</th>
<th>PZT5H</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{11}$ (GPa)</td>
<td>82.68</td>
<td>139.02</td>
<td>286.0</td>
<td>126.0</td>
</tr>
<tr>
<td>$C_{12}$ (GPa)</td>
<td>27.56</td>
<td>77.85</td>
<td>173.0</td>
<td>79.5</td>
</tr>
<tr>
<td>$C_{13}$ (GPa)</td>
<td>27.56</td>
<td>74.33</td>
<td>170.5</td>
<td>84.1</td>
</tr>
<tr>
<td>$C_{22}$ (GPa)</td>
<td>82.68</td>
<td>139.02</td>
<td>286.0</td>
<td>126.0</td>
</tr>
<tr>
<td>$C_{23}$ (GPa)</td>
<td>27.56</td>
<td>74.33</td>
<td>170.5</td>
<td>84.1</td>
</tr>
<tr>
<td>$C_{33}$ (GPa)</td>
<td>82.68</td>
<td>115.45</td>
<td>269.5</td>
<td>126.0</td>
</tr>
<tr>
<td>$C_{44}$ (GPa)</td>
<td>27.56</td>
<td>25.6</td>
<td>45.3</td>
<td>23.3</td>
</tr>
<tr>
<td>$C_{55}$ (GPa)</td>
<td>27.56</td>
<td>25.6</td>
<td>45.3</td>
<td>23.0</td>
</tr>
<tr>
<td>$C_{66}$ (GPa)</td>
<td>27.56</td>
<td>30.6</td>
<td>56.5</td>
<td>23.0</td>
</tr>
</tbody>
</table>

degree rotation around the thickness direction $x_3$ [99]. For this example, the poling direction of the piezoelectric material is assumed to be in $x_1$ direction. The fixed surface is applied with 0 V and the free surface is applied with 500 V. 124,000 SOLID5 coupled brick elements are used in ANSYS 3D analysis while only 200 8-noded quadrilateral elements are used in VABS analysis. The non-zero cross-sectional stiffness properties are listed in Table 4.2. To verify the present model, we compare the displacements along beam span and stress distributions at mid-span ($x_1 = 0.5$ m) recovered by VABS with that obtained from 3D ANSYS multiphysics simulation. Fig. 4.4 compares the transverse centroidal displacements between VABS and ANSYS. The distribution of non-zero stress components at mid-span are plotted in Figs. 4.5-4.7. As one can observe from the above results that an excellent match is found between the present model and 3D results.
Fig. 4.3: Beam sketch of multiphysics Example 1.

Table 4.2: Cross-sectional Constants of Two-layer Smart Beam Under Multiphysics Load Case 1

<table>
<thead>
<tr>
<th>Stiffness</th>
<th>VABS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{11}$ (N)</td>
<td>$6.673 \times 10^8$</td>
</tr>
<tr>
<td>$s_{15}$ (N)</td>
<td>$-5.507 \times 10^9$</td>
</tr>
<tr>
<td>$s_{22}$ (N)</td>
<td>$2.310 \times 10^8$</td>
</tr>
<tr>
<td>$s_{24}$ (N)</td>
<td>$-5.232 \times 10^9$</td>
</tr>
<tr>
<td>$s_{33}$ (N)</td>
<td>$2.713 \times 10^9$</td>
</tr>
<tr>
<td>$s_{44}$ (N m$^2$)</td>
<td>$4.476 \times 10^6$</td>
</tr>
<tr>
<td>$s_{55}$ (N m$^2$)</td>
<td>$5.559 \times 10^9$</td>
</tr>
<tr>
<td>$s_{66}$ (N m$^2$)</td>
<td>$5.560 \times 10^9$</td>
</tr>
<tr>
<td>$\bar{e}_{11}$ (C)</td>
<td>$-9.315 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\bar{e}_{15}$ (C)</td>
<td>$-2.331 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\bar{k}_{77}$ (C m/N)</td>
<td>$-3.111 \times 10^{-11}$</td>
</tr>
<tr>
<td>$\mu_{88}$ (N·m$^2$/N)</td>
<td>$-5.000 \times 10^{-8}$</td>
</tr>
</tbody>
</table>
Fig. 4.4: Transverse deflection along beam span for multiphysics Example 1.

Fig. 4.5: Distribution of $\sigma_{11}$ along thickness at $x_2 = 0$ for multiphysics Example 1.
Fig. 4.6: Distribution of $\sigma_{22}$ along thickness at $x_2 = 0$ for multiphysics Example 1.

Fig. 4.7: Distribution of $\sigma_{33}$ along thickness at $x_2 = 0$ for multiphysics Example 1.
4.5.2 Example 2: Three-layer Beam Under Multiphysics Load Case 1

In the second example, we investigate two 1.0 m long three-layer beams made of piezoelectric PZT-4 and magnetostrictive CoFe$_2$O$_4$. The three layers have equal thickness of 0.03 m (with a total thickness 0.09 m) and the width of the cross-section is 0.1 m. The stack sequences take the form of PZT-4/CoFe$_2$O$_4$/PZT-4 (called P/C/P), and CoFe$_2$O$_4$/PZT-4/CoFe$_2$O$_4$ (called C/P/C), respectively. Fig. 4.8 shows these two cross-sections used in this example. The cross-section is discretized in 360 8-noded quadrilateral elements. The load and boundary condition applied to the beams are the same as that in multiphysics Example 1. The polarization direction of this beam is parallel to the beam reference line. The non-zero stress components of mid-span ($x_1 = 0.5$ m) are plotted in Figs. 4.9-4.11. There are no 3D finite element solutions in these figures (also in Figs. 4.17 and 4.18) due to the fact that the current version ANSYS 13.0 does not have a suitable element that can capture the electric-magnetic-mechanical coupling behavior. It is observed that the stress distributions for these two configurations are dramatically different from each other due to the different stacking sequence. The curves are not symmetric because of the difference between the elastic constants of PZT-4 and CoFe$_2$O$_4$ materials.

Fig. 4.8: Beam cross-sections of multiphysics Example 2.
Fig. 4.9: Distributions of $\sigma_{11}$ along thickness at $x_2 = 0$ for multiphysics Example 2.

Fig. 4.10: Distributions of $\sigma_{22}$ along thickness at $x_2 = 0$ for multiphysics Example 2.
Fig. 4.11: Distributions of $\sigma_{33}$ along thickness at $x_2 = 0$ for multiphysics Example 2.

4.5.3 Example 3: Two-layer Beam Under Multiphysics Load Case 2

The next two examples will investigate the present model for multiphysics load case 2. For this example, we use the same cantilever beam that we used in multiphysics Example 1. The top surface of the PZT-4 layer is prescribed to be 500 V and the interface between the piezoelectric layer and aluminum layer is grounded. The piezoelectric material is polarized along the thickness direction and the applied electric field is parallel to the polarization. We mesh this cross section with 200 8-noded quadrilateral elements (10 elements along the width and the thickness of each layer). To check the accuracy of the present model against 3D finite element analysis, we construct a slender structure of the same geometry in ANSYS. This model uses a total of 14,400 8-noded coupled brick elements. Fig. 4.12 compares the transverse centroidal displacements between VABS and ANSYS. Fig. 4.13 plots the voltage distribution along the thickness of the structures. The non-zero stress components $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{33}$ at mid-span along thickness are plotted in Figs. 4.14, 4.15, and 4.16, respectively. Excellent agreement between VABS and ANSYS can be observed for all these quantities.
Fig. 4.12: Transverse deflection along beam span for multiphysics Example 3.

Fig. 4.13: Voltage distribution along thickness for multiphysics Example 3.
Fig. 4.14: Distribution of $\sigma_{11}$ along thickness at $x_2 = 0$ for multiphysics Example 3.

Fig. 4.15: Distribution of $\sigma_{22}$ along thickness at $x_2 = 0$ for multiphysics Example 3.
Fig. 4.16: Distribution of $\sigma_{33}$ along thickness at $x_2 = 0$ for multiphysics Example 3.

