DEVELOPMENT OF A DRAINAGE FUNCTION FOR THE TRANSIENT CASE, AND A TWO-DIMENSIONAL GROUND-WATER MOUND STUDY TO EVALUATE AQUIFER PARAMETERS

by

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LIST OF SYMBOLS

\( a = \sqrt{V/KD} \)

\( b = \) a constant

\( c = \) a constant

\( d = \) diameter of tiles \((L)\)

\( D = \) the average depth of saturated media \((L)\)

\( D' = \) equal increment of depth of water \((L)\)

\( D_{n}(u) = \) drainage function of \(n\) parallel drains \((\text{dimensionless})\)

\( D_{1}(u) = \) drainage function for the point midway between the drains \((\text{dimensionless})\)

\( \text{erf}(u) = \) error function of \( u = 2/\sqrt{\pi} \int_{0}^{u} e^{-z^2} \, dz \) \((\text{dimensionless})\)

\( \text{erfc}(u) = \) error function complementary of \( u = 2/\sqrt{\pi} \int_{u}^{\infty} e^{-z^2} \, dz \) \((\text{dimensionless})\)

\( h = \) height of the water table above drains \((L)\)

\( h' = \) incremental rise of the water table due to an increment of irrigation \((L)\)

\( h_1 = \) total increment of rise of the water table due to irrigation \((L)\)

\( h_o = \) height of the water table above drains before irrigation \((L)\)

\( H_1 = \) height of the water table above the impermeable layer \((L)\)

\( H_o = \) initial height of a horizontal water table above the drains \((L)\)

\( H'_o = \) height of a disk of water in the soil due to an increment of applied water \((L)\)
\( H_e \) = effective height of water (L)

\( I \) = intake rate of water into the soil (L \( T^{-1} \))

\( k \) = hydraulic conductivity of the saturated media (L \( T^{-1} \))

\( L \) = distance between drains (L) and Lapace transformation

\( m \) = summation index

\( M \) = number of increments \( t_m \) in period \( t \)

\( n \) = summation index

\( N \) = number of drains

\( q \) = rate of flow discharge per unit length of drain (L \( T^{-1} \))

\( q(t) \) = rate of discharge per unit length of time \( t \) (L \( T^{-1} \))

\( q(u_n) \) = drainage discharge function (dimensionless)

\( Q \) = rate of discharge (L \( T^{-1} \))

\( S \) = slope of hydraulic grade line in Manning's formula (dimensionless)

\( S(x, t) \) = drawdown at distance \( x \) and time \( t \) (L)

\( S_0 \) = drawdown at \( x = 0 \) and \( t = 0 \) (L)

\( t \) = time (T)

\( t_1 \) = time after the irrigation (T)

\( t' \) = time from the middle of each period of irrigation to the end of the irrigation period (T)

\( t_e \) = effective time (T)

\( T \) = transmissivity of aquifer = D K (L \( T^{-1} \))

\( u \) = limit of integration (dimensionless)

\( u_1 \) = \( x_1 / \sqrt{4at} \) (dimensionless)
\( u_n = \frac{x_n}{\sqrt{4\alpha t}} \) (dimensionless)

\( v(t) \) = total volume of water discharged to a drain per unit length \((L^2)\)

\( V \) = specific yield (dimensionless)

\( x \) = horizontal distance \((L)\)

\( x_1 \) = distance to first drain \((L)\)

\( x_n \) = distance to \( n \)th drain \((L)\)

\( z \) = variable of integration (dimensionless)

\( \alpha \) = \( T/V \) \((L^2 T^{-1})\)

\( \lambda \) = variable of integration

\( \pi \) = 3, 14159, 26535, 89693, 2

\( \phi \left( \frac{x^2}{\sqrt{4\alpha t}} \right) = \sqrt{\pi} \int_x^\infty \frac{e^{-u^2}}{u^2} \, du \)

\( \psi = \frac{2}{\sqrt{\pi}} \int_0^u e^{-z^2} \, dz \)

\( \Sigma \) = summation sign
ABSTRACT

Development of a Drainage Function for the Transient Case, and a Two-Dimensional Ground-Water Mound Study to Evaluate Aquifer Parameters

by

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A drainage function, \( D(u_n) \), is developed to describe the rate of fall and the shape of the water table between drains for an infinite series of parallel drains. The drainage function is evaluated and presented graphically for design purposes. The solution of the drainage function is compared with the solution of Glover's transient state drainage equation. The two equations agreed very well, but the agreement was not as good for very high values of \( \sqrt{\alpha t / L} \).

The drainage function, \( D(u_n) \), was solved for the point midway between drains and the solution was presented graphically for design purposes.

A drainage discharge function, \( q(u_n) \), is developed to describe the rate of discharge into a drain of an infinite series of parallel drains. A dimensionless curve of the drainage discharge function is presented.
A method is presented to evaluate the rate of the water table recession between drains for any initial water table condition.

The effect of irrigation water on the water table between drains is determined, and the rate of recession of the water table after irrigation is solved.

A field method is presented to determine the integrated values of transmissivity and specific yield of an area to be drained.
**INTRODUCTION**

**Importance and need for drainage**

Drainage is one of the most important problems in irrigated areas. Many ancient civilizations were ruined because of the lack of appreciation for the importance of drainage in these regions.

In both humid and arid areas, when precipitation is higher than the evapotranspiration, some of the excess water penetrates into the soil. If there is not enough natural drainage, the water table builds up and may reach the root zone. When precipitation is less than the evapotranspiration, other factors, such as irrigation, may contribute to the drainage problem.

Irrigation is not a perfect operation, and in every system there are some losses. Conveyance losses, distribution losses, and the seepage from canals and upper lands contribute a great deal to the waterlogging problem.

Where the soil is salty and contains a considerable amount of soluble salts, the osmotic pressure of the soil solution is high, and hinders the growth of plants. To have a good environment for the plant growth, the excess salt in the soil should be leached out.

In most irrigated regions, the irrigation water is not always of good quality and carries some amount of soluble salts. When this water is applied to the soil, some of it evaporates and leaves the
soluble salts in the soil. The transpiration process and use of water by crops also leaves most of the salt in the soil. As a result, after a few years of irrigation, even though the soil may have been very good to begin with, it may contain considerable amounts of soluble salt which crops can hardly tolerate. Excess salt may come from other sources such as the upward movement of salty water from an artesian aquifer, seepage of salty water from upper lands, etc.

In all of the above mentioned cases, the excess and leached water should be drained away from the region to protect the area from being destroyed. The Bureau of Reclamation has recently recommended that the drainage system should be designed along with the irrigation system when a new project is initiated.

**Previous assumptions and designs**

Designs of drainage systems have customarily been based on one of two important assumptions: (1) Steady state condition, and (2) transient condition. Most of the early designs, especially in the humid regions, were worked on the steady state assumption. In arid and semi-arid regions, where the recharging period is very short in respect to the non-recharging period, steady state would only approximate the real conditions and the transient case should be applied. Even in humid regions the flow of water in the soil is not at a steady rate, and when it rains the free water surface moves up and then recedes when the rain stops.
Design of a drainage system for the transient case could be approached from two points of view: (1) A horizontal initial water table above the drains, and (2) an initial water table where shape is represented by a fourth-degree polynomial.

**Present interest and contents of this dissertation**

In this presentation only the transient state case with no upward or lateral flow toward the area is of primary interest. The horizontal initial water table case is a valid assumption only for conditions at the time of installing the drains or when they have been closed for some period of the season. In irrigated regions, when the recharging period has ceased, the shape of the water table depends primarily on the location of natural or artificial drains, the amount of deep percolation, length of non-recharging period, initial shape, height of the water table, hydraulic conductivity of the soil, etc. In previous developments, for ease of mathematical treatment, the shape of the water table has been assumed to be a fourth-degree polynomial (Dumm 1964). However, this particular assumption is not used in this analysis. A method will be developed to predict the rate of rise or fall of water table between drains for a given initial water table condition. A field experiment was designed to study a two-dimensional ground-water mound to evaluate the transmissivity and the specific yield of the aquifer. The results of this experiment will be presented.

The effect of the amount of irrigation water and its application
interval has an important effect on the fluctuation of the water table between drains. In this study this effect will be considered and the design of the drainage system will include the irrigation parameter.

Specific yield or drainable porosity is an important, but controversial parameter. Most authors assume that it is constant while others contend that it cannot be assumed to be constant because its amount changes with depth to the water table. Luthin (1957) showed, with a laboratory study, that the shapes of the experimental water table are similar to the theoretical shapes which are based on the constant drainable porosity, but the rate of fall does not accurately match the theory.

Depth of the impermeable layer is hard to measure, if not impossible. Laboratory measurement of hydraulic conductivity gives a value which is not very reliable for the field use. A field test is needed to evaluate the integrated values of hydraulic conductivity, k, depth to the impermeable layer, D, and the specific yield, V. In practice, k and D do not need to be determined separately, but their product or the transmissivity of the aquifer \( T = kD \) should be determined.

To determine the transmissivity and the specific yield of the area, a small trench was dug and a constant recharge, q, per unit length was introduced into the trench. The rise of the water table, h, at different distances, x, from the trench was measured for different times, t. The values of \( h/qx \) were plotted vs. \( x^2/t \) on log-log paper and
matched with the curve of the seepage from canals to determine $T$ and $V$.

The calculated values of transmissivity and specific yield found from the matching method are assumed to be the integrated values of $T$ and $V$ which directly effect the fluctuation of the water table. Determined in this manner, they do not have the disadvantages of the laboratory measurements.
The review of literature will be divided into two parts:

1. Build-up of the water table due to spreading of the water on the land, and

2. Rate of rise and the recession of the water table with the presence of drains.

**Spreading of the water on the land**

The continuity equation describing the flow of water in the soil for a transient state (Glover 1964) is:

\[
\frac{\partial}{\partial x} \left( k h_1 \frac{\partial h_1}{\partial x} \right) = V \frac{\partial h_1}{\partial t}
\]  

(1)

where

- \( k \) is the hydraulic conductivity of the media
- \( h_1 \) is the height of the water table above the impermeable layer
- \( x \) is the distance from the origin
- \( V \) is the specific yield
- \( t \) is the time

Equation (1) is a non-linear partial differential equation and its solution is rather involved. To solve Equation (1), \( h_1 \) may be written as (see Figure 1)
\[ h_1 = D + h \]

where

\( D \) is the depth of impermeable layer to the origin or the average thickness of the aquifer

\( h \) is the height of the water table above the origin

If it is assumed that \( h \) is very small with respect to \( D \), then Equation (1) could be linearized as follows:

\[
kD \frac{\partial^2 h}{\partial x^2} = V \frac{\partial h}{\partial t} \quad (2)
\]

Most of the equations of the flow through porous media are the solutions of the linearized differential, Equation (2), for different boundary and initial conditions.

