3-14-2012

All homogeneous pure radiation spacetimes satisfy the Einstein–Maxwell equations

Charles G. Torre
Department of Physics, Utah State University

All homogeneous pure radiation spacetimes satisfy the Einstein–Maxwell equations

2012 Class. Quantum Grav. 29 077001

(http://iopscience.iop.org/0264-9381/29/7/077001)

Class. Quantum Grav. **29** (2012) 077001 (2pp) doi:10.1088/0264-9381/29/7/077001

NOTE

All homogeneous pure radiation spacetimes satisfy the Einstein–Maxwell equations

C G Torre

Department of Physics, Utah State University, Logan, UT 84322-4415, USA

Received 8 February 2012
Published 14 March 2012
Online at stacks.iop.org/CQG/29/077001

Abstract
It is shown that all homogeneous pure radiation solutions to the Einstein equations admit electromagnetic sources. This corrects an error in the literature.

PACS numbers: 04.20.−q, 04.20.Jb, 04.40.Nr

A spacetime (M, g) is said to be a homogeneous pure radiation solution of the Einstein equations if it admits a transitive group of isometries and if there exists a function Φ and a 1-form k such that the Einstein tensor G is of the form

$$G_{ab} = \Phi^2 k_a k_b, \quad \text{where} \quad g^{ab}k_a k_b = 0. \tag{1}$$

A classification of homogeneous pure radiation solutions is given in [1] based upon results of [2] and [3]. Such solutions are all pp waves, and they are either plane waves or are diffeomorphic to

$$g = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}u \odot \mathrm{d}v - 2\,\mathrm{e}^{2\rho x}\,\mathrm{d}u \otimes \mathrm{d}u, \quad \rho = \text{const.} \tag{2}$$

The plane waves are known to arise from electromagnetic sources. It is asserted in [1, 3], based upon results of [4], that the metric (2) does not arise from an electromagnetic source. However, the metric (2) *does* arise from an electromagnetic source. In fact, infinitely many electromagnetic fields can serve as the source for the spacetime defined by (2).

For any function $f(u)$ the null 2-form

$$F = 2\rho\,\mathrm{e}^{\rho x}[\cos(\rho y + f(u))\,\mathrm{d}u \wedge \mathrm{d}x - \sin(\rho y + f(u))\,\mathrm{d}u \wedge \mathrm{d}y], \tag{3}$$

its Hodge dual $\tilde{F}$ and the metric (2) satisfy the source-free Maxwell equations $dF = 0 = d\tilde{F}$ and the Einstein equations:

$$G_{ab} = \tfrac{1}{2}\left(F_{ac}F_b{}^c + \tilde{F}_{ac}\tilde{F}_b{}^c\right) = 4\rho^2\,\mathrm{e}^{2\rho x}\nabla_a u \nabla_b u. \tag{4}$$

The electromagnetic field given in (3) is 'non-inheriting', that is, it is not invariant under the isometry group of (M, g). For example, F is not invariant under translations in y for any choice of $f(u)$.

Thus, granted theorem 12.6 of [1], all homogeneous pure radiation solutions of the Einstein equations arise from electromagnetic sources.

Acknowledgments

I thank Ian Anderson for discussions. These results were obtained using the *DifferentialGeometry* package in *Maple*.

References

[1] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press)
[2] Wils P 1989 *Class. Quantum. Grav.* **6** 1243
[3] Steele J 1990 *Class. Quantum. Grav.* **7** L81
[4] Sippel R and Goenner H 1986 *Gen. Rel. Grav.* **18** 1229