4.5.4 Example 4: Three-layer Beam Under Multiphysics Load Case 2

Now, we study the cantilever beam that we used in Example 2 with stacking sequence of P/C/P. The interfaces between PZT-4 and CoFe$_2$O$_4$ are grounded and the magnetic potential is prescribed as 0 C/s. The top and bottom surfaces are prescribed to be 500 V. Both piezoelectric and piezomagnetic materials are polarized along thickness direction. Since there is not available results in literature known to the author, here we only plot the results obtained by VABS. Also we do not plot the stress distributions here because the curves are similar to those of P/C/P in Figs. 4.9-4.11 although the values are different. Figs. 4.17 and 4.18 show the variations of the electric and magnetic potentials along the thickness direction of the beam. The magnitude of electric (magnetic) potential is zero in magnetostrictive CoFe$_2$O$_4$ (PZT-4) layer due to the fact that for the CoFe$_2$O$_4$ (PZT-4) material, the piezoelectric coefficients $e_{ij}$ (piezomagnetic $q_{ij}$) are zero. It is also observed that the distribution of electric potential in the PZT-4 layer is linear while the distribution of induced magnetic potential in the mid layer is not linear because of the coupling effects.
Fig. 4.17: Distribution of electric potential along thickness for multiphysics Example 4.

Fig. 4.18: Distribution of magnetic potential along thickness for multiphysics Example 4.
4.5.5 Example 5: Shear Actuation

The shear actuation of adaptive sandwich structure has been proposed by Sun et al. [100, 101]. In this case, the piezoelectric material is polarized along the axial $x_1$ direction while the applied electric field is perpendicular to the polarization direction, see Fig. 4.19. An example of a sandwich beam given in Ref. [101] has been taken, where top and bottom layers are made of aluminium and a PZT5H layer is sandwiched in the middle. Table 4.3 compares the VABS result with 3D finite element result obtained by ANSYS and analytical solution [101]. 219,600 SOLID5 brick elements are used in the 3D analysis. It can be seen that there is a good correlation between results obtained by these three methods.

4.5.6 Example 6: Thermal Effects in Multiphysics Analysis

This example is to study the effects of thermal load applied to the structure in multiphysics analysis. A cantilevered smart beam of configuration $[0_8/p]$ is presented, see Fig. 4.20, where $p$ is the piezoelectric layer. The beam is $L = 25.4$ cm long and $b = 2.54$ cm wide, and consists of eight graphite/epoxy layers and a piezoelectric layer where each layer

<table>
<thead>
<tr>
<th>Displacement $u_3$(m)</th>
<th>3D FEA</th>
<th>Analytical</th>
<th>VABS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1.1691 \times 10^{-7}$</td>
<td>$1.1961 \times 10^{-7}$</td>
<td>$1.20724 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
has a thickness of $t = 0.0127$ cm. Representative material properties of a graphite/epoxy composite are used in this case as $E_{11} = 39$ GPa, $E_{22} = E_{33} = 8.6$ GPa, $G_{12} = G_{13} = 3.8$ GPa, $G_{23} = 3.07$ GPa, $\nu_{12} = \nu_{13} = 0.28$, $\nu_{23} = 0.4$, $\alpha_{11} = 7.0 \times 10^{-6}/^\circ\text{C}$, $\alpha_{22} = \alpha_{33} = 21.0 \times 10^{-6}/^\circ\text{C}$. The piezoelectric layer is made of PZT-4 material and the properties are listed in Table 4.1. The thermal properties for PZT-4 are $\alpha_{11} = \alpha_{22} = 3.8 \times 10^{-6}/^\circ\text{C}$, $\alpha_{33} = 1.2 \times 10^{-6}/^\circ\text{C}$, and $p_1 = p_2 = 0$, $p_3 = -1.7 \times 10^{-4}$ C/m$^2$·K where $p_i$ are pyroelectric constants. A uniform temperature load of 100$^\circ$C is applied to this beam. The interface between piezoelectric layer and graphite/epoxy layers is grounded and three different electric potentials, 0 V, 100 V, and 300 V, are applied to the top surface of the piezoelectric layer. Fig. 4.21 shows the transverse displacements caused by thermal and electric loads. These results indicate that the thermally induced deformation can be compensated by piezoelectric actuators when electric loads apply to it. When the piezoelectric layer works in the sensory mode, the deformation of structure can be monitored by the electric quantities generated in the layer. Fig. 4.22 shows the electric displacements detected in the piezoelectric layer under different loads. Firstly, the top surface of the PZT-4 layer is prescribed with a 100 V electric potential and the interface is grounded. The curve and data points labeled “Electric Load” are calculated for this load case from ANSYS 3D coupling analysis and VABS, respectively. Excellent agreement can be observed. Next, a uniform temperature change of 100$^\circ$C is applied to this beam in addition to the electric load. The curve labeled “Electric-Thermal Load” shows the electric displacement in the PZT-4 layer calculated by VABS. Finally, the
Fig. 4.21: Transverse displacement of a smart beam under thermal-electric loads.

electric displacement considering pyroelectric effect under the same electro-thermal load is also plotted in this figure with the label of “Pyroelectric.”
Fig. 4.22: Distribution of electric displacement in the piezoelectric layer for multiphysics Example 6.
Chapter 5
Conclusions and Recommendations

The current research presents new beam models for thermoelastic and multiphysics analysis based on the framework of the variational-asymptotic method. Its focus is on the 2D cross-sectional analysis aspect of beam theory. To be more specific, it focuses on the cross-sectional analysis of composite beams under multiphysics environment including thermal, electric, magnetic loads in addition to the traditional mechanical load. This work is an extension of previous research conducted on composite beam structures. This chapter reviews the main accomplishments and lists recommendations for future work.

5.1 Accomplishments

The theory for the cross-sectional analysis of beams based on VAM is extended to incorporate thermoelastic analysis and multiphysics. The quasisteady theory of linear thermoelasticity, which neglects the temperature changes due to deformations, is adopted to avoid the fully-coupled thermoelasticity problem. Under this situation, the thermal problem separates into two problem to be solved consecutively: the heat conduction problem to solve for the thermal field and the one-way coupled thermoelastic problem for the structure under a prescribed thermal field.

A heat conduction beam model is constructed first to obtained the thermal field, and then this model is refined to handle the beams with initial twist and curvatures. Two load cases of the prescribed temperature fields have been treated. A 1D heat conduction analysis exists only if the temperature is not prescribed at any point of the cross-section along the span except the end surfaces (thermal load case 1). The recovery relation of thermal field has been derived. A discussion shows that the current model is also able to handle convection heat transfer problem. Using the solved thermal field as input loads, a
composite beam model for thermoelastic analysis is developed. The stiffness constants of classical model obtained from a Euler-Bernoulli beam model are suitable for long beams. To deal with moderate to thick beams, a generalized Timoshenko beam model has been proposed by introducing transverse shear strains as additional DOFs. This refined model also accounts for the effects due to initial twist and curvatures.

Then the VAM is applied to the multiphysics beam analysis. Starting from the energy functionals that govern the elastic behavior of the 3D solids, the variational-asymptotic method is applied to (1) rigorously decouple the original 3D multiphysics elasticity problem into a global 1D analysis and a 2D cross-sectional, (2) state the 2D cross-sectional analysis as a constrained minimization problem, and (3) solve the resulting constrained minimization problem. This results in obtaining an asymptotically correct energy expression. However, it is not easy to use this model directly due to the derivatives of the generalized classical strain measures in the expression. Making use of the equilibrium equations, these derivative terms are replaced with new variables, and the original energy functional is transformed into a generalized Timoshenko model which is easy for practical use. A detailed description of the units used in the multiphysics modeling are provided. In order to avoid ill-conditioned matrix, a scaling method is also presented.

Two other issues regarding thermal analysis are included in the current VABS model. One issue is the thermoelastic beam modeling under large temperature changes. The previous thermoelastic beam model is extended for composite materials which removed the restriction on temperature variations and added the dependence of material properties with respect to temperature based on the Kovalenoko’s small-strain thermoelasticity theory. Another issue is that the couplings between thermal field and electric and magnetic fields, the pyroelectric and pyromagnetic effects, in the multiphysics model are also taken into consideration. Two numerical examples illustrate these issues clearly.

All these newly developed beam models are numerically implemented by using finite element method. VABS now is capable of handling thermal and multiphysics problems of composite beams composed of arbitrary materials and geometries. The recovery of 3D field
quantities in terms of 1D variables has been derived so that the cross-sectional distributions of displacements, strains, stresses, and electric and magnetic field quantities can be obtained. These results obtained by the current beam models have been extensively compared with those obtained by 3D analysis, analytical solutions, and those available in the literature. In comparison with other methods, the present models provide an efficient and rigorous means to design and analyze thermal and multiphysics problems of composite beams without significant loss of accuracy.