For build-up of the water mound due to water spreading on a strip of land of finite width with an infinite length, Glover (1960) solved Equation (2) and developed the following formula:

\[
h = \frac{H}{\sqrt{\pi}} \int_{u_1}^{u_2} e^{-u^2} \, du \quad (3)
\]

where

\( h \) is the height of water mound in feet

\( H \) is the height of a cube of water due to an instantaneous percolation of water into the soil. The amount of \( H \) is
equal to $I t / V$, where $I$ is the intake rate, $t$ is the period of the application, and $V$ is the fillable porosity

$u$ is the variable of integration

$$u_1 = \frac{x - \frac{w}{2}}{\sqrt{4 \alpha t}}$$

$$u_2 = \frac{x + \frac{w}{2}}{\sqrt{4 \alpha t}}$$

$w$ is the width of water spreading area

$$\alpha = \frac{kD}{V}$$

$D$ is the depth of saturated aquifer

Hantush (1963) derived a formula for the growth of ground-water ridge in response to a uniform rate of deep percolation on a limited area as follows:

$$h_1^2(x, t) = h_0^2 + \frac{2 w v t}{k} \left\{ 1 - \frac{1}{2} \left[ 4 i^2 \text{erfc} \left( \frac{L - x}{\sqrt{4 v t}} \right) \right] \right\}$$

$$+ 4 i^2 \text{erfc} \left( \frac{L + x}{\sqrt{4 v t}} \right) \right\}$$

and

$$h_2^2(x, t) = h_0^2 + \frac{w v t}{k} \left[ 4 i^2 \text{erfc} \left( \frac{x - L}{\sqrt{4 v t}} \right) \right]$$
where

\[ h_1 \text{ and } h_2 \text{ are the height of water above the base of the aquifer} \]

\[ \text{after the water has been introduced to the area, under the} \]

\[ \text{percolation zone, and beyond that zone respectively} \]

\[ h_0 \text{ is the initial height of the water table} \]

\[ w \text{ is the uniform rate of percolation per unit area} \]

\[ v = kh_0 / \varepsilon \]

\[ \varepsilon \text{ is the specific yield of the aquifer} \]

\[ \text{erfc} (x) \text{ is the second repeated integral of the error function,} \]

\[ \text{values for which are available in tabular form (Carslaw} \]

\[ \text{and Jaeger, 1959)} \]

\[ 2L \text{ is the width of spreading area} \]

Glover (1964), for the case of the water table build-up due to constant seepage from a canal, proposed the following formula:

\[ h = \frac{q x}{2 \pi k D} \sqrt{\pi} \int_x^\infty \frac{e^{-u^2}}{u^2} \frac{du}{\sqrt{4 \alpha t}} \]

\[ (6) \]

where

\[ h \text{ is the rise of water table at the distance } x \text{ from the} \]

\[ \text{canal due to the seepage rate of } q \text{ per unit length of} \]

\[ \text{the canal} \]
Bittinger (1965) proposed a formula for a periodic instantaneous release of disks of ground water, each having a height of \( H \) and a radius of \((a)\) as follows:

\[
\frac{h_n}{H} = \frac{1}{2} e^{-\frac{r^2}{4\alpha t}} \int_0^a e^{-\frac{r'^2}{4\alpha t}} I_o \left( \frac{r r'}{2\alpha t} \right) r' dr'
\]

where

- \( h_n \) is the water rise due to the release of disk number \( n \)
- \( H \) is the height of the ground-water disk due to the rate of seepage, \( Q \), for a period of \( t_p \), \((H = Q t_p / \pi a^2 V)\)
- \( V \) is the specific yield
- \( t_n \) is the time since instantaneous release of disk number \( n \)
- \( r \) is the radial distance from the center of the spreading basin
- \( r' \) is the variable of integration running from 0 to \( a \)
- \( I_o \) is the modified bessel function of the first kind and zero order

Marino (1967) proved the formula which was introduced by Hantush (1963) with the Hele-shaw model. He also presented a formula for the case of the decay of the ground-water ridge after the cessation of the recharge, which is as follows:

\[
h_{1,2}^2(x, t) = h_i^2 \left( 1 + 2 \frac{w \bar{v} t}{k} - \frac{w \bar{v} t}{k} \left[ 4 i^2 \text{erfc} \left( \frac{L - x}{\sqrt{4 \bar{v} t}} \right) \right] \right)
\]
\[ + \left[ 4 i^2 \text{erfc}\left(\frac{L + x}{\sqrt{4v't}}\right) \right] - \frac{2w\sqrt{t'}}{k} - \frac{2w\sqrt{t'}}{k} \]

\[ \left[ 4 i^2 \text{erfc}\left(\frac{L - x}{\sqrt{4v't'}}\right) + 4 i^2 \text{erfc}\left(\frac{L + x}{\sqrt{4v't'}}\right) \right] \] \hspace{1cm} (8)

and

\[ h_2^2(x, t) = h_i^2 + \frac{w\sqrt{t}}{k} \left[ 4 i^2 \text{erfc}\left(\frac{x - L}{\sqrt{4v't}}\right) \right. \]

\[ - 4 i^2 \text{erfc}\left(\frac{x + L}{\sqrt{4v't}}\right) \left. - \frac{w\sqrt{t'}}{k} \left[ 4 i^2 \text{erfc}\left(\frac{x - L}{\sqrt{4v't'}}\right) \right] \right. \]

\[ - 4 i^2 \text{erfc}\left(\frac{x + L}{\sqrt{4v't'}}\right) \] \hspace{1cm} (9)

where

\[ t' = t - t_o \]

\[ t \] is the time from which flow started

\[ t_o \] is the time from which the flow has ceased

\[ h_i \] is the initial depth of unconfined saturated aquifer

\[ L \] is the width of recharging strip

\[ w \] is the uniform rate of vertical percolation per unit area

\[ \overline{v} = K \overline{h}/G \]
Dagan (1967) for a free surface ground-water flow due to a uniform recharge over a strip of land of finite width, proposed the following formula:

\[
\eta_1 (x, t) = -\frac{1}{\pi} \left[ \frac{x + 1}{2} \ln \frac{(x + 1)^2}{(x + 1)^2 + t^2} + \frac{x - 1}{2} \right] + \ln \frac{(x - 1)^2 + t^2}{(x - 1)^2} + t \arctan \frac{2t}{1 - x^2 - t^2} \tag{10}
\]

where

\[\eta_1 (x,t)\] is the height of the water table build-up above the initial water table

\[x = x'/L\]

\[2L\] is the width of water spreading strip

\[t = t' k/nL\]

\[n\] is the effective porosity

\[t'\] is the time after application of the water

Dagan also analyzed the water mound build-up on an initially sloping water table. From his analysis, he concluded that the difference in the elevation for the case of the initial slope of \(\beta = 0, \beta = 1,\) and \(\beta = 2\) are insignificant, and suggested that at such slopes the solution for \(\beta = 0\) could be adopted without large errors for \(\beta \neq 0.\)
Fluctuation of the water table with the presence of drains

Ferris (1950) solved Equation (2) for the case of drawdown of water table due to a single drain which has penetrated all the way through the aquifer. He assumed that the out-flow from the drain per linear foot is constant, and the initial water table is horizontal. He proposed his equation as follows

\[ S = \frac{q \cdot x}{2 \cdot T \cdot \sqrt{\pi}} \int_{x}^{\infty} \frac{e^{-u^{2}}}{u^{2}} \, du \]  

(11)

where

- \( S \) is the drawdown of the water table at the distance \( x \),
- and for the time, \( t \), after opening of drains

\[ T = kD \]

Dumm (1954) reported that Glover developed a formula for the transient case of the water table height between drains. He assumed that the water table is initially horizontal, and the height of the water table above the drains is very small with respect to the depth of the aquifer. His formula is as follows:

\[ y = y_{o} \left[ 4 \cdot \frac{\pi}{\alpha} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \cdot e^{-\frac{L^{2}}{2 \cdot \pi t}} \cdot \sin \frac{n\pi x}{L} \right] \]

(12)

where

- \( y \) is the height of the water table above the drains at the
distance $x$ from the drain, and after the time, $t$

$y_o$  is the initial height of water table above the drains

$L$  is the drain spacing

$a = KD_a/V$

$D_a = d + y_o/2$

$d$  is the depth of aquifer below the drain

For the case where $at/L^2$ approaches 0.025, all the terms of the above series, but the first one, go to zero. If $x = L/2$ in the above equation, the height of the water table midway between drains would be:

$$y_c = y_o \left( \frac{4}{\pi} \right) e^{\frac{-\alpha \pi^2 t}{L^2}}$$  \hspace{1cm} (13)

Solving Equation (13) for $L$

$$L = \pi \sqrt{\frac{k D_a t}{V \ln \left( \frac{4}{\pi} \frac{y_o}{y_c} \right)}}$$  \hspace{1cm} (14)

where

$y_c$  is the height of water table above the drains at the midpoint

Van Schilfgaarde et al (1956) reported that Glover solved the continuity Equation (1) for the case where $d = 0$, and came up with the following equation:
Solving this equation for $S$ yields the following equation:

$$S = \left[ \frac{9 k y_0 t}{2 f \left( \frac{y_0}{S} - 1 \right)} \right]^{1/2}$$  \hspace{1cm} (16)

where

$S$ is the drain spacing

$f$ is the drainable porosity

Peterson (1957) used the solution of the drawdown of the piezometric pressure of an aquifer due to a constant pumping rate from a well as follows:

$$S = \frac{Q}{4 \pi T} \left[ \int_{\frac{r^2}{4 T t}}^{\infty} \frac{e^{-u}}{u} \ du \right]$$  \hspace{1cm} (17)

where

$Q$ is the constant discharge from the well

$S$ is drawdown

$r$ is the distance of the point to the well

$s$ is specific yield of the aquifer
He suggested that the drainage of an area could be accomplished by pumping from a battery of wells in that area.

Luthin (1959a) proposed a formula for the falling water table with the following assumptions:

1. The rate of flow in the tile line is proportional to the distance of the water table above the drain \((y)\), \(q = cky\).

2. The rate of flow in the tile line is independent of the spacing of the tile drains.

3. The rate of flow is independent of the tile diameter.

4. In the first development it was assumed that the water table is flat, but an elliptical shape of water table was assumed in the second development. For the case of flat water table, he proposed the following formula:

\[
S = \frac{ck(t_2 - t_1)}{f \ln \left( \frac{y_1}{y_2} \right)}
\]  

(18)

where

- \(S\) is the drain spacing
- \(C\) is a constant and is equal to 1
- \(y_1\) is the height of the water table between drains at time, \(t_1\)
- \(y_2\) is the height of the water table at time, \(t_2\)
- \(f_a\) is the average drainable porosity

Luthin (1959b), as a result of personal communication with T. Talsma and H. C. Hasken (Luthin 1959b), found that the relationship
between the outflow from tile drains and the height of the water table at midpoint is not linear, but rather polynomial as follows:

\[ q = a y + b y^2 \quad (19) \]

where

- \( q \) is the rate of the discharge into the tile line
- \( a \) is a constant
- \( b \) is a constant
- \( y \) is the height of the water table at midpoint

Then he proposed the following formula for the tile spacing:

\[
S = \frac{a k (t_2 - t_1)}{f \ln \left( \frac{y_2}{y_1} \right)} = \frac{a k (t_2 - t_1)}{f \ln \left( \frac{a + by_1}{y_1} \right)} = \frac{a k (t_2 - t_1)}{f \ln \left( \frac{a + by_2}{y_2} \right)}
\quad (20)
\]

Isherwood (1959) solved the differential equation for the water table recession between two parallel drains by the relaxation method. He used a high speed digital computer for programming the fall of the water table. The result of the computation was presented graphically in his paper.

