All homogeneous pure radiation spacetimes satisfy the Einstein–Maxwell equations
NOTE

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Abstract
It is shown that all homogeneous pure radiation solutions to the Einstein equations admit electromagnetic sources. This corrects an error in the literature.

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A spacetime \((M, g)\) is said to be a homogeneous pure radiation solution of the Einstein equations if it admits a transitive group of isometries and if there exists a function \(\Phi\) and a 1-form \(k\) such that the Einstein tensor \(G\) is of the form

\[
G_{ab} = \Phi^2 k_a k_b, \quad \text{where} \quad g^{ab} k_a k_b = 0.
\]

(1)

A classification of homogeneous pure radiation solutions is given in [1] based upon results of [2] and [3]. Such solutions are all pp waves, and they are either plane waves or are diffeomorphic to

\[
g = dx \otimes dx + dy \otimes dy + du \otimes dv - 2 e^{2\varphi} du \otimes du, \quad \rho = \text{const}.
\]

(2)

The plane waves are known to arise from electromagnetic sources. It is asserted in [1, 3], based upon results of [4], that the metric (2) does not arise from an electromagnetic source. However, the metric (2) does arise from an electromagnetic source. In fact, infinitely many electromagnetic fields can serve as the source for the spacetime defined by (2).

For any function \(f(u)\) the null 2-form

\[
F = 2 \rho e^{\rho \pi} [\cos(\rho y + f(u)) du \wedge dx - \sin(\rho y + f(u)) du \wedge dy],
\]

(3)

its Hodge dual \(\tilde{F}\) and the metric (2) satisfy the source-free Maxwell equations \(dF = 0 = d\tilde{F}\) and the Einstein equations:

\[
G_{ab} = \frac{1}{2} \left( F_{ac} F_b^c + \tilde{F}_{ac} \tilde{F}_b^c \right) = 4 \rho^2 e^{2\rho \pi} \nabla_a u \nabla_b u.
\]

(4)

The electromagnetic field given in (3) is ‘non-inheriting’, that is, it is not invariant under the isometry group of \((M, g)\). For example, \(F\) is not invariant under translations in \(y\) for any choice of \(f(u)\).

Thus, granted theorem 12.6 of [1], all homogeneous pure radiation solutions of the Einstein equations arise from electromagnetic sources.
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References