5.2 Recommendations for Future Work

As stated already, a beam analysis includes a 2D cross-sectional analysis and a 1D global analysis. The focus of the current research is on the 2D cross-sectional analysis of beam structures, so the need for a general 1D beam solver which is capable of handling thermal and multiphysics effects persists. The global behavior of the beams like the axial distributions of displacements, rotation, moment, and stress resultants plays a more important role than the cross-sectional results in some situations, for example, in the aeroelastic analysis. Moreover, the recovery of cross-sectional results also needs information from 1D analysis including 1D strains and displacements. The validation examples in this dissertation only require simply 1D beam analysis like a cantilever beam so that the problem can be solved analytically. However, for beam problems with complicated boundary conditions and load cases, for example a beam with joints and convection in heat conduction analysis, one needs to rely on a general 1D solver to obtain the correct solutions. The author suggests that two more modules, one for 1D thermal analysis and the other for multiphysics analysis, can be developed and implemented into a recently developed powerful 1D beam solver GEBT (Geometrically Exact Beam Theory) [102,103].

Another recommendation for future work is the beam modeling with spanwise nonuniformity. The present beam model is only valid for beams with uniform cross-sections. However, most of the active materials applications in modern aerospace structures are distributed systems. For example, the piezoelectric patches are more likely used as actuators and sensors than a full layer made of piezoelectric materials. In addition, control algorithms
for smart structures can be developed in the future.

Finally, the frame of current work is linear elasticity in the dimensional reduction. This author would recommend to develop models based on material nonlinearity in the 2D cross-sectional analysis. Moreover, some damage effects such as delamination at the interface of two layers which could lead to degeneration of stiffness should be taken into consideration in the beam modeling.
References


Appendices
Appendix A

Transformation to Generalized Timoshenko Model

The strain energy including the load-related bilinear terms of the generalized Timoshenko model can be written as

\[ 2\Pi = \epsilon^T X \epsilon + 2\epsilon^T Y \gamma_s + \gamma_s^T G \gamma_s - 2\epsilon^T F \epsilon - 2\gamma_s^T F \gamma \]  

(A.1)

To ensure that the generalized Timoshenko model in Eq. (A.1) represents the original asymptotical correct model in Eq. (3.71) as accurately as possible, we must make use of all the known information between these two models. From Eq. (2.32), we know that the classical strain measures (\( \bar{\epsilon} \)) can be expressed in terms of the Timoshenko strain measures (\( \epsilon \) and \( \gamma_s \)), which implies that we can express Eq. (3.71) in terms of Timoshenko strain measures and their derivatives. To arrive at the functional form in Eq. (A.1), we also need to express the Timoshenko strain derivatives in terms of the strains themselves. To simplify the procedure for obtaining these relations, let us assume that the quadratic terms, and the bilinear terms in the asymptotic energy in Eq. (3.71) can be packed into the quadratic terms, and bilinear terms of the Timoshenko energy form in Eq. (A.1), respectively. We can use the equilibrium equations to achieve derive the relations between strains and their derivatives. As we will show later, the nonlinear 1D equilibrium equations of the Timoshenko beam model for initially curved and twisted beams without distributed forces, both applied and inertial (the loads are neglected because our purpose here is to repack the quadratic asymptotically correct energy into the counterparts of the Timoshenko form), can be written as

\[ F' + \tilde{K} F = 0 \]  
\[ M' + \tilde{K} M + (\tilde{e}_1 + \tilde{\gamma}) F = 0 \]  

(A.2)

Here, \( F \) is the column matrix of the cross-sectional stress resultant vector in the \( B_x \) basis,
$M$ is the column matrix of the cross-sectional moment resultant vector in the $B_i$ basis. In our asymptotic analysis, terms of order $\mu \varepsilon^3$ and $\mu \varepsilon^2 \tilde{h}^3$ are neglected in the strain energy which leads to the estimation $\varepsilon = O(\tilde{h}^3)$. Thus, the nonlinear terms in the equilibrium equations will not affect the strain energy which is only asymptotically correct up to the second order of $\tilde{h}$. Only linear equations are useful for the present purpose of creating a generalized Timoshenko model. For multiphysics load case 1, where two degree of freedom are introduced for electric and magnetic fields, two more 1D electro-magneto-mechanical equilibrium equations can be derived as

$$\bar{F}_E' = 0$$
$$\bar{F}_H' = 0$$

Here $\bar{F}_E$ and $\bar{F}_H$ come as conjugates to the 1D electric and magnetic degree of freedom in the 1D variational statement. The explicit definitions of the cross-sectional stress resultants are given as

$$\frac{\partial U_{1D}}{\partial \gamma_{11}} = F_1$$
$$\frac{\partial U_{1D}}{\partial \gamma_{1\alpha}} = F_\alpha$$
$$\frac{\partial U_{1D}}{\partial \kappa_i} = M_i$$

$$-\frac{\partial U_{1D}}{\partial E_{1D}} = \bar{F}_E \quad \text{(For multiphysics load case 1)}$$
$$-\frac{\partial U_{1D}}{\partial H_{1D}} = \bar{F}_H \quad \text{(For multiphysics load case 1)}$$

Neglecting the nonlinear terms, Eq. (A.2) can be rewritten more explicitly as

$$\begin{pmatrix} F_2' \\ F_3' \end{pmatrix} + D_1 \begin{pmatrix} \bar{F}_2 \\ \bar{F}_3 \end{pmatrix} + D_2 \begin{pmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = 0$$

(A.5)
\[
\begin{align*}
\begin{bmatrix}
F'_1 \\ M'_1 \\ M'_2 \\ M'_3 
\end{bmatrix} + D_3 \begin{bmatrix}
F_1 \\ M_1 \\ M_2 \\ M_3 
\end{bmatrix} + D_4 \begin{bmatrix}
F_2 \\ F_3 
\end{bmatrix} &= 0 \\
\text{(A.6)}
\end{align*}
\]

where

\[
\begin{align*}
D_1 &= \begin{bmatrix}
0 & -k_1 \\
k_1 & 0 
\end{bmatrix} \\
D_2 &= \begin{bmatrix}
k_3 & 0 & 0 & 0 \\
-k_2 & 0 & 0 & 0 
\end{bmatrix} \\
D_3 &= \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -k_3 & k_2 \\
0 & k_3 & 0 & -k_1 \\
0 & -k_2 & k_1 & 0 
\end{bmatrix} \\
D_4 &= Q - D_2^T \\
\text{(A.7)}
\end{align*}
\]

If we just focus on the quadratic terms first, the forces and moments in Eqs. (A.5) and (A.6) are defined through the quadratic terms of the generalized Timoshenko model as conjugates to the generalized strains, that is, we have

\[
\begin{align*}
\begin{bmatrix}
F_1 \\ M_1 \\ M_2 \\ M_3 \\ F_2 \\ F_3 
\end{bmatrix} &= \begin{bmatrix}
X & Y \\
Y^T & G 
\end{bmatrix} \begin{bmatrix}
\epsilon \\ \gamma_s 
\end{bmatrix} \\
\text{(A.8)}
\end{align*}
\]

It is pointed out that for multiphysics load case 1, the dimensions of \(D_2, D_3, D_4\) matrices are different from those in Eq. (A.7) because of the introduction of two new degrees of
freedom. For this case, the explicit form of Eq. (A.2) together with Eq. (A.3) should be written as

\[
\begin{align*}
\begin{bmatrix} F_2' \\ F_3' \end{bmatrix} + D_1 \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} + D_2^* \begin{bmatrix} M_2 \\ M_3 \\ \tilde{F}_E \\ \tilde{F}_H \end{bmatrix} &= \begin{bmatrix} F_1 \\ M_1 \end{bmatrix} \\
\begin{bmatrix} F_1' \\ M_1' \\ M_2' \\ M_3' \\ \tilde{F}_E' \\ \tilde{F}_H' \end{bmatrix} + D_3^* \begin{bmatrix} F_2 \\ M_2 \\ M_3 \\ \tilde{F}_E \\ \tilde{F}_H \end{bmatrix} + D_4^* \begin{bmatrix} F_3 \\ M_3 \\ \tilde{F}_E \\ \tilde{F}_H \end{bmatrix} &= \begin{bmatrix} F_1 \\ M_1 \end{bmatrix}
\end{align*}
\]

(A.9)

where

\[
\begin{align*}
D_2^* &= \begin{bmatrix} k_3 & 0 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 & 0 \end{bmatrix} \\
D_3^* &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & k_2 & 0 & 0 \\ 0 & k_3 & 0 & -k_1 & 0 & 0 \\ 0 & -k_2 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
D_4^* &= Q^* - D_2^{*T}
\end{align*}
\]

(A.11)
Thus Eq. (A.8) for multiphysics load case 1 can be written as

\[
\begin{bmatrix}
F_1 \\
M_1 \\
M_2 \\
M_3 \\
\bar{F}_E \\
\bar{F}_H \\
F_2 \\
F_3
\end{bmatrix} =
\begin{bmatrix}
X^* & Y^*
\end{bmatrix}
\begin{bmatrix}
\epsilon \\
\gamma_s
\end{bmatrix}
\]

(A.12)

In the following derivation, we will use the same notations as those used in thermoelastic modeling (matrices without an asterisk) with the understanding that the dimensions of these matrices are different for multiphysics load case 1.