Maasland (1959) derived the following formulas for the case of a water table build-up due to rainfall between drains. If the rain intensity is \( P \), then the water table induced due to this recharge would be:
\[
    z = h_1 + \frac{P_c}{2} \left\{ x (L - x) - \left( \frac{8L^2}{3\pi} \right) \sum_{n=0}^{\infty} (2n+1)^3 \sin (2n+1) \pi \frac{x}{L} \exp (-\mu t) \right\}
\]

where

\[
    z \quad \text{is the height of water table above the impermeable layer}
\]

\[
    P_c = \frac{P}{kD}
\]

\[
    P \quad \text{is the constant rate of recharge}
\]

\[
    \mu = k (2n+1)^2 \left( \frac{\pi}{2} \right)^2
\]

\[
    h_1 \quad \text{is the height of the water table in the drain to the impermeable layer}
\]

Beginning with a water table which is in equilibrium with the rainfall, and suddenly the rain stops at \( t = 0 \), then the recession of the water table would be:

\[
    z = h_1 + \frac{P_c}{2} \left( \frac{8L^2}{3\pi} \right) \sum_{n=0}^{\infty} (2n+1)^3 \sin (2n+1) \pi \frac{x}{L} \exp (-\mu t)
\]

When the water table has not reached the equilibrium state with
the rainfall and at the time \( t_1 \) the rain stopped, then the height of the water table for \( t > t_1 \) would be:

\[
z = h_1 + \frac{P_c}{2} \left( \frac{8 L^2}{\pi^2} \right) \sum_{n=0}^{\infty} \frac{(2n+1)^3}{n!} \sin(2n+1) \cdot \pi \left( \frac{x}{L} \right) \cdot \left[ \exp(-\mu(t-t_1)) - \exp(-\mu t) \right]
\]

(23)

If a constant recharge \( P \) starts at \( t = 0 \), stops at \( t = t_1 \), starts at \( t = T > t_1 \), stops at \( t = T + t_1 \), and so on, the solution of the height of the water table would be:

\[
z = h_1 + \frac{P_c}{2} \left( \frac{x}{L} \right) \cdot \left[ \alpha t - \left( \frac{8 L^2}{\pi^2} \right) \sum_{n=0}^{\infty} \frac{(2n+1)^3}{n!} \sin(2n+1) \cdot \pi \left( \frac{x}{L} \right) \cdot (Q - R) \right]
\]

(24)

where

- \( \alpha, Q, \) and \( R \) are functions of \( t, t_1, T, \) and \( \mu \)
- \( r \) is the number of the periods.

Brooks (1961) presented a solution for the non-linear differential equation (1) describing unsteady flow toward an array of parallel drains. He started with an initial horizontal water table. His solution is quite involved and will not be presented. He also presented a graphical
solution for the height of the water table with time at the midpoint between drains.

For the case of tile drains located at the water table and the rise of the water between drains is due to intermittent irrigation water, Maasland (1961) proposed the following formula:

\[
h(x, t) = \left( \frac{P_c L^2}{8} \right) \frac{32}{3} \sum_{n=0}^{\infty} (2n + 1)^{-3} \sin(2n + 1)
\]

\[
\pi \left( \frac{x}{L} \right) \cdot \alpha_n \exp(-\mu_n T \xi)
\]

where

\[
P_c = P/kD
\]

\[P\] is the uniform application rate per unit area

\[
\alpha_n = (\exp \mu_n T \xi - 1)/(\exp \mu_n T - 1)
\]

\[T\] is the time interval between the consecutive recharge

\[
\xi = t/T
\]

\[
\mu_n = kD/\epsilon (2n + 1)^2 (\pi/L)^2
\]

\[\epsilon\] is the specific yield

Stallman (1962) solved the equation of drawdown of the water table in the embankment of a stream when the water in the stream suddenly drops to \(S_o\) at \(t = 0\). He assumed a horizontal water table for the initial condition in the derivation of his formula. His formula is as follows:
where

\[ S_{o} - S = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} \, du \]  

(26)

\[ S_{o} - S = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} \, du \]  

(26)

where

- \( S \) is the amount of drawdown for the distance \( x \) from the stream and time \( t \).

- \( S_{o} \) is the initial drawdown at \( x = 0 \) and \( t = 0 \).

- Other symbols as previously defined.

Van Schilfgaarde (1963) suggested the following formula for drain spacing for the falling water table case:

\[ S = 3A \left[ \frac{k (d + m) (d + m_{o}) t}{2f (m_{o} - m)} \right]^{1/2} \]  

(27)

where

- \( S \) is the drain spacing

- \( A \) is the function which needs a numerical evaluation and may be estimated from the following formula within \( \pm 3 \% \):

\[ A = \left[ 1 - \left( \frac{d}{y_{o}} \right)^2 \right]^{1/2} \]  

(28)

where

- \( m \) is the height of the water table above drains at the midpoint and time \( t \).
m_o is the height of the water table above drains for the mid-
point at t = 0

y_o = d + m_o

d is the depth of the drains to the impermeable layer

In his development, the equivalent depth (Equation (45)) which was pro-
posed by Hooghoudt (1937) was used to take care of the convergence
near the drains.

Kirkham (1964) used a physical model consisting of fictitious
membranes which were located along the streamlines in the soil to
analyze the fluctuation of the water with time. He assumed that the
water table was initially in equilibrium with the rainfall, and suddenly
the rain stops. The formula is for the fall of the water table after
cessation of rainfall and is as follows:

\[ t = \frac{\left[ (b_o - b) + 2SF(x, o) \ln \left( \frac{b_o}{b} \right) \right]}{k/f} \]  

(29)

where

t is the time after the rain has stopped

b_o is the equilibrium height of the water table at t = 0

b is the height of the water table for time t

2S is the drain spacing

F(x, o) is a function of (x, L, and r)

r is the radius of the tile drains

f is the drainable porosity
Kirkham found that the above formula is not exact, but it is on the safe side (t is greater than the actual time).

Dumm (1964) reported that Glover assumed an initial water table between drains which corresponds with a fourth-degree polynomial:

\[ y(x, o) = \frac{8H}{L^4} (L^3 x - 3L^2 x^2 + 4L x^3 - 2 x^4) \]  

(30)

where

\( H \) is the height of the water table at midpoint

Glover's formula for the recession of the water table at midpoint is:

\[
\frac{y(c, t)}{y(c, o)} = \frac{192}{\pi^3} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{n-1}{2} (-1)^{(n-1)} \frac{n^2 - 8}{n^5} \exp\left(\frac{2 \frac{2}{n} \pi \alpha t}{L^2}\right)
\]

(31)

where

\( y(c) \) is the height of the water table above the drains at midpoint

Dagan (1964) presented an approximation method for the evaluation of the water table induced by a variable discharge. In his presentation he assumed that the drains are set at the initial water table and the intensity of rainfall has been step-wise with time. A graphical solution of the water table at the midpoint with respect to
time was presented.

Ligon et al (1964) extended the steady-state water table solution of Kirkham (1958 and 1960) to the falling water table case. Kirkham's (1960) solution is:

\[ z = \frac{RS}{k(1 - R/k)} F(x, o) \]  

where

- \( z \) is the height of the water table above the drain
- \( R \) is the intensity of the rain
- \( 2S \) is the drain spacing
- \( F(x, o) \) is a function of \((x, S, r, \text{and } h_l)\)
- \( 2r \) is the bottom width of the ditch
- \( h_l \) is the height of the water in the ditch

If the rain stops after the steady-state water table has been established, Ligon assumed that the water table will fall, between the fictitious frictionless strips of thin metal which are located at the stream lines, would be at the rate of:

\[ \frac{dz}{dt} = \frac{R}{f} \]  

where

- \( f \) is the drainable pore space

Using these two equations, (32) and (33), Ligon came up with
Glover (1964) proposed the following equation for the water table between the parallel drains beginning with a horizontal water table:

\[
h = \frac{4H}{\pi} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{1}{n} e^{-\frac{n^2 \pi^2 \alpha t}{L^2}} \sin \left( \frac{n \pi}{L} x \right)
\]  

(35)

where

- \( h \) is the height of the water table above the drains
- \( x \) is the distance perpendicular to the direction of the drains
- \( H \) is the initial height of the water table above the drains

The above formula is for the case where the drains have been located above the impermeable layer. For the limit case, when the drains are located on the impermeable layer, the continuity condition is:

\[
\frac{\partial}{\partial x} \left( k h \frac{\partial h}{\partial x} \right) = V \frac{\partial h}{\partial t}
\]  

(36)

If \( H \) is the value of \( h \) for \( x = L/2 \) and \( t = 0 \), let

\[
U = \frac{h}{H}, \quad \xi = \frac{x}{L}, \quad \text{and} \quad \eta = \frac{k H t}{V L^2}
\]
then, the above differential equation becomes:

$$\frac{\partial}{\partial \xi} U \frac{\partial U}{\partial \xi} = \frac{\partial U}{\partial \eta}$$  \hspace{.5cm} (37)

A solution for Equation (37) is:

$$U = W Y$$  \hspace{.5cm} (38)

$W$ can be determined by:

$$\xi = C \int_0^W \frac{wdw}{\sqrt{1 - w^3}}$$  \hspace{.5cm} (39)

where

$$C = \frac{\Gamma(7/6)}{\sqrt{\pi} \Gamma(5/3)} = .5798$$  \hspace{.5cm} (40)

and $Y$ is:

$$Y \approx \frac{1}{9/2 \frac{\alpha t}{L^2} + 1}$$  \hspace{.5cm} (41)

A dimensionless plot of Equations (38) and (41) was presented by Glover. Van Schilfgaarde (1965) reported that Kirkham (1964) derived a falling water table equation from the potential theory as follows:
where

\[ F \] is an infinite series function of \( r/S \) and \( d/S \)
\[ r \] is the radius of the tile drain
\[ S \] is the drain spacing
\[ f \] is the drainable porosity
\[ \log \] is the natural logarithm
\[ m_o \] is the initial water table above drains
\[ m \] is the height of the water table midpoint between tiles
\[ d \] is the depth of the drains to the impermeable layer

Kirkham developed his theory on the model which involved some fictitious frictionless membranes along the initial streamlines.

