

What is the Spacetime Geometry of an Electromagnetic Wave?

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Supported by NSF Grant OCI-1148331

Electrovacua: Einstein-Maxwell Equations

Einstein-Maxwell Equations:

$$R_{ab} - \frac{1}{2}R g_{ab} = \kappa \left(F_{ac} F_b{}^c - \frac{1}{4}g_{ab} F_{cd} F^{cd} \right)$$

$$\nabla^a F_{ab} = 0 = \nabla_{[a} F_{bc]}$$

Example: Reissner-Nordstrom

Load in required packages.

```
[> with(DifferentialGeometry): with(Tensor):
```

Initialize manifold and define the metric.

```
[> DGsetup([t, r, theta, phi], M);  
frame name: M (1.1.1)
```

```
M > f := 1-2*m/r + q^2/r^2;  
f := 1 - \frac{2m}{r} + \frac{q^2}{r^2} (1.1.2)
```

```
M > g := evalDG(- f*dt &t dt + 1/f*dr &t dr + r^2*(dtheta &t  
dtheta + sin(theta)^2*dphi &t dphi));  
g := \frac{2mr - q^2 - r^2}{r^2} dt \otimes dt - \frac{r^2}{2mr - q^2 - r^2} dr \otimes dr + r^2 d\theta \otimes d\theta  
+ r^2 \sin(\theta)^2 d\phi \otimes d\phi (1.1.3)
```

Define the electromagnetic field.

```
M > F := evalDG(-q*sqrt(2)/r^2*dt &w dr);
```

$$F := -\frac{q\sqrt{2}}{r^2} dt \wedge dr \quad (1.1.4)$$

Verify the Einstein equations.

M > G := EinsteinTensor(g);

$$G := -\frac{q^2}{r^2(2mr - q^2 - r^2)} D_t \otimes D_t + \frac{q^2(2mr - q^2 - r^2)}{r^6} D_r \otimes D_r \quad (1.1.5)$$

$$+ \frac{q^2}{r^6} D_\theta \otimes D_\theta + \frac{q^2}{r^6 \sin(\theta)^2} D_\phi \otimes D_\phi$$

M > T := EnergyMomentumTensor("Electromagnetic", g, F);

$$T := -\frac{q^2}{r^2(2mr - q^2 - r^2)} D_t \otimes D_t + \frac{q^2(2mr - q^2 - r^2)}{r^6} D_r \otimes D_r \quad (1.1.6)$$

$$+ \frac{q^2}{r^6} D_\theta \otimes D_\theta + \frac{q^2}{r^6 \sin(\theta)^2} D_\phi \otimes D_\phi$$

M > evalDG(G - T);

$$0 \quad (1.1.7)$$

Verify the Maxwell Equations.

M > g1 := InverseMetric(g);

M > C := Christoffel(g);

$$\nabla_a F_{bc}$$

M > DF := CovariantDerivative(F, C);

$$DF := \frac{2q\sqrt{2}}{r^3} dt \otimes dr \otimes dr + \frac{(2mr - q^2 - r^2)q\sqrt{2}}{r^3} dt \otimes d\theta \otimes d\theta \quad (1.1.8)$$

$$+ \frac{(2mr - q^2 - r^2)\sin(\theta)^2 q\sqrt{2}}{r^3} dt \otimes d\phi \otimes d\phi - \frac{2q\sqrt{2}}{r^3} dr \otimes dt \otimes dr$$

$$- \frac{(2mr - q^2 - r^2)q\sqrt{2}}{r^3} d\theta \otimes dt \otimes d\theta$$

$$- \frac{(2mr - q^2 - r^2)\sin(\theta)^2 q\sqrt{2}}{r^3} d\phi \otimes dt \otimes d\phi$$

$$\nabla^b F_{bc}$$

$$\left[\begin{array}{l} \mathbf{M} > \text{ContractIndices}(\mathbf{g1}, \mathbf{DF}, [[1,2], [2,3]]); \\ \qquad \qquad \qquad 0 \, dt \end{array} \right. \quad (1.1.9)$$

$$\nabla_{[a} F_{bc]}$$

$$\left[\begin{array}{l} \mathbf{M} > \text{SymmetrizeIndices}(\mathbf{DF}, [1,2,3], \text{"SkewSymmetric"}); \\ \qquad \qquad \qquad 0 \, dt \otimes dt \otimes dt \end{array} \right. \quad (1.1.10)$$

Non-null electrovacua: Rainich Geometry

The Rainich Conditions:

$$R_h^i R_j^h - \frac{1}{4} \delta_j^i R^{hk} R_{hk} = 0, \quad R_i^i = 0, \quad R_{ij} t^i t^j > 0$$

$$\nabla_{[j} \alpha_{i]} = 0, \quad \text{where} \quad \alpha_i = \frac{\epsilon_{ijhk} R_m^j \nabla^h R^{mk}}{R_{cd} R^{cd}}$$

- Necessary and sufficient for the existence of a non-null solution of the Einstein-Maxwell solutions.
- Proof includes a construction of the EM field.