Using Eqs. (A.5), (A.6), and (A.8), one may express the derivatives of the strain measures as

\[
\begin{bmatrix}
\epsilon \\
\gamma_s
\end{bmatrix}' = -[Z] \begin{bmatrix}
\epsilon \\
\gamma_s
\end{bmatrix}
\]

(A.13)

where

\[
[Z] = \begin{bmatrix}
[Z]_{11} & [Z]_{12} \\
[Z]_{21} & [Z]_{22}
\end{bmatrix} =
\begin{bmatrix}
-N^{-1}A_4 & -N^{-1}A_3 \\
G^{-1}(Y^TN^{-1}A_4 + D_1 Y^T + D_2 X) & G^{-1}(Y^TN^{-1}A_3 + D_1 G + D_2 Y)
\end{bmatrix}
\]

(A.14)

along with

\[
A_3 = (YG^{-1}D_1 - D_4)G + (YG^{-1}D_2 - D_3)Y = -QG + \phi G + \psi Y
\]

(A.15)

\[
A_4 = A_3 G^{-1}Y^T + (YG^{-1}D_2 - D_3)N = A_3 G^{-1}Y^T + \psi N = -QY^T + \phi Y^T + \psi X
\]

(A.16)

\[
N = X - YG^{-1}Y^T
\]

(A.17)

and \(\phi = YG^{-1}D_1 + D_2^T\) and \(\psi = YG^{-1}D_2 - D_3\). \(\phi\) and \(\psi\) may be termed explicitly linear.
with respect to \( k_i \) if its dependence on unknown quantities are viewed as being implicit. This terminology for \textit{explicit} dependence will be repeated. Note, starting from Eq. (A.13), due to the complexity of derivation, notations and symbols used here are not related to the rest of the report except for self-evident exceptions. They are specifically used for the transformation into a generalized Timoshenko model.

From Eqs. (2.32) and (A.13), one may write

\[
\bar{\epsilon} = (\Delta_4 - Q[Z]_{21})\epsilon + (P - Q[Z]_{22})\gamma_s \equiv \alpha\epsilon + \beta\gamma_s \tag{A.18}
\]

Differentiating both sides of the above equation with respect to \( x_1 \) yields

\[
\bar{\epsilon}' = \alpha\epsilon' + \beta\gamma'_s = -(\alpha[Z]_{11} + \beta[Z]_{21})\epsilon - (\alpha[Z]_{12} + \beta[Z]_{22})\gamma_s \equiv \alpha'\epsilon + \beta'\gamma_s \tag{A.19}
\]

Note that \( \alpha' \) and \( \beta' \) are not derivatives of \( \alpha \) and \( \beta \), but are rather new symbols defined from the above equation. Similarly, one may derive

\[
\bar{\epsilon}'' = \alpha\epsilon'' + \beta\gamma''_s = (\alpha[Z]_{11} + \beta[Z]_{21})\epsilon + (\alpha[Z]_{12} + \beta[Z]_{22})\gamma_s \equiv \alpha''\epsilon + \beta''\gamma_s \tag{A.20}
\]

After some algebraic manipulations, the following useful identities are established:

\[
Z_{11} - Z_{12}G^{-1}Y^T = -N^{-1}\psi N \tag{A.21}
\]

\[
Z_{21} - Z_{22}G^{-1}Y^T = G^{-1}MN \tag{A.22}
\]

\[
Z_{11}N^{-1}\psi - Z_{12}G^{-1}M = -N^{-1}(QD_2 - \phi D_2 - \psi D_3) \tag{A.23}
\]

\[
Z_{21}N^{-1}\psi - Z_{22}G^{-1}M = G^{-1}[Y^TN^{-1}QD_2 - (LD_2 + MD_3)] \tag{A.24}
\]

with \( L = Y^TN^{-1}\phi + D_1 \) and \( M = Y^TN^{-1}\psi + D_2 \). \( L \) and \( M \) are \textit{explicitly} linear with respect to \( k_i \). These identities help to derive the following expressions:

\[
\alpha = \beta G^{-1}Y^T + W \quad \alpha' = \beta' G^{-1}Y^T + W' \quad \alpha'' = \beta'' G^{-1}Y^T + W'' \tag{A.25}
\]
where

\[ W = \Delta_4 - PG^{-1}Y^T - QG^{-1}MN \equiv \Delta_4 + W_1 \]  
\[ W' = (N^{-1}\psi - QG^{-1}Y^TN^{-1}QD_2)N + [QG^{-1}(LD_2 + MD_3) - PG^{-1}M] \equiv W_1' + W_2' \]  
\[ W'' = N^{-1}QD_2N + [(QG^{-1}Y^TN^{-1}QD_1 - N^{-1}\phi + W_1N^{-1}Q)D_2 - N^{-1}\psi D_3] \equiv W_1'' + W_2'' \]

\[ W_1, W_1', \text{and } W_1'' \] are explicitly linear with respect to \( k_i \). \( W_2' \) and \( W_2'' \) are explicitly quadratic with respect to \( k_i \). Some terms in \( W'' \) are explicitly cubic with respect to \( k_i \) and are purposely ignored as they will not affect the perturbation solution up to the second order of \( k_i \).

Substituting Eqs. (A.18)-(A.20) into the quadratic terms of Eq. (3.71) and equating with the quadratic terms of the generalized Timoshenko energy of Eq. (A.1), the following matrix equations are obtained:

\[ X = \alpha^T A\alpha + \alpha'^T Ca' + \alpha'^T (Ba' + Da'') + (\quad)_T \]  
\[ Y = \alpha^T A\beta + \alpha'^T C\beta' + \alpha'^T B\beta' + \alpha'^T B^T \beta + \alpha D\beta'' + \alpha''D^T \beta \]  
\[ G = \beta^T A\beta + \beta'^T C\beta' + \beta'^T (B\beta' + D\beta'') + (\quad)_T \]

Here \((\quad)_T\) is used to denote the transpose of the preceding underlined term. Using the relations in Eq. (A.25) and Eq. (A.31), one may rewrite Eq. (A.30) as

\[ 0 = W^T (A\beta + B\beta' + D\beta'') + W'^T (B^T \beta + C\beta') + W''^T D^T \beta \]

Using the above equation and the relations in Eq. (A.25), one may rewrite Eq. (A.29) as

\[ X - YG^{-1}Y^T = W^T AW + W'^T CW' + W'^T (BW' + DW'') + (BW' + DW'')^T W \]

Now the problem is to solve \( X, Y, \) and \( G \) from Eqs. (A.31)-(A.33), which in principal can
be solved numerically as pointed out in Ref. [76] but for most cases the numerical technique cannot find a converged solution. Hence, we propose to solve $X$, $Y$, and $G$ analytically by perturbation methods with accuracy up to second-order with respect to $k_i$.