The Bureau of Reclamation Staff (Luthin, 1965) proposed a formula in which the initial water table has a shape that corresponds with a fourth-degree polynomial, Equation (30). The transient equation for the height of the water table \( y \) at the distance \( x \) and the time \( t \) between two drains is as follows:

\[
y = \frac{192 H}{5 \pi} \sum_{m=0}^{\infty} \frac{(2m + 1)^2 \pi^2}{(2m + 1)^5} - 8 \exp \left( -\frac{(2m + 1)^2 \pi^2 \alpha t}{L^2} \right) \sin \left( \frac{(2m + 1) \pi x}{L} \right)
\]
Mood (1966) reported the solution of the continuity Equation (2) for the case in which the initial water table was a fourth-degree polynomial. He proposed the following formula:

\[
V = \frac{192}{\pi^5} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2n+1}{n} \right)^2 \frac{\pi^2}{5} \frac{2}{(2n+1)^5} \exp\left[-\left(\frac{2n+1}{2}\right) \frac{W}{T}\right] \cos\left(\frac{2n+1}{\pi} \right) U
\]

where

\[
V = \frac{h}{H}
\]

\[
U = \frac{x}{L}
\]

\[
W = \frac{kDt}{S L^2}
\]

\[S\] is the storage coefficient

For the depth of the aquifer, he used the equivalent depth which was proposed by Hooghoudt (1937). The equivalent depth was proposed to take care of the losses due to the convergence of the flow to drains as follows:

\[
\hat{d} = \left[1 + \frac{d}{L} \left(\frac{8}{\pi} \ln \frac{d}{a} - 3.4\right)\right]^{-1} \quad \text{for } 0 \leq \frac{d}{L} \leq .03
\]

where

\[
a\] is the radius of the tile drains
d and \( \hat{d} \) are the depth and equivalent depth of the aquifer, respectively.

Moody solved the non-linear differential equation (1) numerically and drew the following three kinds of curves relating:

1. Maximum water table height between drains,
2. rate of discharge to the drains, and
3. the volume of water removed from the region between two drains and the time.

Dylla (1966) used the Donnon steady-state equation of drain spacing:

\[
S^2 = \frac{4 \, P \, (b^2 - a^2)}{Q_d} \tag{46}
\]

where

- \( S \) is the drain spacing
- \( P \) is the permeability of the saturated media
- \( b \) is the depth of the water table to the impermeable layer
- \( a \) is the depth of the drains to the impermeable layer
- \( Q_d \) is the drainage requirement (volume of water per unit area per unit time) which was assumed to be equal to:

\[
Q_d = f \, \frac{dm}{dt} \tag{47}
\]

- \( m \) is the height of the water table above the drains
- \( f \) is the drainable porosity
Dylla combined Equations (46) and (47) and integrated to get the following formula:

\[
    t = \frac{2.3 \cdot S^2 \cdot f}{8 \cdot a \cdot P} \log_{10} \left[ \frac{m_o \cdot (m + 2m)}{m \cdot (m_o + 2a)} \right]
\]

(48)

where

\(m_o\) is the height of the water table midway between the drains at \(t = 0\)

\(m\) is the height of the water table at midpoint at time \(t\)
DERIVATION OF THE THEORY

Solution of the boundary value problem

Taking the general continuity equation of the flow of the fluid through porous media:

\[
\frac{\partial}{\partial x} \left( k H_1 \frac{\partial H_1}{\partial x} \right) = V \frac{\partial H_1}{\partial t} \tag{1}
\]

where

- \( k \) is the hydraulic conductivity
- \( H_1 \) is the height of the water table above the impermeable layer
- \( V \) is the specific yield of the aquifer
- \( t \) is the time

Assume that there is a line sink located below and close to the water table (Figure 1) in a quite deep aquifer. If the height of the water table (Figure 1) in a quite deep aquifer. If the height of the water table

![Figure 1. Drawdown of the water table due to a single drain](image-url)

Figure 1. Drawdown of the water table due to a single drain
table above the line sink, \( h \), is very small with respect to the depth of aquifer, \( D \), then \( H_1 = D + h \) and the continuity Equation (1) with the boundary and initial conditions becomes:

\[
kD \frac{\partial^2 h}{\partial x^2} = V \frac{\partial h}{\partial t}
\]  

(49)

I. C. \( h(x, 0) = H_0 \)

B. C. \[
\begin{align*}
&h(0, t) = 0 \\
&h(\infty, t) = H_0 
\end{align*}
\]

let \( S = H_0 - h \), then Equation (49) becomes:

\[
kD \frac{\partial^2 S}{\partial x^2} = V \frac{\partial S}{\partial t}
\]  

(50)

I. C. \( S(x, 0) = 0 \)

B. C. \[
\begin{align*}
&S(0, t) = H_0 \\
&S(\infty, t) = 0 
\end{align*}
\]

letting \( V/kD = a^2 \) and taking the Laplace transformation of Equation (50) with the use of the initial condition, one obtains:

\[
\frac{\partial^2}{\partial x^2} L \left[ S(x, t) \right] = a^2 s L \left[ S(x, t) \right]
\]  

(51)

the solution of Equation (51) is:

\[
L \left[ S(x, t) \right] = A(s) e^{-a\sqrt{s} x} + B(s) e^{a\sqrt{s} x}
\]  

(52)
Because \( S(x, t) \) is bounded (second boundary condition), then \( B(s) = 0; \) using the first B. C.:

\[
L \left[ S(o, t) \right] = \frac{H_0}{s} = A(s)
\]

(53)

Substitution Equation (53) into Equation (52) results in

\[
L \left[ S(x, t) \right] = \frac{H_0}{s} e^{-a\sqrt{s}x}
\]

(54)

The convolution theorem is used to solve the Equation (54):

\[
L \left[ f(t) \right] L \left[ g(t) \right] = L \left[ \int_0^t f(t - \lambda) g(\lambda) d\lambda \right]
\]

(55)

\[
L \left[ f(t) \right] = \frac{H_0}{s}
\]

(56)

\[
L \left[ g(t) \right] = e^{-a\sqrt{s}x}
\]

(57)

To find the inverse transform Equation (57), the following equation is used (Wiley, p. 402):

\[
L^{-1} \left[ e^{-b\sqrt{s}} \right] = \frac{b e^{-b^2/4t}}{2\sqrt{\pi} t^{3/2}}
\]

(58)

Using Equations (55), (56), (57), and (58), Equation (54) becomes:
\[ S(x, t) = \frac{H_0}{2\sqrt{\pi}} \int_0^t \frac{ax}{\lambda^{3/2}} e^{- \frac{a^2 x^2}{4\lambda}} d\lambda \]  

(59)

Letting \( z = \frac{a^2 x^2}{4\lambda} \), Equation (50) becomes:

\[ S(x, t) = \frac{2H_0}{\sqrt{\pi}} \int_{\frac{ax}{\sqrt{4t}}}^{\infty} e^{-z^2} dz \]  

(60)

Let

\[ a^2 = \frac{V}{KD} = \frac{1}{\alpha} \]  

(61)

Then Equation (60) becomes:

\[ S(x, t) = \frac{2H_0}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty} e^{-z^2} dz \]  

(62)

Defining:

\[ \frac{x}{\sqrt{4\alpha t}} = u \]

\[ \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-z^2} dz = \text{erfc}(u) \]

\[ \frac{2}{\sqrt{\pi}} \int_{0}^{u} e^{-z^2} dz = \text{erf}(u) \]
\[ \text{erfc}(u) = 1 - \text{erf}(u) \] (63)

Using Equations (63) and Equation (62) becomes:

\[ S(x, t) = H \text{erfc}(u) \] (64)

If

\[ h(x, t) = H - S(x, t) \] (65)

the formula for the height of the water above a single drain becomes:

\[ h(x, t) = H_o - H \text{erfc}(u) \]

or

\[ h(x, t) = H_o \text{erf}(u) \] (66)

Values of \( \text{erf}(u) \) have been tabulated (National Bureau of Standards, 1941) for different values of \( u \) and presented in Figure 2.

Consideration of having infinite series of parallel drains at distances \( L \)

Equation of drawdown for a line sink was found to be:
Figure 2. The relationship between \( \text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_{0}^{u} e^{-z^2} \, dz \) and \( u \).
\[ S (x, t) = H_0 \ \text{erfc} \ (u) \]  \hspace{1cm} (64)

To determine the drawdown of the water table at any point for an infinite series of parallel drains, the drawdown due to each drain at the point should be considered (see Figure 3), or in other words:

\[ S (x, t) = H_0 \ \sum_{n=1}^{\infty} \ \text{erfc} \ (u_n) \]  \hspace{1cm} (67)

where

\[ u_n = \frac{x_n}{\sqrt{4 \alpha t}} , \]

\[ x_n \] is the distance from the \( n \)th drain to the point considered.

As \( n \) goes to infinity, \( \text{erfc} \ (u_n) \) goes to zero. Calculation of Equation (67) shows that for \( n > 4 \) \( \text{erfc} \ (u_n) \) does not have any significant effect on \( S (x, t) \), and, therefore, \( \text{erfc} \ (u_n) \) for \( n > 4 \) will be eliminated from Equation (67) to get:

\[ S(x_1, t) = H_0 \ \sum_{n=1}^{4} \ \text{erfc} \ (u_n) \]  \hspace{1cm} (68)

Using Equations (65) and (68), the height of the water table at any point \( x_1 \) from a drain for an infinite series of parallel drains will be:

\[ h (x_1, t) = H_0 \ \left[ \text{erf} \ (u_1) - \ \sum_{n=2}^{4} \ \text{erfc} \ (u_n) \right] \]  \hspace{1cm} (69)

To simplify Equation (69), the term
Figure 3. Showing the position of an infinite series of parallel drains and their effects on the draw-down of the water table between drain numbers (1) and (2)
will be called the drainage function, \( D(u_n) \) and \( h(x, t) \) will be written as \( h(x, t) \), where \( x \) is the distance of the point to the first drain. Therefore Equation (69) becomes

\[
h(x, t) = H_0 D(u_n)
\]  

(70)

Numerical values for Equation (70) are computed and presented in Figure 4. In Figure 4, the values of \( D(u_n) \) are plotted vs. \( x/L \) for different values of \( \sqrt{4at/L} \). The solution of Equation (70) is plotted on log-log paper (Figure 5). Since Figure 5 is a dimensionless curve, it may be used in the design of drain systems.

For any given shape of the water table between drains, the initial height of a horizontal water table and the time required to reach the given shape may be found by matching of the curve of \( h \) (given height of the water table) versus \( x/L \) with Figure 5. The height of the horizontal initial water table, and the time, \( t \), required to reach the given height of the water table is called the effective initial height and the effective time for the given water table respectively.

Fluctuation of the water table at mid-point between drains may be found by substituting \( x = L/2 \) in Equation (70) to obtain:

\[
h(L/2, t) = H_0 D_1(u_n)
\]  

(71)
Figure 4. Variation of drainage function $D(u_n)$ with distance ratio $x/L$ for different values of $\sqrt{4\alpha t}/L$. 
Figure 5. Curves showing the relationship between the drainage function $D(u_n)$ and the distance ratio $x/L$ for different values of $\sqrt{\frac{4at}{L}}$. 