M > MetricSearch();

Example: An erroneous formula from "Exact Solutions..."

Equations (12.21) and (12.22) from *Exact Solutions of Einstein's Field Equations* for a non-inheriting electrovacuum.

$$ds^2 = a^2 x^{-2} (dx^2 + dy^2) + x^2 d\phi^2 - (dt - 2y d\phi)^2$$

$$F_{tx} = \frac{F_{xy}}{2y} = \cos\left(\frac{2 \ln(x)}{x}\right), F_{y\phi} = -\sin(2 \ln(x)).$$

- Not a solution of the Einstein-Maxwell equations.

> DGsetup([t, phi, x, y], M);
frame name: M (2.1.1)

M > omega := evalDG(dt - 2*y*dphi);
 $\omega := dt - 2y d\phi$ (2.1.2)

M > g := evalDG(a^2*x^(-2)*(dx &t dx + dy &t dy) + x^2*dphi &t dphi - omega &t omega);
 $g := -dt \otimes dt + 2y dt \otimes d\phi + 2y d\phi \otimes dt + (x^2 - 4y^2) d\phi \otimes d\phi + \frac{a^2}{x^2} dx \otimes dx$ (2.1.3)

$$+ \frac{a^2}{x^2} dy \otimes dy$$

M > RainichConditions(g);
true (2.1.4)

M > F0:=RainichElectromagneticField(g);
 $F_0 := \frac{\cos(2 \ln(x) + _CI) \sqrt{2} \sqrt{\operatorname{csgn}\left(\frac{1}{a^2}\right) + 1} \operatorname{csgn}\left(\frac{1}{x}\right)}{x} dt \wedge dx$ (2.1.5)
 $- \frac{2 \cos(2 \ln(x) + _CI) \sqrt{2} \sqrt{\operatorname{csgn}\left(\frac{1}{a^2}\right) + 1} \operatorname{csgn}\left(\frac{1}{x}\right) y}{x} d\phi \wedge dx$

$$-\frac{\sin(2 \ln(x) + _CI) \sqrt{\frac{x^2}{a^4}} a^2 \sqrt{2} \sqrt{\operatorname{csgn}\left(\frac{1}{a^2}\right) + 1} \operatorname{csgn}\left(\frac{1}{x}\right)}{x} d\phi \wedge dy$$

M > F := simplify(F0) assuming a > 0, x > 0;

$$F := \frac{2 \cos(2 \ln(x) + _CI)}{x} dt \wedge dx - \frac{4 \cos(2 \ln(x) + _CI) y}{x} d\phi \wedge dx - 2 \sin(2 \ln(x) + _CI) d\phi \wedge dy \quad (2.1.6)$$

• The "Exact solutions..." EM field is off by some signs, factors of 2, and component labelings.

$$F_{tx} = \frac{F_{xy}}{2y} = 2 \cos\left(\frac{2 \ln(x)}{x}\right), F_{y\phi} = -2 \sin(2 \ln(x))$$

Null Electrovacuum Geometry

$$\text{Null Maxwell Field: } F_{ab} F^{ab} = 0 = F_{ab} \star F^{ab}$$

$$\text{Electrovacuum: } R_{ab} R^{bc} = 0.$$

Are there analogs of the Rainich conditions for solutions of the Einstein-Maxwell equations with a null electromagnetic field?

cf. Jordan and Kundt (1961), Robinson (1961), Geroch (1966), Ludwig (1970).

Main Result

Theorem: *Necessary and sufficient conditions on a metric for it to define a null electrovacuum.*

- $R_{ab} R^{bc} = 0 \Leftrightarrow R_{ab} = k_a k_b$ where $k_a k^a = 0$. ("Pure radiation" spacetime.)
- k is tangent to a shear-free, null, geodesic congruence. (cf. Robinson 1961)
- $k \wedge dk \neq 0$ (twisting/rotating) - 3 additional conditions on the congruence

(CGT, 2012)

or

- $k \wedge dk = 0$ (twist-free/irrotational) - 1 additional condition on the congruence (CGT, 2012)

Examples: Twist-free case

- One additional condition on the shear-free, null geodesic congruence.
- Introduce any null tetrad adapted to the congruence: (k, l, m, \bar{m}) .
- In terms of the Newman-Penrose formalism:

$$\Re \left[(\delta + \beta - \bar{\alpha})(\bar{\tau} - 2\bar{\beta}) - \frac{1}{2}(\epsilon - \bar{\epsilon})(\mu - \bar{\mu}) \right] = 0$$

- This condition is independent of the choice of null tetrad.