### A.1 Solving $X$, $Y$, $G$ Using Perturbation Method

To avoid the difficulties of solving these equations numerically, an approximate solution is sought using the perturbation method with $k_i$ being the small perturbation parameter. For convenience, $\beta$, $\beta'$, and $\beta''$ are decomposed into orders as

$$\beta = \beta_0 + \beta_1 \quad \beta' = \beta'_0 + \beta'_1 + \beta'_2 \quad \beta'' = \beta''_1 + \beta''_2$$

(A.34)

with

$$\beta_0 = QG^{-1}Y^TN^{-1}QG \quad \beta_1 = P - QG^{-1}(LG + MY)$$

(A.35)

$$\beta'_0 = -N^{-1}QG$$

(A.36)

$$\beta'_1 = N^{-1}(\phi G + \psi Y) - QG^{-1}Y^TN^{-1}Q(D_1G + D_2Y) + (PG^{-1}Y^TN^{-1}Q - QG^{-1}Y^TN^{-1}QG)QG$$

(A.37)

$$\beta'_2 = QG^{-1}[L(D_1G + D_2Y) + M(D_3Y - D_2^T G)] - PG^{-1}(LG + MY)$$

(A.38)

$$\beta''_1 = N^{-1}[Q(D_1G + D_2Y) - \psi QG]$$

(A.39)

$$\beta''_2 = (Q^{-1}Y^TN^{-1}QD_1 - N^{-1}\phi - PG^{-1}Y^TN^{-1}Q - QG^{-1}M Q)(D_1G + D_2Y)$$

$$+ (Q^{-1}Y^TN^{-1}QD_2 - N^{-1}\psi)(D_3Y - D_2^T G) - QG^{-1}(LD_2 + MD_3)QG + PG^{-1}MQG$$

(A.40)

The subscripts of the components of $\beta$, $\beta'$, and $\beta''$ indicate the order of its explicit dependence on $k_i$. Some terms in $\beta''$, which have explicit cubic dependence on $k_i$, are purposely ignored as they will not affect the perturbation solution up to the second order of $k_i$.

A perturbation solution is sought in the following form:

$$G = G_0 + G_1 + G_2 \quad Y = Y_0 + Y_1 + Y_2 \quad N = N_0 + N_1 + N_2$$

(A.41)
Due to Eq. (A.17), knowledge of $N$ leads to $X$. $N$, $Y$, and $G$ are decomposed in the above equation into its zeroth-, first-, and second-order components with respect to $k_i$. In Eqs. (A.31)-(A.33), $A$, $B$, $C$, and $D$ are viewed as given constants, so it is unnecessary to expand them. The perturbation expansions of the inverse of $G$ and $N$ are also needed. They can be derived from the definition of matrix inverse such that

$$G^{-1} = G_0^{-1} - G_0^{-1}G_1G_0^{-1} + G_0^{-1}(G_1G_0^{-1}G_1 - G_2)G_0^{-1} \equiv G_0^{-1}(\Delta_2 + g_1 + g_2) \quad \text{(A.42)}$$

$$N^{-1} = N_0^{-1} - N_0^{-1}N_1N_0^{-1} + N_0^{-1}(N_1N_0^{-1}N_1 - N_2)N_0^{-1} \equiv (\Delta_4 + n_1 + n_2)N_0^{-1} \quad \text{(A.43)}$$

The validity of these expressions are verified by $G_0^{-1}(\Delta_2 + g_1 + g_2)(G_0 + G_1 + G_2) = \Delta_2$ and $(\Delta_4 + n_1 + n_2)N_0^{-1}(N_0 + N_1 + N_2) = \Delta_4$.

Before proceeding to solve the system of equations given by Eqs. (A.31)-(A.33), these equations are written in perturbed form as

$$G_0 + G_1 + G_2 = S_0 + S_1 + S_2 \quad \text{(A.44)}$$

$$0 = F_0 + F_1 + F_2 \quad \text{(A.45)}$$

$$N_0 + N_1 + N_2 = A + A_1 + A_2 \quad \text{(A.46)}$$
with

\[ S_0 = \beta_0^T A \beta_0 + \beta_0^T C \beta_0 + \beta_0^T B \beta_0 + B^T \beta_0 \quad (A.47) \]
\[ S_1 = \beta_1^T A \beta_0 + \beta_0^T C \beta_1 + \beta_0^T (B \beta_1 + D \beta_1' + \beta_1^T B \beta_0') + (A.48) \]
\[ S_2 = \beta_1^T A \beta_1 + \beta_1^T C \beta_1 + \beta_1^T (B \beta_2 + D \beta_2') + \beta_1^T (B \beta_1' + D \beta_1'') + (A.49) \]
\[ F_0 = A \beta_0 + B \beta_0' \quad (A.50) \]
\[ F_1 = A \beta_1 + B \beta_1' + W_1^T (A \beta_0 + B \beta_0') + W_1^T (B^T \beta_0 + C \beta_0') + W_1''^T D^T \beta_0 \quad (A.51) \]
\[ F_2 = B \beta_2 + B \beta_2' + W_1^T (A \beta_1 + B \beta_1' + D \beta_1'') + W_1''^T (B^T \beta_1 + C \beta_1') \]
\[ + W_2^T (B^T \beta_0 + C \beta_0') + W_1''^T D^T \beta_1 + W_2''^T D^T \beta_0 \quad (A.52) \]
\[ A_1 = A W_1 + B W_1' + D W_1'' + (A.53) \]
\[ A_2 = W_1^T A W_1 + W_1''^T C W_1' + B W_2' + D W_2'' + W_1^T (B W_1' + D W_1'') + (A.54) \]

The subscripts of S, F, and A indicate the order of its *explicit* dependence on \( k_i \). Terms, which are *explicitly* cubic of \( k_i \), are again ignored here as they will not affect the perturbation solution up to second order.

From Eqs. (A.26)-(A.28) and (A.35)-(A.40), it is clear that \( W, W', W'' \) and all components of \( \beta, \beta', \) and \( \beta'' \) are functions of \( G_0, G_1, G_2, Y_0, Y_1, Y_2, N_0, N_1, \) and \( N_2 \). To consider the effects due to the perturbation of unknown variables, another subscript is introduced to indicate the order due to different orders of the unknown perturbed variables. For example, \( F_0 \) is decomposed as

\[ F_0 = F_{00} + F_{01} + F_{02} \quad (A.55) \]

with \( F_{00}, F_{01}, \) and \( F_{02} \) indicating the zeroth-, first-, and second-order terms, respectively, due to the perturbation of \( G, Y, \) and \( N \) from Eq. (A.41). These expressions are derived as

\[ F_{00} = A \beta_{00} + B \beta_{00}' \quad F_{01} = A \beta_{01} + B \beta_{01}' \quad F_{02} = A \beta_{02} + B \beta_{02}' \quad (A.56) \]
where the subscripts for $\beta$ are also defined similarly such that

$$\beta_{00} = QG_0^{-1} Y_0^T N_0^{-1} QG_0$$  \hfill (A.57)

$$\beta_{01} = QG_0^{-1} [(g_1 Y_0^T + Y_1^T + Y_0^T n_1) N_0^{-1} QG_0 + Y_0^T N_0^{-1} QG_1]$$  \hfill (A.58)

$$\beta_{02} = QG_0^{-1} [(g_2 Y_0^T + Y_2^T + Y_0^T n_2) N_0^{-1} QG_0 + Y_0^T N_0^{-1} QG_2]$$

$$+ QG_0^{-1} (g_1 Y_0^T + Y_1^T + Y_0^T n_1) N_0^{-1} QG_1 + QG_0^{-1} \left[(Y_1^T + g_1 Y_0^T) n_1 + g_1 Y_1^T\right] N_0^{-1} QG_0$$  \hfill (A.59)

With such a notation, any symbol with the added subscript of 0 can be directly evaluated using the original formula with the unknown functions substituted using the zeroth-order solution. For example, $\phi_0 = Y_0 G_0^{-1} D_1 + D_2^T$ and $\psi_0 = Y_0 G_0^{-1} D_2 - D_3$.