Solution: $h(x, t) = H_0 \cdot D(u_n)$.
Solution of:
\[ h(L/2, t) = H_0 D_1(u_n) \]
for midway of drains

Figure 6. Curve showing the relationship between the drainage function \( D_1(u_n) \) and \( \sqrt{4at/L} \) at the midpoint between drains
Values of $D_1 (u_{n_1})$, computed from Equation (71) are plotted versus $\sqrt{4\alpha t/L}$ and are shown in Figure 6.

**Determination of flow rate from drains**

The volume of the water coming out per unit length of the drain $(v_t)$ is equal to twice of the shaded area ABCD (see Figure 7).

$$v(t) = 2 \int_0^{L/2} V S(x, t) \, dx \quad (72)$$

From another point of view, the total volume of the water is equal to the integration of the discharge per unit length of drain with respect to time:
\[ v(t) = \int_{0}^{t} q(t) \, dt \quad (73) \]

where

\[ q(t) \]

is the rate of discharge per unit length of drain

Equation (72) is equal to Equation (73):

\[ 2 \int_{0}^{L/2} V S(x, t) \, dx = \int_{0}^{t} q(t) \, dt \quad (74) \]

taking the derivative of both sides of Equation (74) respect to time:

\[ q(t) = 2 \frac{d}{dt} V \int_{0}^{L/2} S(x, t) \, dx \quad (75) \]

The time derivative can be taken inside of the integration sign because the integration is with respect to \( x \) only:

\[ q(t) = 2 V \int_{0}^{L/2} \frac{d}{dt} S(x, t) \, dx \quad (76) \]

From the previous development we have:

\[ S(x, t) = H \sum_{n=1}^{4} \text{erfc}(u_n) \quad (68) \]

\[ = H \left[ 4 - \frac{2}{\sqrt{\pi}} \int_{0}^{u_1} e^{-z^2} \, dz - \frac{2}{\sqrt{\pi}} \int_{u_1}^{u_2} e^{-z^2} \, dz \right] \]
Taking the derivative of Equation (68),

\[
\frac{d}{dt} \int_{0}^{u_{3}} e^{-z^2} \, dz = \int_{0}^{u_{4}} e^{-z^2} \, dz
\]

Inserting Equation (77) into Equation (76), and integrating with respect to \( x \), then the equation of the discharge per linear foot of drain would result as follows:

\[
q(t) = H_0 \sqrt{\frac{4VT}{\pi t}} \left[ -2 e^{-\frac{L^2}{16\alpha t}} + 2 e^{-\frac{L^2}{4\alpha t}} - \frac{9L^2}{16\alpha t} + e^{-\frac{L^2}{\alpha t}} \right]
\]
To eliminate time, $t$, from the term outside of the brackets, Equation (78) will be multiplied and divided by $\sqrt{4\alpha t/L}$ to get:

$$q(t) = H_0 \frac{4T}{\sqrt{\pi L}} \left[ \frac{L}{\sqrt{4\alpha t}} \left( 1 - 2e^{\frac{L^2}{16\alpha t}} + 2e^{\frac{L^2}{4\alpha t}} ight) ight]$$

(79)

The terms inside of the brackets of Equation (79) will be called the "drainage discharge function, $q(u_n)$." Equation (79) becomes:

$$q(t) = H_0 \frac{4T}{\sqrt{\pi L}} q(u_n)$$

(80)

The solution of Equation (80) for given values of $\sqrt{4\alpha t/L}$ is shown in Figures 8 and 9. These figures enable one to determine the flow rate from a given length of drain after the elapse of a given time.

**Determination of effect of irrigation on the water table between drains**

1. Assuming that irrigation water is applied with a constant application rate of (I) ft/sec., and for a period of $t$, if the period $t$ is divided into $M$, equal intervals of $t_1$, then the height of the cube of water in the soil due to irrigation of period $t_1$ would be:

$$H_0' = \frac{I t_1}{V}$$
Solution of:

\[ q(t) = \frac{4}{\sqrt{\pi}} \frac{H_0 T}{L} q(u_n) \]

Figure 8. Curve showing the relationship between the drainage discharge function \( q(u_n) \) and \( \sqrt{4\alpha t/L} \)
Solution:

\[ q(t) = \frac{4}{\sqrt{\pi}} \frac{H_0 T}{L} q(u_n) \]

Figure 9. Curve showing the relationship between the drainage discharge function \( q(u_n) \) and \( \sqrt{4at/L} \).
If this depth of water is assumed to be applied to the soil instantaneously, and at the middle of the period $t_1$, then the problem will be changed to the recession of the water table between drains of the height $H_0'$ according to Equation (70) or:

$$ (h')_m = H_0' \left[ D(u_n) \right]_m $$

(81)
If this depth of water is assumed to be applied to the soil instantaneously, and at the middle of the period $t_1$, then the problem will be changed to the recession of the water table between drains of the height $H_0^1$ according to Equation (70) or:

$$ (h')_m = H_0^1 \left[ D(u_n) \right]_m $$

(81)

where

$(h')_m$ is the increase of the height of the water table due to the irrigation of $m$th period $t_1$

$u_n = x_n / \sqrt{4 \alpha t'}$

$t'$ is the time from the middle of the $m$th period to the end of the irrigation.

The total increase in height of the water table due to the irrigation for the whole period of $t$ would be the summation of the increments of the water table rise for different consecutive periods of $t_1$ or:

$$ h_1 = \sum_{m=1}^{M} (h')_m $$

(82)

where

$h_1$ is the total effect of irrigation on the height of the water table between drains

The height of the water table, $h$, after irrigation will be equal to the height of the water table, $h_0$, before irrigation plus $(h_1)$, the
effect of the irrigation.

The recession of the water table from then on will be according to Equation (70) in which:

\[ H_0 = H_e \]

where

- \( H_e \) is an imaginary initial water table height from which the present water table condition has developed after a time \( t_e \); \( H_e \) will be called the effective height of the water table.

If \( t_e \) is the time required for the water table to drop from \( H_e \) to the present water table and is called the effective time and if \( t_1 \) is the time after the end of irrigation, then

\[ t = t_1 + t_e \]

\( H_e \) and \( t_e \) may be found by matching of the curve of \( h \) vs. \( x/L \) for the situation after irrigation with Figure 5. When the two curves are matched, the ordinate gives the value of \( (H_e) \) and from the corresponding value of \( \sqrt{4\alpha t/L} \) by having \( \alpha \) and \( L \), the value of \( t_e \) could be found.

2. Assuming that the intake rate is variable and of the form:

\[ I(t) = c t^b \]
where

\[ I(t) \] is the rate of infiltration of the water into the soil

\[ t \] is the time

\[ b \] is a constant

\[ c \] is a constant

If a total depth of excess water, \( D \), was to be applied to the soil, then \( D \) would be equal to:

\[
D = \int_{t_1}^{t_2} c \, t^b \, dt
\]

where

\[ t_1 \] is the time required to supply the moisture deficiency of the soil profile

\[ t_2 \] is the total time of irrigation application

The effect of the irrigation water on the water table is:

\[
h_1 = \frac{1}{V} \int_{t_1}^{t_2} \frac{I(t)}{V} \, D \left( u_n \right) \, dt
\]  \hspace{1cm} (83)

where

\[ u_n = \frac{x_n}{\sqrt{4\alpha(t_2 - t_1)}} \]

Equation (83) may be integrated numerically.

Assuming that \( D \) is applied as \( M \) equal incremental depths of water \( (D') \) in \( M \) unequal time intervals of \( t_1', t_2', t_3', \ldots, t_m' \), and \( (D') \) is applied instantaneously at the middle of each period \( t_m' \). Then the
increase of the water table due to increment, $D'$, would be:

$$(h')_m = \frac{D'}{V} \left[ D \left( u_n \right) \right]_m$$

where

$(h')_m$ is the rise of the water table due to $m$th interval of irrigation

$$u_n = x_n / \sqrt{4 \alpha t'}$$

$t'$ is the time from the middle of $m$th period to the end of the irrigation period, $t$.

The total effect of irrigation water will be:

$$h_1 = \sum_{m=1}^{M} (h')_m$$

This depth of the water, $h_1$, will be added to the depth of the water table before irrigation, $h_0$, to obtain the present height of the water table, $h$:

$$h = h_0 + h_1$$

The balance of the procedure will be identical with section 1.
PROCEDURE AND DESIGN OF THE EXPERIMENT

A line source was created by constructing a 200 foot long shallow trench at the drainage experimental farm (19 T, 12 North R 1 East, Logan, Utah). To obtain a uniform application of water throughout the length of the trench, a perforated pipe was installed at the bottom of the trench and covered with pea-size gravel. The water was supplied to the perforated pipe from a nearby artesian well.

During the summer the piezometric pressure of the artesian aquifer decreased, which resulted in a decrease of outflow from the well. To control the flow and provide a constant discharge, a regulating mechanism had to be devised for the system. An overflow riser with two regulatory valves on each side was placed on the system to control the outflow (see Figure 10). The pressure head of the system could be fixed by selecting a proper height for the riser. The two valves were used to regulate minimum possible outflow from the riser. A water meter was installed in the system to record the amount of discharge.

One gallon per minute discharge was set by selecting the proper height of the riser and regulating the two valves. This device worked very well, and the rate of outflow was quite constant during the experiment.

Three rows of piezometers were installed in batteries at depths of 2, 3, 4, 5, and 9 feet at distances of 5, 10, 20, 40, and 80 feet on
Figure 10. Illustration showing the arrangement of the valves, riser, and the water meter in the system.
each side of the trench (see Figure 11). Standard 3/8 inch black iron pipe was used for the piezometers. The piezometers were installed by driving them down into the soil. A glass marble was used at the end of each piezometer to stop the soil from entering into the pipe as it was driven down. After the piezometers had reached the designed depth, the marble was pushed down 4 inches below the bottom of the piezometer by a steel rod.

A set of 9-feet deep auger holes was dug parallel to and on one side of the middle row of piezometers. The auger holes were lined with perforated plastic material to protect them and prevent caving.

Before starting the experiment, the piezometers were flushed out with a hand pump with the outlet tube extending down to the bottom of the piezometers. The pumping was continued until clear water was coming out of the piezometers. This action was accomplished to create a cavity at the bottom of the piezometers and also to open the soil pores which had been compacted during the installation of piezometers.

Initially the depth of the water table was read by an electric water level indicator, model DR-762A, soil test company. This equipment was not very satisfactory. The needle frequently getting stuck during the operation and the apparatus had to be cleaned out before each use, requiring considerable time. By using a piece of hard plastic tubing, an accurate determination of the water table could be obtained. Depth to the water table was determined by slowly blowing into the tubing which was lowered in the piezometer or in the auger hole and measuring
Figure 11. Layout of the piezometer and auger hole sets in the experimental plot.
the length of the tubing when the sound of bubbling was heard due to hitting the water table.

To begin with, the data were taken every day, but it was found that a daily reading was not necessary during the whole experiment. For the first two to four days of the experiment, a daily reading was taken, but as the experiment proceeded, the reading interval was lengthened to once or twice a week.
ANALYSIS OF DATA

Depth to the water table was measured at the piezometer, and the auger hole sets after the water was turned in the ditch in the summer of 1966 (see Tables 1 and 2). For most of the time during the experiment, the shallower piezometers were dry; so the data for the 9 feet piezometers (Table 1) were used for the analysis. The recharge to the trench was calculated from the water meter and amounted to 1.13346 x 10^{-5} cfs per linear foot of the trench.