Example 1: A homogeneous null electrovacuum

Originally thought to be the only homogeneous, pure radiation spacetime NOT to admit an electromagnetic source.

Define the chart and metric.

```
> DGsetup([u, v, x, y], M);  
frame name: M (3.2.1.1)
```

```
> g := evalDG(dx &t dx + dy &t dy + 2*du &s dv - 2*exp(2*s*x)*  
du &t du);  
g := -2 e2*s*x du ⊗ du + du ⊗ dv + dv ⊗ du + dx ⊗ dx + dy ⊗ dy (3.2.1.2)
```

The Ricci tensor is null, $R_{ab}R^{ac} = 0$.

```
M > Ricci := RicciTensor(g);  
Ricci := 4 s2 e2*s*x du ⊗ du (3.2.1.3)
```

```
M > TensorInnerProduct(g, Ricci, Ricci, tensorindices = [1])  
;  
0 du ⊗ du (3.2.1.4)
```

The Ricci tensor can be expressed as $R_{ab} = \frac{1}{4}k_a k_b$ for a null vector field k .

$$\begin{aligned} \text{M} > \mathbf{K} &:= \text{DGzip}([\mathbf{K1}, \mathbf{K2}, \mathbf{K3}, \mathbf{K4}], [\mathbf{du}, \mathbf{dv}, \mathbf{dx}, \mathbf{dy}], \text{"plus"}); \\ &K := K1 du + K2 dv + K3 dx + K4 dy \end{aligned} \quad (3.2.1.5)$$

$$\begin{aligned} \text{M} > \mathbf{EQTensor} &:= \text{evalDG}(\text{Ricci} - 1/4*\mathbf{K} \&t \mathbf{K}); \\ \mathbf{EQTensor} &:= - \left(\frac{K1^2}{4} - 4 s^2 e^{2sx} \right) du \otimes du - \frac{K1 K2}{4} du \otimes dv \end{aligned} \quad (3.2.1.6)$$

$$- \frac{K1 K3}{4} du \otimes dx - \frac{K1 K4}{4} du \otimes dy - \frac{K1 K2}{4} dv \otimes du - \frac{K2^2}{4} dv \otimes dv$$

$$- \frac{K2 K3}{4} dv \otimes dx - \frac{K2 K4}{4} dv \otimes dy - \frac{K1 K3}{4} dx \otimes du$$

$$- \frac{K2 K3}{4} dx \otimes dv - \frac{K3^2}{4} dx \otimes dx - \frac{K3 K4}{4} dx \otimes dy - \frac{K1 K4}{4} dy \otimes du$$

$$- \frac{K2 K4}{4} dy \otimes dv - \frac{K3 K4}{4} dy \otimes dx - \frac{K4^2}{4} dy \otimes dy$$

$$\begin{aligned} \text{M} > \mathbf{DGsolve}(\mathbf{EQTensor}, \mathbf{K}, \{\mathbf{K1}, \mathbf{K2}, \mathbf{K3}, \mathbf{K4}\}); \\ &\{4 \text{RootOf}(_Z^2 - e^{2sx}) s du\} \end{aligned} \quad (3.2.1.7)$$

$$\begin{aligned} \text{M} > \mathbf{Kdown} &:= \text{simplify}(\text{allvalues}(\mathbf{DGsolve}(\mathbf{EQTensor}, \mathbf{K}, \{\mathbf{K1}, \\ &\mathbf{K2}, \mathbf{K3}, \mathbf{K4}\}))[1], \text{symbolic}))[1]; \\ &Kdown := 4 e^{sx} s du \end{aligned} \quad (3.2.1.8)$$

$$\begin{aligned} \text{M} > \mathbf{Kup} &:= \text{RaiseLowerIndices}(\text{InverseMetric}(\mathbf{g}), \mathbf{Kdown}, [1]); \\ &Kup := 4 e^{sx} s D_y \end{aligned} \quad (3.2.1.9)$$

Verify the result.

$$\begin{aligned} \text{M} > \text{evalDG}(\text{Ricci} - 1/4*\mathbf{Kdown} \&t \mathbf{Kdown}); \\ &0 du \otimes du \end{aligned} \quad (3.2.1.10)$$

$$\begin{aligned} \text{M} > \text{TensorInnerProduct}(\mathbf{g}, \mathbf{Kdown}, \mathbf{Kdown}); \\ &0 \end{aligned} \quad (3.2.1.11)$$

It now follows that

$$f_{ab} = k_{[a} s_{b]}$$

has the correct energy-momentum tensor, where s_a is any unit vector orthogonal to k_a .