### A.2 Zeroth-order Solution

The zeroth-order equations of Eqs. (A.44)-(A.46) are

$$G_0 = \beta_{00}^T A \beta_{00} + \beta_{00}' C \beta_{00}' + \beta_{00}' B \beta_{00} + \beta_{00}' B^T \beta_{00}$$  \hfill (A.60)

$$0 = A \beta_{00} + B \beta_{00}'$$  \hfill (A.61)

$$N_0 = A$$  \hfill (A.62)

Equation (A.60), with usage of Eq. (A.61), can be simplified as

$$G_0 = \beta_{00}^T (C \beta_{00}' + B^T \beta_{00}) = \beta_{00}^T (C - B^T A^{-1} B) \beta_{00}'$$  \hfill (A.63)

Substituting Eq. (A.36) into the above equation results in

$$G_0^{-1} = Q^T N_0^{-1} (C - B^T A^{-1} B) N_0^{-1} Q = Q^T A^{-1} (C - B^T A^{-1} B) A^{-1} Q$$  \hfill (A.64)

Substituting Eqs. (A.35) and (A.36) into Eq. (A.61) results in

$$0 = (A Q G_0^{-1} Y_0^T - B) N_0^{-1} Q G_0$$  \hfill (A.65)
As this equation should be valid for arbitrary $N_0^{-1}QG_0$, the following statement must hold:

$$AQG_0^{-1}Y_0^T = B$$  \hspace{1cm} (A.66)

$A$ is an invertible matrix, so it must be that

$$QG_0^{-1}Y_0^T = A^{-1}B$$  \hspace{1cm} (A.67)

The first two rows of $QG_0^{-1}Y_0^T$ are filled with zeros, which implies that Eq. (A.67) may be used to solve for $Y_0$ exactly only if the first two rows of $A^{-1}B$ are filled with zeros. An approximate solution for $Y_0$ can be obtained by pre-multiplying $GQ^T$ to both sides of Eq. (A.67) and then taking the transpose. The final solution for $Y_0$ becomes

$$Y_0 = B^TA^{-1}QG_0$$  \hspace{1cm} (A.68)

It can be verified that if the first two rows of $A^{-1}B$ are filled with zeros, then Eq. (A.68) is the exact solution satisfying Eq. (A.66). However, in cases where the first two rows of $A^{-1}B$ are not filled with zeros, pre-multiplying Eq. (A.67) by $GQ^T$ actually reduces the dimension of the system so that Eq. (A.68) becomes an approximation.

The zeroth-order solution can now be stated clearly. $G_0$ is found from taking the inverse of Eq. (A.64). $Y_0$ is found from Eq. (A.68). Combining Eqs. (A.17) and (A.62), $X$ is found as $X = A + Y_0G_0^{-1}Y_0^T$.

### A.3 First-order Solution

The first-order equations of Eqs. (A.44)- (A.46) are

$$G_1 = S_{01} + S_{10}$$  \hspace{1cm} (A.69)

$$0 = F_{01} + F_{10}$$  \hspace{1cm} (A.70)

$$N_1 = A_{10} = AW_{10} + BW'_{10} + DW''_{10} + ()^T$$  \hspace{1cm} (A.71)
\( N_1 \) can be directly solved from Eq. (A.71). One needs to solve \( G_1 \) and \( Y_1 \) from Eqs. (A.69) and (A.70). From Eq. (A.47), \( S_{01} \) is identified as

\[
S_{01} = \beta_{01}^T (A \beta_{00} + B \beta'_{00}) + \beta_{01}^T (C \beta'_{00} + B^T \beta_{00}) + (0)^T = \beta_{01}^T (C \beta'_{00} + B^T \beta_{00}) + (0)^T \tag{A.72}
\]

The second equality of Eq. (A.72) is found from usage of Eq. (A.61). From Eq. (A.36), \( \beta'_{00} = -N_0^{-1} Q G_0 \) and \( \beta'_{01} = -n_1 N_0^{-1} Q G_0 - N_0^{-1} Q G_1 \). Using these relations along with Eq. (A.66), the following is derived:

\[
S_{01} = 2G_1 + G_0 Q^T N_0^{-1} n_1^T (C - B^T A^{-1} B) N_0^{-1} Q G_0 + (0)^T \tag{A.73}
\]

Substituting the above equation into Eq. (A.69) results in

\[
G_1 = G_0 Q^T N_0^{-1} n_1^T (B^T A^{-1} B - C) N_0^{-1} Q G_0 + (0)^T - S_{10} \tag{A.74}
\]

where \( S_{10} \) can be obtained from Eq. (A.48) and (A.66) as

\[
S_{10} = \beta_{00}^T C \beta'_{10} + \beta_{00}^T (B \beta'_{10} + D \beta''_{10}) + (0)^T \tag{A.75}
\]

\( \beta'_{10} \) and \( \beta''_{10} \) are evaluated from Eqs. (A.37) and (A.39) with the unknown stiffness terms replaced by the zeroth-order solutions.

To solve for \( Y_1 \), expressions for \( F_{01} \) and \( F_{10} \) are needed. From Eqs. (A.42), (A.50), and (A.66), \( F_{01} \) is found as

\[
F_{01} = A \beta_{01} + B \beta'_{01} = A Q G_0^{-1} (Y_1^T + g_1 Y_0^T) N_0^{-1} Q G_0 \tag{A.76}
\]
Using Eq. (A.61), $F_{10}$ can be simplified from Eq. (A.51) as

$$F_{10} = A\beta_{10} + B\beta_{10}' + D\beta_{10}'' + W_{10}^{TT}(B^T\beta_{00} + C\beta_{00}') + W_{10}^{TT}D^T\beta_{00}$$

$$= A\beta_{10} + \frac{D}{\Delta^2}Q^{-1} - AQQG_{0}^{-1} - BQQG_{0}^{-1}Y_0^T N_0^{-1}Q)(D_1G_0 + D_2Y_0) + BPPG_{0}^{-1}Y_0^T N_0^{-1}QG_0$$

$$+ N_0 [(\psi_0^T N_0^{-1} - D_2^T Q^T N_0^{-1} Y_0^T G_0^{-1} Q^T)(B^T QG_0^{-1} Y_0^T - C) + D_2^T Q^T N_0^{-1} D^T QG_0^{-1} Y_0^T]$$

$$N_0^{-1}QG_0 + (BQQG_{0}^{-1}Y_0^T - D)N_0^{-1}\psi_0QG_0$$

$$\equiv AQG_{0}^{-1} F_{10}^* N_0^{-1} QG_0$$

(A.77)

The last equality is introduced so that $Y_1$ can be solved following a similar procedure as the one used from solving $Y_0$. In view of the fact that $Q^TQ = \Delta_2$, it is helpful to write $G_0$ as

$$G_0 = Q^TQG_0 = Q^T N_0^{-1} QG_0$$

(A.78)

Along with Eq. (A.68), $F_{10}^*$ is found as

$$F_{10}^* = G_0 Q^T P G_0^{-1} Q^T A + (G_0 Q^T A^{-1} D A^{-1} Q - \Delta_2 - Y_0^T QG_0^{-1} Y_0^T A^{-1} Q)(D_1Q^T A + D_2B^T)$$

$$+ G_0 Q^T \psi_0^T A^{-1} (B^T QG_0^{-1} Y_0^T - C) + (G_0 Q^T A^{-1} D - Y_0^T QG_0^{-1} Y_0^T) A^{-1} D_3A + Y_0^T P G_0^{-1} Y_0^T$$

(A.79)

If $B$ is the first symbol in a term from Eq. (A.79), then Eq. (A.66) is used to extract $AQG_{0}^{-1}$ out of $B$. This is employed so that $Y_1$ is solved analytically within the approximation already introduced in the zeroth-order solution for cases where the first two rows of $A^{-1}B$ contain nonzero components. Substituting Eqs. (A.76) and (A.77) into Eq. (A.70), one can solve $Y_1$ from

$$Y_1^T = -g_1 Y_0^T - F_{10}^*$$

(A.80)

Finally, the first-order solution of $X$ can be found from Eq. (A.17).
A.4 Second-order Solution

The second-order equations of Eqs. (A.44)-(A.46) are

\[
\begin{align*}
G_2 &= S_{02} + S_{11} + S_{20} \\ 0 &= F_{02} + F_{11} + F_{20} \\ N_2 &= A_{11} + A_{20}
\end{align*}
\] (A.81) (A.82) (A.83)

\(N_2\) can be directly evaluated from Eq. (A.83). \(A_{20}\) is found from Eq. (A.54) with the values of \(G\), \(Y\), and \(N\) taken from the zeroth-order solutions. From Eq. (A.53), \(A_{11}\) is

\[
A_{11} = AW_{11} + BW^\prime_{11} + DW^\prime\prime_{11} + ()_T
\] (A.84)

where

\[
W_{11} = -PG^{-1}_0(g_1 Y^T_0 + Y^T_1) - QG^{-1}_0(g_1 M_0 N_0 + M_1 N_0 + M_0 N_1)
\] (A.85)

\[
W^\prime_{11} = [n_1 N_0^{-1} \psi_0 + N_0^{-1} \psi_1 - QG^{-1}_0(g_1 Y^T_0 + Y^T_1 + Y^T_0 n_1) N_0^{-1} Q D_2] N_0
\] (A.86)

\[
W^\prime\prime_{11} = N_0^{-1} Q D_2 N_1 + n_1 N_0^{-1} Q D_2 N_0
\] (A.87)

with \(\psi_1 = Y_1 G_0^{-1} D_2 + Y_0 G_0^{-1} g_1 D_2\) and \(M_1 = Y^T_1 N_0^{-1} \psi_0 + Y^T_0 n_1 N_0^{-1} \psi_0 + Y^T_0 N_0^{-1} \psi_1\).