Values of h/qx, and x^2/t were calculated for piezometers in Table 3, and for auger holes in Table 4. The results are plotted on log log paper as shown in Figures 12 and 13.

Because of the low permeability, intake rate, and the stratification of the soil, the flow pattern required some time to become established. As a result, the points corresponding to the first days of the recharge did not follow the general trend of the water table rise, and were ignored in fitting the curves.

According to Glover (1964) the height of the water table build up due to a line source is:

\[ h = \frac{qx}{2 \pi T} \sqrt{\pi} \int_{x}^{\infty} \frac{e^{-u^2}}{u^2} \, du \]  \hspace{1cm} (6)
Table 1. Depth to the water table for the 9 foot deep piezometers on the north side of the trench for different distances, x, after the water has been turned in.

<table>
<thead>
<tr>
<th>Date</th>
<th>Time</th>
<th>x = 5'</th>
<th>x = 10'</th>
<th>x = 20'</th>
<th>x = 40'</th>
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</tr>
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<td>5.46</td>
</tr>
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Table 2. Depth to the water table for the auger holes north side of trench, and for different distances, x, after the water has been turned in

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<th>Time</th>
<th>Depth to the water table in feet</th>
<th></th>
<th></th>
<th></th>
</tr>
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Table 3. Values of $h/q_x$ and $x^2/t$ for the piezometers

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</tr>
<tr>
<td></td>
<td></td>
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<td>Read $x^2/t \cdot 10^5$</td>
<td>Read $x^2/t \cdot 10^5$</td>
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<tr>
<td>h/q_x</td>
<td>h/q_x</td>
<td>h/q_x</td>
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Table 4. Values of $h/q_x$, and $x^2/t$ for the auger holes

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<th>Time Elapsed Sec.</th>
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Figure 12. Curve showing the relationship between $h/qx$ and $x^2/t$ for 9 foot deep piezometers.
Figure 13. Curve showing the relationship between \( \frac{h}{qx} \) and \( \frac{x^2}{t} \) for 9 foot deep auger holes.
rearranging Equation (6), we get:

\[ 2\pi T \frac{h}{qx} = \sqrt{\pi} \int_{x}^{\infty} \frac{e^{-u^2}}{u^2} \, du \]

let

\[ \phi \left( \frac{x^2}{4\alpha t} \right) = \sqrt{\pi} \int_{x}^{\infty} \frac{e^{-u^2}}{u^2} \, du \]

Selected numerical values of \( \phi \) for different values of \( x/\sqrt{4\alpha t} \) are given in Appendix (Table 12).

Values of the function \( \phi \) which are equal to \( 2\pi T h/qx \) were plotted against \( x^2/4\alpha t \) on the log log paper (Figure 14). A portion of Figure 14 which correlates with the selected type of soil was replotted on Figure 15.

Using the matching curve technique similar to that developed by Theis (1935) the values of \( \alpha \) and \( T \) were found by matching Figures 12 and 13 with Figure 15. The values of \( x^2/t \) and \( h/qx \) from Figure 12, and the corresponding values of the matched points from Figure 15 were recorded in Table 5. From the following relationship \( \alpha \), \( T \), and \( V \) are found:

\[ \alpha = \frac{(x^2/t)}{4\left(\frac{x^2}{4\alpha t}\right)} \]
\[ \phi = \sqrt{\pi} \int_{x/\sqrt{4\alpha t}}^{\infty} e^{-u^2/u^2} du \]
Figure 15. Selected section of Figure 14
Table 5. Values of $T$, and $V$ for the piezometers obtained from matching the curves

<table>
<thead>
<tr>
<th>Item</th>
<th>$\frac{x^2}{t}$</th>
<th>$\frac{x^2}{4\alpha t}$</th>
<th>$\frac{h}{q x}$</th>
<th>$2\pi T \cdot \frac{h}{q x}$</th>
<th>$\alpha = \frac{(x^2/t)}{4\left(\frac{x^2}{4\alpha t}\right)} \times 10^3$</th>
<th>$T = \left(\frac{2\pi T \cdot h}{2q x}\right) \times 10^4$</th>
<th>$V = \frac{T}{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.002</td>
<td>.362</td>
<td>475</td>
<td>.82</td>
<td>1.382</td>
<td>2.75</td>
<td>.1998</td>
</tr>
<tr>
<td>2</td>
<td>.00032</td>
<td>.058</td>
<td>2730</td>
<td>4.75</td>
<td>1.38</td>
<td>2.765</td>
<td>.200</td>
</tr>
<tr>
<td>3</td>
<td>.000027</td>
<td>.0049</td>
<td>13000</td>
<td>22.5</td>
<td>1.38</td>
<td>2.752</td>
<td>1.994</td>
</tr>
<tr>
<td>4</td>
<td>.003</td>
<td>.538</td>
<td>2700</td>
<td>.47</td>
<td>1.394</td>
<td>2.77</td>
<td>.1986</td>
</tr>
<tr>
<td>5</td>
<td>.00046</td>
<td>.082</td>
<td>2030</td>
<td>3.55</td>
<td>1.402</td>
<td>2.78</td>
<td>.1982</td>
</tr>
<tr>
<td>6</td>
<td>.000082</td>
<td>.0145</td>
<td>6700</td>
<td>11.5</td>
<td>1.412</td>
<td>2.735</td>
<td>.1938</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8.350</td>
<td>16.552</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.3916</td>
<td>2.758</td>
<td>1.9818</td>
</tr>
</tbody>
</table>
This analysis was repeated for the auger hole data (Table 6). The values of $\alpha$, $T$, and $V$ determined using piezometers and auger holes were found to be very close (see Tables 5 and 6). The average of these two values were used for the rest of the analysis, that is, $\alpha = 1.4147 \times 10^{-3}$ foot square per second, and $T = 2.8055 \times 10^{-4}$ foot square per second.

The average value of the specific yield was obtained by dividing the transmissivity, $T$, by $\alpha$:

$$V = \frac{T}{\alpha} = 0.1983$$
Table 6. Values of $T$ and $V$ for the auger holes obtained from matching the curves

<table>
<thead>
<tr>
<th>Item</th>
<th>$\frac{x^2}{t}$</th>
<th>$\frac{x^2}{4\alpha t}$</th>
<th>$\frac{h}{q\lambda}$</th>
<th>$2\pi \frac{T}{q\lambda}$</th>
<th>$\alpha = \frac{\left(\frac{x^2}{t}\right)}{4 \left(\frac{x^2}{4\alpha t}\right)} \times 10^3$</th>
<th>$T = \frac{2\pi \frac{T}{q\lambda}}{0.1984} \times 10^4$</th>
<th>$V = \frac{T}{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.002</td>
<td>.352</td>
<td>480</td>
<td>.85</td>
<td>1.421</td>
<td>2.82</td>
<td>.1980</td>
</tr>
<tr>
<td>2</td>
<td>.00042</td>
<td>.0805</td>
<td>2000</td>
<td>3.56</td>
<td>1.43</td>
<td>2.83</td>
<td>.198</td>
</tr>
<tr>
<td>3</td>
<td>.000021</td>
<td>.0368</td>
<td>14600</td>
<td>26</td>
<td>1.428</td>
<td>2.835</td>
<td>.1984</td>
</tr>
<tr>
<td>4</td>
<td>.0022</td>
<td>.38</td>
<td>426</td>
<td>.77</td>
<td>1.448</td>
<td>2.88</td>
<td>.1986</td>
</tr>
<tr>
<td>5</td>
<td>.00038</td>
<td>.0655</td>
<td>2320</td>
<td>4.2</td>
<td>1.45</td>
<td>2.88</td>
<td>.1985</td>
</tr>
<tr>
<td>6</td>
<td>.000029</td>
<td>.005</td>
<td>12200</td>
<td>22</td>
<td>1.45</td>
<td>2.87</td>
<td>.198</td>
</tr>
</tbody>
</table>

**TOTAL**  
8.627  
17.115

**Average**  
1.4378  
2.852  
.19835
SOME EXAMPLES OF DRAINAGE DESIGN

Design of depth and spacing of the drains when the initial water table is horizontal

1. Determine the spacing of the tile drains for an irrigated region where the horizontal water table is located 1 foot below the soil surface. It is desired to lower the water table at the point midway between drains to a depth of 4 feet in six days. The depth of the tiling machine, and other limiting factors, have dictated that the depth of drain be 8 feet. Specific yield, $V$, and the transmissivity, $T$, of the aquifer is $0.20$ and $8 \times 10^{-4}$ sq. ft./sec., respectively.

Solution: From the above information, $\alpha = T/V = 4 \times 10^{-3}$ sq. ft./sec., and $D_1(u_n) = h/H_o = 4/7 = 0.571$.

Referring to Figure 6, the relationship between the height of the water table at midpoint between drains with time, for $D_1(u_n) = 0.571$, the corresponding value of $\sqrt{4\alpha t}/L$ is found to be $0.57$. Substituting $t = 6$ days and $\alpha = 4 \times 10^{-3}$, the spacing is obtained as follows:

$$L^2 = \frac{4\alpha t}{(0.57)^2} = \frac{4(4 \times 10^{-3})(6 \times 24 \times 3600)}{0.325} = 25500$$

$L = 159.5$ say $L = 160$ feet

2. Using the above drain spacing, find the time required for
the water table to drop to a depth of 6 feet midway between drains.

Solution:

\[ D_1 \left( \frac{u_n}{h} \right) = \frac{h}{H_o} = \frac{2}{7} = 0.285 \]

From Figure 6 for \( D_1 \left( \frac{u_n}{h} \right) = 0.285 \), the value of \( \sqrt{4\alpha t/L} \) is found to be equal to 0.762. Having \( \alpha \) and \( L \), the time, \( t \), is obtained as follows:

\[
t = (0.762)^2 \frac{L^2}{4\alpha} = 0.58 \frac{25600}{4 \times 4 \times 10^{-3}} = 928,000 \text{ sec.}
\]

\[ t = 10.76 \approx 11 \text{ days} \]

Designing the size of tile drains

For the above example, determine the size of a 1/4 mile long drain which is to be installed in a region having a general slope of one percent.

Solution: By referring to Figure 4, a maximum height of the water table that might occur during the irrigation season is chosen. Assuming that the ratio of \( D_1 \left( \frac{u_n}{h} \right) = h/H_o = 0.96 \), the corresponding value of \( \sqrt{4\alpha t/L} = 0.30 \). By referring to Figure 8, the value of the drainage discharge function \( q(\frac{u_n}{h}) \) is found to be 2.92.