Construct a spacelike unit vector orthogonal to k^a .

$$\begin{aligned} \text{M} > \mathbf{S} &:= \text{DGzip}([\mathbf{s1}, \mathbf{s2}, \mathbf{s3}, \mathbf{s4}], [\mathbf{du}, \mathbf{dv}, \mathbf{dx}, \mathbf{dy}], \text{"plus"}); \\ &S := s1 du + s2 dv + s3 dx + s4 dy \end{aligned} \quad (3.2.1.12)$$

$$\begin{aligned} \text{M} > \mathbf{eq1} &:= \text{TensorInnerProduct}(\mathbf{g}, \mathbf{Kdown}, \mathbf{S}); \\ & \end{aligned} \quad (3.2.1.13)$$

$$eq1 := 4 e^{s x} s s2 \quad (3.2.1.13)$$

$$\begin{aligned} \text{M} > eq2 := \text{TensorInnerProduct}(g, S, S) - 1; \\ eq2 := 2 e^{2 s x} s2^2 + 2 s2 s1 + s3^2 + s4^2 - 1 \end{aligned} \quad (3.2.1.14)$$

$$\begin{aligned} \text{M} > \text{DGsolve}([eq1, eq2], S, [s1, s2, s3, s4]); \\ \{s1 du + \text{RootOf}(_Z^2 + s4^2 - 1) dx + s4 dy\} \end{aligned} \quad (3.2.1.15)$$

A simple solution is therefore:

$$\begin{aligned} \text{M} > S := \text{Tools:-DGsimplify}(\text{eval}(S, \{s1=0, s2=0, s3=1, s4=0\})) \\); \\ S := dx \end{aligned} \quad (3.2.1.16)$$

$$\begin{aligned} \text{M} > f := \text{SymmetrizeIndices}(\text{Kdown \&t } S, [1,2], \\ \text{"SkewSymmetric"}); \\ f := 2 e^{s x} s du \otimes dx - 2 e^{s x} s dx \otimes du \end{aligned} \quad (3.2.1.17)$$

Verify the result.

$$\begin{aligned} \text{M} > \text{EnergyMomentumTensor}(\text{"Electromagnetic"}, g, c*f); \\ 4 c^2 e^{2 s x} s^2 D_v \otimes D_v \end{aligned} \quad (3.2.1.18)$$

$$\begin{aligned} \text{M} > \text{EinsteinTensor}(g); \\ 4 s^2 e^{2 s x} D_v \otimes D_v \end{aligned} \quad (3.2.1.19)$$

- The electromagnetic field, if one exists, must be related to f_{ab} by a "local" duality rotation,

$$\phi : M \rightarrow R,$$

$$F = \cos(\phi) f + \sin(\phi) *f.$$

- The Maxwell equations impose conditions on k and ϕ .
- The conditions on k give the shear-free, geodesic condition.
- The conditions on ϕ are then

$$\frac{1}{i} \delta\phi + \tau - 2\beta = 0, \quad \frac{1}{i} \bar{\delta}\phi - \bar{\tau} + 2\bar{\beta} = 0, \quad \frac{1}{i} D\phi - \varepsilon + \bar{\varepsilon} = 0.$$

The integrability conditions give the remaining conditions on the congruence.

- We now compute the properties of the congruence associated to the vector field k^a .

First we verify the geodesic equation $k \cdot \nabla k = 0$.

$$\text{M} > \text{ContractIndices}(\text{Kup}, \text{CovariantDerivative}(\text{Kup},$$

$$\text{Christoffel}(g), [[1,2]]); \quad 0 D_u \quad (3.2.1.20)$$

Next we check that the congruence is shear-free and twist-free.

$$\begin{aligned} & \mathbf{M} > \text{CongruenceProperties}(g, \text{Kup}); \\ & \text{table}(["\text{ShearNormSquared}" = 0, "\text{Expansion}" = 0, "\text{RotationNormSquared}" = 0, \\ & \quad "\text{Raychaudhuri}" = 0]) \end{aligned} \quad (3.2.1.21)$$

Thus k^a is tangent to a congruence which is geodesic, shear-free, non-expanding, and twist-free. (This can also be checked using the Newman Penrose spin coefficients, computed below.)