To solve for \(G_2\) from Eq. (A.81), expressions for \(S_{02}\), \(S_{11}\), and \(S_{20}\) are needed. \(S_{20}\) is given in Eq. (A.49) with the unknown stiffness terms taken from the zeroth-order solutions. \(S_{11}\) can be obtained from Eq. (A.48) as

\[
S_{11} = \beta^T_{01} (A\beta_{10} + B\beta^\prime_{10} + D\beta^\prime\prime_{10}) + \beta^T_{00} (B\beta^\prime_{11} + D\beta^\prime\prime_{11}) + \beta^T_{01} C\beta^\prime_{11} + \beta^T_{00} C\beta^\prime_{00} + \beta^T_{10} B\beta^\prime_{01} + ()_T
\] (A.88)
where the expressions for $\beta_{11}'$ and $\beta_{11}''$ are obtained from Eqs. (A.37) and (A.39) as

$$
\begin{align*}
\beta_{11}' &= N_0^{-1}(\phi_0G_1 + \phi_1G_0 + \psi_1Y_0 + \psi_0Y_1) + n_1N_0^{-1}(\phi_0G_0 + \psi_0Y_0) \\
&\quad + [PG_0^{-1}(Y_0^TN_0 + Y_1^TN_0^T + g_1Y_0^T)N_0^{-1} + QG_0^{-1}(M_1 + g_1M_0)]QG_0 \\
&\quad + (PG_0^{-1}Y_0^TN_0^{-1} + QG_0^{-1}M_0)QG_1 - QG_0^{-1}Y_0^TN_0^{-1}Q(D_1G_1 + D_2Y_1) \\
&\quad - QG_0^{-1}(Y_0^TN_0 + Y_1^TN_0^T + g_1Y_0^T)N_0^{-1}Q(D_1G_0 + D_2Y_0)
\end{align*}
$$
(A.89)

$$
\beta_{11}'' = N_0^{-1}[Q(D_1G_1 + D_2Y_1) - \psi_1QG_0 - \psi_0QG_1] + n_1N_0^{-1}[Q(D_1G_0 + D_2Y_0) - \psi_0QG_0]
$$
(A.90)

$S_{02}$ is derived from Eq. (A.47) as

$$
S_{02} = \beta_{01}^T A_{01} + \beta_{01}^T C\beta_{01}' + \beta_{01}' B\beta_{01}' + \beta_{02}' (C\beta_{00}' + B\beta_{00}') + (\_\_\_\_\_\_\_)_T
$$
(A.91)

where $\alpha_s$ can already be directly evaluated. From Eq. (A.36), $\beta_{02}'$ is derived as

$$
\beta_{02}' = -N_0^{-1}QG_2 - n_2N_0^{-1}QG_0 - n_1N_0^{-1}QG_1.
$$

Following the procedure used in obtaining Eq. (A.73), $S_{02}$ is found as

$$
S_{02} = \alpha_s + 2G_2 + (n_2N_0^{-1}QG_0 + n_1N_0^{-1}QG_1)(C - B^TA^{-1}B)N_0^{-1}QG_0 + (\_\_\_\_\_\_\_)_T
$$
(A.92)

Summing $S_{20}$, $S_{11}$, and $S_{02}$ yields $G_2$ from Eq. (A.81) as

$$
G_2 = -S_{20} - S_{11} - \alpha_s + (n_2N_0^{-1}QG_0 + n_1N_0^{-1}QG_1)(B^TA^{-1}B - C)N_0^{-1}QG_0 + (\_\_\_\_\_\_\_)_T
$$
(A.93)

To solve for $F_2$ from Eq. (A.82), one needs to first evaluate $F_{02}$, $F_{11}$ and $F_{20}$. $F_{20}$ is given in Eq. (A.52) with the unknown stiffness terms taken from the zeroth-order solutions, so it becomes

$$
F_{20} = AQG_0^{-1}F_{20}N_0^{-1}QG_0
$$
(A.94)
\( F_{11} \) is obtained from Eq. (A.51) as

\[
F_{11} = A\beta_{11} + B\beta_1' + D\beta_0'' + W_{10}^T(A\beta_{01} + B\beta_0') + W_{10}^T(B^T\beta_{01} + C\beta_0') + W_{11}^T(B^T\beta_{00} + C\beta_0') + W_{10}^T D^T\beta_{00} + W_{11}^T D^T\beta_{01} \equiv AQG_0^{-1} F_{11}^* N_0^{-1} QG_0
\]

(A.95)

with

\[
\beta_{11} = -QG_0^{-1} [(L_0 G_1 + L_1 G_0 + M_0 Y_1 + M_1 Y_0) + g_1(L_0 G_0 + M_0 Y_0)]
\]

(A.96)

\( F_{02} \) is obtained from Eq. (A.50) as

\[
F_{02} = A\beta_{02} + B\beta_0' = AQG_0^{-1} \left[ g_2 Y_{0}^T + Y_2^T + g_1 Y_1^T + (Y_1^T + g_1 Y_0^T) n_1 \right] N_0^{-1} QG_0 + (g_1 Y_{0}^T + Y_1^T) N_0^{-1} QG_1
\]

(A.97)

Substituting expressions of \( F_{20}, F_{11}, \) and \( F_{02} \) into Eq. (A.82), \( Y_2^T \) is solved in a procedure similar to the one used to solve for \( Y_1^T \). The result is

\[
Y_2^T = -F_{11}^* - F_{20}^* - g_2 Y_{0}^T - g_1 Y_1^T -(Y_1^T + g_1 Y_0^T) n_1 -(Y_1^T + g_1 Y_0^T) N_0^{-1} QG_1 G_0^{-1} Q^T N_0
\]

(A.98)

Lastly, the second-order solution is completed by evaluating \( X \) from Eq. (A.17). The generalized Timoshenko stiffness matrices have now been solved up to second-order with respect to \( k_i \).

The bilinear terms \( F_\epsilon \) and \( F_\gamma \) in Eq. (A.1) can be easily obtained from Eq. (3.71) using Eqs. (A.18) and (A.19) as

\[
F_\epsilon = \alpha^T F_\epsilon + \alpha' F_{\epsilon'} + \alpha'' F_{\epsilon''} \quad F_\gamma = \beta^T F_\epsilon + \beta' F_{\epsilon'} + \beta'' F_{\epsilon''}
\]

(A.99)

So far, we have constructed a generalized Timoshenko model which is as asymptotically correct as possible.
Appendix B

Corrections on Initial Twist and Curvatures

It is noted that the three terms in operator $\sqrt{g}$ in Eq. (2.17) are of different orders. The last two terms, $-x_2k_3$ and $x_3k_2$, are of the order $O(h/R)$ while the first term is of the order $O(1)$. Therefore, to correctly model the initial twist and curvatures, the difference in orders of terms in $\sqrt{g}$ should be considered in the dimensional reduction process. To be more specific, the matrices used in dimensional reduction in Eq. (3.18), (3.32), (3.49), (3.64), and (4.29) should be expanded into terms up to different orders. For some cases such a change made significant differences for obtaining first and second correction of the stiffness matrix due to initial curvatures and twist.