Having \( T = 8 \times 10^{-4} \text{ sq. ft./sec.}, \ L = 160 \text{ ft.}, \ H_o = 7 \text{ ft.} \), and the length of drain \( x = \frac{1}{4} \text{ mile} \), the total discharge would be:
\[ Q(t) = q(t) \cdot x = \frac{4}{\pi} \frac{H \cdot o \cdot T}{L} q(u_n) \cdot x \]

\[ = \frac{4}{\pi} \frac{7.0 \times 8 \times 10^{-4} \times 2.92}{160} \times \frac{5280}{4} \]

\[ = 0.3045 \text{ cfs} \]

Using Manning's formula:

\[ Q = \frac{1.49}{n} \frac{\pi}{4} \frac{d^{8/3}}{(4)^{2/3}} \frac{S^{1/2}}{4} \]

where

- \( Q \) is the discharge
- \( n \) is Manning's coefficient
- \( d \) is the diameter of the tile
- \( S \) is the slope of the tile line

For \( n = 0.011 \), the designed size of the tile drain becomes \( d = 0.3735 \) ft., (using the 6 inches).

**Assuming a curved initial water table**

If the initial water table is assumed to have some arbitrary function, for example a fourth-degree polynomial of the form:

\[ y(x, 0) = \frac{8H}{4} \left( L^3 x - 3 L^2 x^2 + 4 L x^3 - 2 x^4 \right) \quad (30) \]
Determine the time required for the water table to drop from 1 foot
depth to 4 feet in depth for the above drain spacing.

Solution: Begin with a horizontal water table as in the
previous example. The first step is to determine the time required for
the water to recede from the initial horizontal state to the fourth-degree
polynomial given. This will be determined as follows:

Rearranging Equation (30) to obtain

$$\frac{y(x,0)}{H} = 8 \left[ \left( \frac{x}{L} \right)^2 + 3 \left( \frac{x}{L} \right)^4 \right]$$

Equation (84) is solved and the values of $y(x,0)/H = D(u_n)$ are plotted
vs. $x/L$ on log log paper (Figure 16). By matching the curve of Figure
12, with the appropriate curve of Figure 5, the value of $\sqrt{4at}/L$ is found
to be .180. Using this result, the effective time required for a horizontal
water table to reach this given condition (fourth-degree polynomial) is
found as follows:

$$t_e = \frac{\sqrt{4at}/L}{4\alpha} \cdot L^2 = \frac{(.180)^2 (160)^2}{4 \times 4 \times 10^{-3}} = 51850 \text{ sec.}$$

$$= 14.4 \text{ hours, say 15 hours}$$

The time $t_e = 15$ hours, found above, is the time required for
the water table to reach the curved surface of the fourth-degree poly-
Figure 16. Profile of the hypothetical fourth-degree polynomial water table between drains

Solution of:

\[
y(x, 0) = \frac{8H}{L^4}(L^3x^3 - 3L^2x^2 + 4Lx^3 - 2x^4)
\]
nomial starting from a horizontal water surface. In the previous example the time required for the water to drop from a horizontal water table 1 foot below the soil surface to a depth of 4 feet at the midpoint of drains was six days. Thus the time required to drop for the condition assumed (1 foot to 4 feet) becomes

\[ t_1 = t - t_e = 6 \times 24 - 15 = 129 \text{ hours} \]

where

- \( t \) is the time required for the horizontal water table to drop to the present water table
- \( t_1 \) is the time required for the given water table (fourth-degree polynomial) to drop to the present water table
- \( t_e \) is the time required for the horizontal water table to drop to the given condition (fourth-degree polynomial)

**Effect of irrigation on the rate of rise of the water table between drains**

**Constant application rate.** Assume six days after the installation of the drains, 3.2 inches of water is applied to an area with an application rate of 0.2 inches per hour to leach out the excess salt as well as to supplement the moisture deficiency of the soil. If 50 percent of this water is lost due to deep percolation, then 1.6 inches of the water will be added to the water table. Assume that in the first 8 hours of irrigation the water has been used to bring the soil moisture up to field capacity,
and in the second 8 hours the irrigation has contributed to the water table.

Solution: The 1.6 inches of deep percolation will be divided into four 0.4-inch increments of water, each applied in 2-hour periods. It is assumed that each 0.4 inch of water has been applied instantaneously at the middle of each period. The height of the cube of water in the soil due to this 0.4 inches of water would be:

\[ H_0 = \frac{h}{v} = \frac{0.4}{2} = 2 \text{ inches} \]

Then the opportunity time for each increment cube of water to recede is from the middle of the period to the end of the irrigation period.

The successive increments of rise due to the first increment for a 7 hour interval, the second increment for a 5 hour interval, the third increment for a 3 hour interval, and the fourth increment for 1 hour can be calculated from Equation (70). The computation is shown in Table 7 and the effect of 0.4-inch increment for the different intervals is shown graphically in Figure 17. The total effect of the 1.6-inch application is the sum of the individual effects of each 0.4-inch increment.

The height of the water table six days after the installation of drains (Table 8) is then added to the increment height of water table due to the 1.6 inches of irrigation (Table 7). The accumulated height of the water table, \( h \), is plotted on log log paper (Figure 18) and the curve matched with Figure 5 to find the effective height, \( H_e \), and the
Table 7. Computations for the increase of height of water table between drains due to a constant application rate

<table>
<thead>
<tr>
<th>Time Elapsed (hours)</th>
<th>Height of the water table</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x = 16'</td>
</tr>
<tr>
<td>1</td>
<td>1.9942</td>
</tr>
<tr>
<td>3</td>
<td>1.8284</td>
</tr>
<tr>
<td>5</td>
<td>1.6672</td>
</tr>
<tr>
<td>7</td>
<td>1.4802</td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
<tr>
<td>(inches)</td>
<td>6.9700</td>
</tr>
<tr>
<td>(feet)</td>
<td>.5808</td>
</tr>
</tbody>
</table>

Depth of water table after six days in feet from Table 8

Figure 17. Curves showing the effect of one inch increments of irrigation water on the water table between drains for different time intervals.
Table 8. Computation for the height of the water table between
drains for $H_o = 7$ feet, $\alpha = 4 \times 10^{-3}$ ft²/sec, $L = 160$
ft, $t = 6$ days, and $\sqrt{4\alpha t/L} = .569$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x/L$</th>
<th>$D(u^-_n)$</th>
<th>$h = H_o \cdot D(u^-_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>.1</td>
<td>.1710</td>
<td>1.1970</td>
</tr>
<tr>
<td>32</td>
<td>.20</td>
<td>.3341</td>
<td>2.3387</td>
</tr>
<tr>
<td>40</td>
<td>.25</td>
<td>.4033</td>
<td>2.8231</td>
</tr>
<tr>
<td>48</td>
<td>.30</td>
<td>.4622</td>
<td>3.2354</td>
</tr>
<tr>
<td>64</td>
<td>.40</td>
<td>.5439</td>
<td>3.8073</td>
</tr>
<tr>
<td>80</td>
<td>.50</td>
<td>.5734</td>
<td>4.0138</td>
</tr>
</tbody>
</table>
Figure 18. Curve showing the variation of the water table height, $h$, between drains with $(x/L)$ when a constant application rate is applied.
effective time, $t_e$.

The value of $H_e$ and $(\sqrt{\frac{4\alpha t}{L}})$ was found to be 4.8 and .3150, respectively. Having $\alpha$, and $L$, the effective time, $t_e$, required for the water table to drop from the horizontal effective height, $H_e$, to the present height could be found:

$$t_e = \frac{(0.3150)^2 L^2}{4\alpha} = \frac{(0.0992)(25600)}{4 \times (4 \times 10^{-3})} = 159,000 \text{ sec.}$$

$$= 44.2 \text{ hours} \approx 44 \text{ hours}$$

To determine the depth of the water table after irrigation, in practice, the above calculation is not needed. A few auger holes dug perpendicular to the direction of the drains can be used to measure the water table, $h$. By plotting $(h)$ vs. $x/L$ on log log paper, $t_e$, and $H_e$ can be found as in the above example.

The recession of the water table, after irrigation, therefore, will follow equation 70 with $H_0 = H_e$ and $t = t_1 + t_e$ with $t$, $t_1$, and $t_e$ as defined earlier.

Non-uniform intake rate case. If the equation for the intake rate of the soil is:

$$D = .168 \ t^5 \quad (85)$$

where
\( D \) is the total depth of infiltrated water in inches
\( t \) is the time in minutes.

The previous example may be solved for the effective height of the water table, and the effective time as follows.

Solution: According to the above infiltration equation, the time required for 4, 5, 6, 7, and 8 increments of water to penetrate into the soil is

\[
T_4 = (1.6/0.168)^2 = 90.5 \text{ minutes}
\]

\[
T_5 = (2/0.168)^2 = 142.0 \text{ minutes}
\]

\[
T_6 = (2.4/0.168)^2 = 204.0 \text{ minutes}
\]

\[
T_7 = (2.8/0.168)^2 = 277.5 \text{ minutes}
\]

\[
T_8 = (3.2/0.168)^2 = 362.5 \text{ minutes}
\]

Then the time taken for the 5th, 6th, 7th, and 8th increment of water to penetrate into the soil would be:

\[
T_{4-5} = 142 - 90.5 = 51.5 \text{ minutes}
\]

\[
T_{5-6} = 204 - 142 = 62 \text{ minutes}
\]
As in the above example, if it is assumed that each depth of water was applied instantaneously at the middle of the period, then the elapsed time for each depth of water would be 42.5, 121.75, 190.5, and 246.25 minutes, respectively. The effect of irrigation and the accumulative depth of water table are shown in Table 9 and Figure 19.

The procedure for finding $H_e$ and $T_e$ is similar to the previous example. $H_e$ and $(\sqrt{4\alpha t/L})$ are found to be 4.8 and .315, respectively, from which $t_e$ can be calculated as follows:

$$t_e = \frac{(0.3150)^2 L^2}{4\alpha} = \frac{0.0992 \times (25600)}{4 \times 4 \times 10^{-3}} = 159,000 \text{ sec.}$$

$$= 44.2 \text{ hours} \approx 44 \text{ hours}$$
Table 9. Computations for the increase in height of the water table between drains due to a variable intake rate

<table>
<thead>
<tr>
<th>Time Elapsed (min.)</th>
<th>x = 16'</th>
<th>x = 32'</th>
<th>x = 40'</th>
<th>x = 48'</th>
<th>x = 64'</th>
<th>x = 80'</th>
</tr>
</thead>
<tbody>
<tr>
<td>42.50</td>
<td>1.9992</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>121.75</td>
<td>1.9275</td>
<td>1.9998</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>190.50</td>
<td>1.8124</td>
<td>1.9982</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>246.25</td>
<td>1.7172</td>
<td>1.9934</td>
<td>1.9999</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Total</td>
<td>7.4562</td>
<td>7.9914</td>
<td>7.9999</td>
<td>8.0</td>
<td>8.0</td>
<td>8.0</td>
</tr>
<tr>
<td>(inches)</td>
<td>.6213</td>
<td>.6659</td>
<td>.6666</td>
<td>.6666</td>
<td>.6666</td>
<td>.6666</td>
</tr>
<tr>
<td>(feet)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Depth of water table after six days in feet from Table 8

<table>
<thead>
<tr>
<th></th>
<th>x = 16'</th>
<th>x = 32'</th>
<th>x = 40'</th>
<th>x = 48'</th>
<th>x = 64'</th>
<th>x = 80'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8183</td>
<td>3.0036</td>
<td>3.4897</td>
<td>3.9020</td>
<td>4.4739</td>
<td>4.6804</td>
<td></td>
</tr>
</tbody>
</table>
Figure 19. Curve showing the variation of the water table height, $h$, between drains with $x/L$ for a variable intake rate.
RESULTS AND DISCUSSION

With the initial condition being a horizontal water table, an equation was developed for the height of water table between drains as follows:

\[ h(x, t) = H_0 \left( \text{erf}(u_L) - \sum_{n=2}^{4} \text{erfc}(u_n) \right) \]  \hspace{1cm} (69)

or

\[ h(x, t) = H_0 \, D(u_n) \]  \hspace{1cm} (70)

\( D(u_n) \) is called the drainage function and is equal to the terms in the brackets of Equation (69).

