Appendix A. Taylor's Theorem and Taylor Series

Taylor's theorem and Taylor's series constitute one of the more important tools used by mathematicians, physicists and engineers. They provides a means of approximating a function in terms of polynomials. To begin, we present *Taylor's theorem*, which is an identity satisfied by any function $f(x)$ that has continuous derivatives of, say, order $(n+1)$ on some interval $a \leq x \leq b$. Taylor's theorem asserts that

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_{n+1}, \quad (A.1)$$

where R_{n+1} – the *remainder* – can be expressed as

$$R_{n+1} = \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(\xi), \quad (A.2)$$

for some ξ with $a \leq \xi \leq b$. Here we are using the notation

$$f^{(k)}(c) = \frac{d^k f}{dx^k}\Big|_{x=c}. \quad (A.3)$$

The number ξ is not arbitrary; it is determined (though not uniquely) via the mean value theorem of calculus. For our purposes we just need to know that it lies between a and b. The equation (A.1) is an identity; it involves no approximations.

The idea is that for many functions the value of n can be chosen so that the remainder is sufficiently small compared to the polynomial terms that we can omit it. In this case we get Taylor's approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n. \quad (A.4)$$

Typically, the approximation is reasonable provided x is close enough to a and none of the derivatives of f get too large in the region of interest. As you can see, if $(x-a)$ is small, *i.e.,* $x - a << 1$, successive powers of $(x-a)$ become smaller and smaller so that one need only keep a few terms in the polynomial expansion to get a good approximation.

If you can prove that

$$\lim_{n\to\infty} R_n = 0, \quad (A.5)$$

then it makes sense to consider expressing $f(x)$ as a *power series*:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(a)(x-a)^n, \quad (A.6)$$

which is known in this context as the *Taylor series* for f. (Note that here we use the definitions $0! = 1$ and $f^{(0)}(x) = f(x)$.) Normally the Taylor series of a function will converge in some neighborhood of $x = a$ and diverge outside of this neighborhood.* In any case, for a sufficiently "well-behaved" function, one can get a good approximation to it using (A.4) by keeping n relatively small. How small n needs to be depends, in large part, on how big $(x - a)$ is. Often times one can get away with just choosing $n = 1$ or perhaps $n = 2$ for x sufficiently close to a.

As a simple example, consider the sine function $f(x) = \sin(x)$. Let us approximate the sine function in the vicinity of $x = 0$, so that we are taking $a = 0$ in the above formulas. The *zeroth-order* approximation amounts to using $n = 0$ in (A.4). We get

$$\sin(x) \approx \sin(0) = 0. \tag{A.7}$$

This is obviously not a terribly good approximation. But you can check (using your calculator *in radian mode*) that if x is nearly zero, so is $\sin(x)$. A better approximation, the *first-order* approximation, arises when $n = 1$ in (A.4). We get (exercise)

$$\sin(x) \approx \sin(0) + \cos(0)x = x. \tag{A.8}$$

Again, you can check this approximation on your calculator. If x is kept sufficienty small (in radians), this approximation does a pretty good job. As x gets larger the approximation gets less accurate. For example, at $x = 0.1$ the error in the approximation is about 0.2%. At $x = 0.75$, the error is about 10%. The *second-order* approximation is identical to the first-order approximation, as you can check explicitly (exercise). The third-order approximation (exercise),

$$\sin(x) \approx x - \frac{1}{6}x^3 \tag{A.9}$$

is considerably better than the first-order approximation. It gives good results out to, say, $x = 1.7$, where the error is about 11%. Incidentally, the remainder term for the sine function satisfies (A.5) (exercise), and we can represent the analytic function $\sin(x)$ by its (everywhere convergent) Taylor series.

* Functions that can be represented by a convergent Taylor series are called *analytic*.