The expression of $\sqrt{g}$ can be rewritten as

$$\sqrt{g} = 1 - x_2k_3 + x_3k_2 \equiv g_0 + g_1$$

where $g_0 = 1$ and $g_1 = -x_2k_3 + x_3k_2$. In this section, number in the subscript indicates the order of the operator or matrix. By Taylor expansion, the $\frac{1}{\sqrt{g}}$ is expressed as

$$\frac{1}{\sqrt{g}} = 1 + (x_2k_3 - x_3k_2) + (x_2k_3 - x_3k_2)^2 + O(\kappa^3)$$

$$\equiv rg_0 + rg_1 + rg_2 + O(\kappa^3)$$
Here the terms up to the second order of $\kappa_i$ are kept. For simplicity, the operators in Eqs. (2.37), (2.38), and (2.39) are expressed as follows

$$\Gamma_\epsilon = \frac{1}{\sqrt{g}} \bar{\Gamma}_\epsilon$$ (B.3)
$$\Gamma_R = \frac{1}{\sqrt{g}} \bar{\Gamma}_R$$ (B.4)
$$\Gamma_l = \frac{1}{\sqrt{g}} \bar{\Gamma}_l$$ (B.5)

Using these expressions along with the relation $\langle \langle \bullet \rangle \rangle = \langle \bullet \sqrt{g} \rangle$, the matrices in Eqs. (3.18) and (3.32) can be expressed as

$$E_T = \langle \langle [\Gamma_T S]^T K [\Gamma_T S] \rangle \rangle = \langle [\Gamma_T S]^T K [\Gamma_T S] \sqrt{g} \rangle$$
$$= \langle [\Gamma_T S]^T K [\Gamma_T S] g_0 \rangle + \langle [\Gamma_T S]^T K [\Gamma_T S] g_1 \rangle$$
$$\equiv E_{T0} + E_{T1}$$ (B.6)

$$D_T = \langle \langle [\Gamma_T S]^T K e_1 \rangle \rangle = \langle [\Gamma_T S]^T K \bar{e}_1 \rangle$$
$$\equiv D_{T0}$$ (B.7)

$$\bar{K} = \langle \langle \frac{1}{g} K_{11} \rangle \rangle = \langle \frac{1}{\sqrt{g}} K_{11} \rangle$$
$$= \langle K_{11} r g_0 \rangle + \langle K_{11} r g_1 \rangle + \langle K_{11} r g_2 \rangle$$
$$\equiv \bar{K}_0 + \bar{K}_1 + \bar{K}_2$$ (B.8)

$$D_{Re} = \langle \langle [\Gamma_{RT} S]^T K e_1 \rangle \rangle$$
$$= \langle [\Gamma_{RT} S]^T K \bar{e}_1 r g_0 \rangle + \langle [\Gamma_{RT} S]^T K \bar{e}_1 r g_1 \rangle$$
$$\equiv D_{Re1} + D_{Re2}$$ (B.9)

$$D_{RT} = \langle \langle [\Gamma_{RT} S]^T K [\Gamma_T S] \rangle \rangle = \langle [\Gamma_{RT} S]^T K [\Gamma_T S] \rangle$$
$$\equiv D_{RT1}$$ (B.10)
\[ D_{RR} = \left\langle \left\langle \left[ \Gamma_{RT} S \right]^T K \left[ \Gamma_{RT} S \right] \right\rangle \right\rangle = \left\langle \left[ \Gamma_{RT} S \right]^T K \left[ \Gamma_{RT} S \right] \right\rangle \]
\[ \equiv D_{RR2} \] 

(B.11)

where \( \bar{e}_1 = [1 \ 0 \ 0]^T \). Other terms in Eqs. (3.49), (3.64), and (4.29) can be expanded up to different orders following the same philosophy. In order to avoid confusion, the explicit form of these expanded matrices are provided here. Note that these expressions are valid for both thermoelastic and multiphysics modeling.

\[ E = \left\langle \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_a S \right] \right\rangle \right\rangle \]
\[ = \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_a S \right] g_0 \right\rangle + \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_a S \right] g_1 \right\rangle \]
\[ \equiv E_0 + E_1 \] 

(B.12)

\[ D_{a\epsilon} = \left\langle \left\langle \left[ \Gamma_a S \right]^T D \bar{\Gamma}_{\epsilon} \right\rangle \right\rangle = \left\langle \left[ \Gamma_a S \right]^T D \bar{\Gamma}_{\epsilon} \right\rangle \]
\[ \equiv D_{a\epsilon 0} \] 

(B.13)

\[ D_{\epsilon\epsilon} = \left\langle \left\langle \left[ \Gamma_{\epsilon} \right]^T D \left[ \Gamma_{\epsilon} \right] \right\rangle \right\rangle \]
\[ = \left\langle \left[ \Gamma_{\epsilon} \right]^T D \left[ \Gamma_{\epsilon} \right] r g_0 \right\rangle + \left\langle \left[ \Gamma_{\epsilon} \right]^T D \left[ \Gamma_{\epsilon} \right] r g_1 \right\rangle + \left\langle \left[ \Gamma_{\epsilon} \right]^T D \left[ \Gamma_{\epsilon} \right] r g_2 \right\rangle \]
\[ \equiv D_{\epsilon\epsilon 0} + D_{\epsilon\epsilon 1} + D_{\epsilon\epsilon 2} \] 

(B.14)

\[ D_{aR} = \left\langle \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_R S \right] \right\rangle \right\rangle = \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_R S \right] \right\rangle \]
\[ \equiv D_{aR1} \] 

(B.15)

\[ D_{RR} = \left\langle \left\langle \left[ \Gamma_R S \right]^T D \left[ \Gamma_R S \right] \right\rangle \right\rangle = \left\langle \left[ \Gamma_R S \right]^T D \left[ \Gamma_R S \right] r g_0 \right\rangle \]
\[ \equiv D_{RR2} \] 

(B.16)

\[ D_{al} = \left\langle \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_l S \right] \right\rangle \right\rangle = \left\langle \left[ \Gamma_a S \right]^T D \left[ \Gamma_l S \right] \right\rangle \]
\[ \equiv D_{al1} \] 

(B.17)
\[ D_{ll} = \langle [\Gamma_l S]^T D [\Gamma_l S] \rangle \]
\[ = \langle [\Gamma_l S]^T D [\Gamma_l S] \, rg_0 \rangle + \langle [\Gamma_l S]^T D [\Gamma_l S] \, rg_1 \rangle + \langle [\Gamma_l S]^T D [\Gamma_l S] \, rg_2 \rangle \]
\[ \equiv D_{l0} + D_{l1} + D_{l2} \quad (B.18) \]

\[ D_{le} = \langle [\Gamma_l S]^T D [\Gamma_e] \rangle \]
\[ = \langle [\Gamma_l S]^T D [\bar{\Gamma}_e] \, rg_0 \rangle + \langle [\Gamma_l S]^T D [\bar{\Gamma}_e] \, rg_1 \rangle \]
\[ \equiv D_{le1} + D_{le2} \quad (B.19) \]

\[ D_{Re} = \langle [\Gamma_R S]^T D [\Gamma_e] \rangle \]
\[ = \langle [\bar{\Gamma}_R S]^T D [\bar{\Gamma}_e] \, rg_0 \rangle + \langle [\bar{\Gamma}_R S]^T D [\bar{\Gamma}_e] \, rg_1 \rangle \]
\[ \equiv D_{Re1} + D_{Re2} \quad (B.20) \]

\[ D_{Rl} = \langle [\Gamma_R S]^T D [\Gamma_l S] \rangle = \langle [\bar{\Gamma}_R S]^T D [\bar{\Gamma}_l S] \, rg_0 \rangle \]
\[ \equiv D_{Rl2} \quad (B.21) \]

\[ \alpha_a = \langle [\Gamma_a S]^T D\alpha \Delta T \rangle = \langle [\Gamma_a S]^T D\alpha \Delta T g_0 \rangle + \langle [\Gamma_a S]^T D\alpha \Delta T g_1 \rangle \]
\[ \equiv \alpha_{a0} + \alpha_{a1} \quad (B.22) \]

\[ \alpha_e = \langle [\Gamma_e]^T D\alpha \Delta T \rangle = \langle [\bar{\Gamma}_e]^T D\alpha \Delta T \rangle \]
\[ \equiv \alpha_{e0} \quad (B.23) \]

\[ \alpha_l = \langle [\Gamma_l S]^T D\alpha \Delta T \rangle = \langle [\bar{\Gamma}_l S]^T D\alpha \Delta T \rangle \]
\[ \equiv \alpha_{l0} \quad (B.24) \]

\[ \alpha_R = \langle [\Gamma_R S]^T D\alpha \Delta T \rangle = \langle [\bar{\Gamma}_R S]^T D\alpha \Delta T \rangle \]
\[ \equiv \alpha_{R1} \quad (B.25) \]
Vita

About the author

Qi Wang was born in Kaifeng, Henan Province, China, on March 15, 1983. He received his B.S. degree from Huazhong University of Science and Technology majoring in Engineering Mechanics with a minor in Computer Science and Technology. Then he received his M.S. degree from Dalian University of Technology in December, 2007. Upon his graduation, he moved to Utah State University where he began his Ph.D. studies.

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