Equation (69) contains the error function which has been tabulated by the National Bureau of Standards (1941). In many respects the equation is similar to the nonequilibrium well equation. Therefore, many of the techniques developed for the flow of water to wells may be modified so as to apply to the flow of water to drains. Values of \( D(u_n) \) for different values of \( x/L \) and \( \sqrt{4at/L} \) have been computed and has been presented graphically in Figures 4 and 5. These figures may be used for design purposes.
ASSUMPTIONS

1. Soil is homogeneous and isotropic.
2. The specific yield of the media is constant.
3. Fillable and drainable porosities are equal.
4. Dupuit Forchheimer assumptions are valid.
5. The flow is laminar (Darcy's law is applicable).
6. The origin of coordinates is at the bottom of the drain or at the center of a tile drain.
7. The flow condition is a transient state.
8. The flow consists of gravitational water from dewatering of the soil voids.
9. Drains are parallel with spacing of L.
10. The water is homogeneous and incompressible.
11. The temperature is constant.
12. The aquifer is incompressible.
Dumm (1954) reported that Glover developed a formula for the height of the water table between drains beginning with a horizontal water table as follows:

\[
h = Y \frac{4}{\pi} \sum_{n=1, 3, 5, \ldots}^{\infty} \frac{1}{n} e^{-\frac{\alpha n^2 \pi^2 t}{L^2}} \sin \frac{n \pi x}{L}
\]  

Equation (12) contains a series of exponential and trigonometric functions which makes its solution rather time consuming and difficult.

The solutions of Equation (70) was compared with the solutions from Glover's formula (Equation (12)). A good agreement was found between the two formulas. The agreement was not so good for very high values of \( \sqrt{4\pi t/L} \) as is shown in Table 10.

Fluctuation of the water table at midpoint of drains may be found by setting \( x = L/2 \) in Equation (70) to get:

\[
h_{(1/2, t)} = H_o D_{1/2} (u_n)
\]  

Equation (71) is solved and plotted in Figure 6.

Glover (Luthin, 1965) solved the equation for the falling water table between drains for the case in which the initial water table was a fourth-degree polynomial. The height of the water table at the midpoint was plotted versus time for design purposes.

In general, the initial water table is not a fourth-degree polynomial. In fact, in most cases, it is different. The shape of the initial water table is not horizontal either. But by using Equation (70) and
Figure 5, it is possible to analyze any given water table by determining the effective time required to obtain the given water table starting from a horizontal water table. This can be done by matching the curve of \( h \) vs. \( x/L \) with Figure 5 to obtain the effective height \( H_e \) and the effective time \( t_e \). Then the recession of the water table will follow Equation (70) with \( H_0 = H_e \) and \( t = t_l + t_e \cdot t_1 \) is the time after measuring of the water table height, \( h \).

Flow to a drain of a parallel system was found to be:

\[
q(t) = H_o \frac{4T}{\sqrt{\pi L}} \left[ \frac{L}{\sqrt{4\alpha t}} \left( 1 - 2e^{-\frac{L^2}{16\alpha t}} \right) 
\right.
\]

\[
\left. + 2e^{-\frac{L^2}{4\alpha t}} - 2e^{-\frac{9L^2}{16\alpha t}} + e^{-\frac{L^2}{\alpha t}} \right] \quad (79)
\]

or

\[
q(t) = H_o \frac{4T}{\sqrt{\pi L}} q(u_n) \quad (80)
\]

where

\( q(u_n) \) is the drainage discharge function and is equal to the terms in brackets of Equation (79)

Drainage discharge function \( q(u_n) \) was solved for different values of \( \sqrt{4\alpha t}/L \) and presented graphically in Figures 8 and 9.

If \( L = \infty \), then Equation (79) becomes:
\[ q(t) = H_o \frac{2KD}{\sqrt{\pi \alpha t}} \quad (86) \]

and for the flow rate from one side of the drain, \( q'(t) \) is:

\[ q'(t) = H_o \frac{KD}{\sqrt{\pi \alpha t}} \quad (87) \]

Glover (1964) reported that the flow from an embankment to a reservoir when the water in the reservoir suddenly drops to \( H_o \) below the initial water surface is:

\[ F_o = \frac{H_o KD}{\sqrt{\pi \alpha t}} \quad (88) \]

which is a special form of Equation (80) or the same as Equation (87).

For design purposes, the solution of Equation (80) is presented graphically in Figures 8 and 9. The size of drains, also, can be designed for a given length of drain and slope of ground from Equation (80), or Figure 8.

The incremental rise of the water table due to irrigation was found from the equation:

\[ (h')_m = H_o' \left[ D (\omega_n) \right]_m \quad (81) \]

and
Table 10. Solutions of Glover's formula compared with the solutions of the newly developed drainage function, $D(u_n)$

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<th>New developed formula b</th>
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a Equation (12)
b Drainage function, $D(u_n)$
By superimposing this incremental rise on the preirrigation water table height, the final elevation of the water table after irrigation is determined. By matching the resultant height of water table between drains, after irrigation with Figure 5 the effective height of the initial water table, and the effective time, may be determined. The water table recession may be found from the following equation:

\[ \frac{h}{H_0} = D (u_n) \]

(70)

where

\[ H_0 = H_e \]
\[ u_n = x_n / \sqrt{4\alpha t} \]
\[ t = t_1 + t_e \]
\[ t_1 \] is the time after irrigation
\[ t_e \] is the effective time

From the analysis of data the average values of \(\alpha\), \(T\), and \(V\) were found to be equal to \(1.4147 \times 10^{-3}\), \(2.8055 \times 10^{-4}\), and \(0.1983\), respectively. Luthin (1957) stated that the assumption of a constant specific yield leads to a solution in which the shape of the water table is similar to the experiment, but the rate of the fluctuation of the water table is different. In this study, the actual fluctuation of the
water table was used to determine the specific yield of the media, and as a result, it would be the effective specific yield.
SUMMARY AND CONCLUSIONS

A drainage function $D(u_n)$ was developed to describe the shape of the water table with time between drains for an infinite series of parallel drains. The drainage function was evaluated and presented graphically for design purposes. The solution of the drainage function was compared with the solution of Glover transient state drainage equations. The two equations agreed very well, but the agreement was not as good for high values of $\sqrt{4\alpha t}/L$.

A drainage discharge function $q(u_n)$ was developed to represent the rate of discharge in a drain of an infinite series of parallel drains. A dimensionless curve of the drainage discharge function was presented for design purposes.

A method was presented to evaluate the rate of water table recession between drains for any initial water table condition.

The effect of irrigation water on the water table between drains...
SUGGESTIONS FOR FURTHER STUDY

1. An intensive field study is needed to develop the practical applications of the theoretical formulas.

2. Because the proposed field method of determining transmissivity and specific yield gives integrated values, it seems that this method could be used for any drain location above a barrier. A test needs to be made to determine the limitations in the method proposed.

3. Either a line source or a line sink may be created to find the aquifer properties. An experiment needs to be conducted to determine if the line sink parameters are more adaptable to drainage formulas.
BIBLIOGRAPHY


Table 11. Values of $\psi = \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{4at}}} e^{-u^2} du$ for given values of $\frac{x}{\sqrt{4at}}$

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a Taken from Glover (1964, pp. 49-65)
VITA

Seid Abdollah Jenab

Doctor of Philosophy

Dissertation: Development of a Drainage Function for the Transient Case, and a Two-Dimensional Ground-Water Mound Study to Evaluate Aquifer Parameters

Major Field: Irrigation and Drainage Engineering

Biographical Information:

Personal Data: Born at Esfahan, Iran, March 1936, son of Mohammad and Soghra Beighom Taghi.

Education: B.S., University of Tehran (Karaj College) Agricultural Engineering, 1959; M.S., Utah State University, Logan, Utah, Irrigation and Drainage Engineering, 1962; Ph.D., Utah State University, Logan, Utah, Irrigation and Drainage Engineering, 1967.

Teaching: Taught the following courses at the University of Tehran (Karaj College) for two years: 1. Drainage Engineering and Reclamation of Saline and Alkali Soils; 2. Land Leveling and Designing of the Irrigation Layouts; 3. Engineering Equipment and Slide Rule; and 4. Conducting the Irrigation Engineering Laboratory. Assistant Professor in the Department of Agricultural and Irrigation Engineering at Utah State University, Logan, Utah, teaching the following courses: 1. Advanced Drainage Engineering; and 2. Design of Irrigation Systems.

Research: 1. Research on M.S. thesis, "Drainage from Unsaturated Soil Profile;" 2. Evaluation and calibration of the neutron scattering moisture meter for the Karaj area; 3. Determination of flow of canals with radioisotope injection (with cooperation of the Nuclear Center, University of Tehran); 4. Evaluation of the cost and the duration of different canal lining materials;
5. Consumptive use and fertility study for different crops; and 6. Research on Ph.D. dissertation, "Development of a Drainage Function for the Transient case, and a Two Dimensional Ground-Water Mound Study to Evaluate Aquifer Parameters."

Practical Experience: 1. Surveying and planning the road and irrigation layouts of the Karaj Experiment Farm; 2. Designing and performing land leveling of Karaj Experimental Farm; 3. Conducting scientific tours for the senior students around the country (Iran); 4. Conducting research at the Utah State University Drainage Farm for the Department of Agricultural and Irrigation Engineering; 5. Evaluate the fluctuation of the ground water, hydraulic conductivity measurement, etc., of Cache Valley, Utah, with a piezometer set at the Utah State Drainage Experimental farm; 6. Setting and running of the sprinkler system of the Utah State University Experimental Drainage Farm; and 7. Evaluation of the ground-water characteristics of Lewiston for drainage design purposes.

Honors and Awards: 1. Received a four-year student fellowship at Karaj College for the B.S. degree; 2. Three years on an international cooperation administration fellowship at Utah State University for the M.S. degree; 3. Received a graduate research assistantship at Utah State University from September 1964 to June 1967; and 4. Was an Iranian delegate at the 1964 Irrigation Seminar held at New Delhi, India.

Publications: Have written six papers, am now preparing three papers, and have presented two papers at national and international affairs.