We now consider the final condition on the congruence. We begin by constructing a null tetrad whose first leg is k^a and then compute its Newman-Penrose spin coefficients and directional derivatives.

$$\begin{aligned} & \mathbf{M} > \text{OT} := \text{DGGramSchmidt}([D_v, D_u, D_x, D_y], g, \text{signature}=[\\ & \quad [1,-1], 1, 1]); \\ & \text{OT} := \left[\frac{\sqrt{2}}{2} D_u + \frac{\sqrt{2} (e^{2sx} + 1)}{2} D_v, -\frac{\sqrt{2}}{2} D_u \right. \\ & \quad \left. - \frac{\sqrt{2} (e^{2sx} - 1)}{2} D_v, D_x, D_y \right] \end{aligned} \quad (3.2.1.22)$$

$$\begin{aligned} & \mathbf{M} > \text{NT0} := \text{NullTetrad}([\text{OT}[2], \text{OT}[3], \text{OT}[4], \text{OT}[1]]); \\ & \text{NT0} := \left[D_v, -D_u - e^{2sx} D_v, \frac{\sqrt{2}}{2} D_x + \frac{1}{2} \sqrt{2} D_y, \frac{\sqrt{2}}{2} D_x \right. \\ & \quad \left. - \frac{1}{2} \sqrt{2} D_y \right] \end{aligned} \quad (3.2.1.23)$$

$$\begin{aligned} & \mathbf{M} > \text{NT} := \text{NullTetradTransformation}(\text{NT0}, "\text{boost}", 4*\exp(s*x)* \\ & \quad s); \\ & \text{NT} := \left[4 e^{sx} s D_v, -\frac{e^{-sx}}{4s} D_u - \frac{e^{sx}}{4s} D_v, \frac{\sqrt{2}}{2} D_x + \frac{1}{2} \sqrt{2} D_y, \right. \\ & \quad \left. \frac{\sqrt{2}}{2} D_x - \frac{1}{2} \sqrt{2} D_y \right] \end{aligned} \quad (3.2.1.24)$$

Verify that NT is a null tetrad.

$$\begin{aligned} & \mathbf{M} > \text{TensorInnerProduct}(g, \text{NT}, \text{NT}); \\ & \end{aligned} \quad (3.2.1.25)$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (3.2.1.25)$$

Compute the Newman-Penrose spin coefficients and directional derivatives.

```
M > NPS:=NPSpinCoefficients(NT);
NPS := table  $\left( \begin{array}{l} \text{"sigma"} = 0, \text{"gamma"} = 0, \text{"nu"} = \frac{\sqrt{2}}{16s}, \text{"tau"} = 0, \text{"mu"} = 0, \text{"pi"} \\ = 0, \text{"rho"} = 0, \text{"epsilon"} = 0, \text{"alpha"} = \frac{s\sqrt{2}}{4}, \text{"lambda"} = 0, \text{"kappa"} = 0, \\ \text{"beta"} = \frac{s\sqrt{2}}{4} \end{array} \right)$  (3.2.1.26)
```

```
M > NPD := NPDirectionalDerivatives(NT):
```

Check the integrability condition:

$$\operatorname{Re} \left[(\delta + \beta - \bar{\alpha})(\bar{\tau} - 2\bar{\beta}) - \frac{1}{2}(\epsilon - \bar{\epsilon})(\mu - \bar{\mu}) \right] = 0.$$

```
M > alpha :=NPS["alpha"]: beta:=NPS["beta"]: epsilon:=NPS
["epsilon"]: mu:=NPS["mu"]: tau := NPS["tau"]:
M > X := simplify(tau - 2*beta) assuming s::real:
M > X1 := simplify(conjugate(X)) assuming s::real:
M > Y := simplify(beta - conjugate(alpha)) assuming s::real:
M > Z := simplify((epsilon - conjugate(epsilon))*(mu -
conjugate(mu))) assuming s::real:
```

```
M > NPD["delta"](X1) + Y*X1 - Z;
0 (3.2.1.27)
```

To illustrate the invariance of the integrability condition with respect to the choice of tetrad adapted to k , we consider a new tetrad obtained from NT by a combination of a spatial and null rotation.

```
M > NT1 := convert(NullTetradTransformation
(NullTetradTransformation(NT, "null rotation", u, "L"),
"spatial rotation", x + y), exp) assuming u::real;
```

We check that NT1 is in fact a null tetrad.

```
M > TensorInnerProduct(g, NT1, NT1);
```

We verify the integrability conditions exactly as before.

```

M > NPS1:=NPSpinCoefficients(NT1);
M > NPD1 := NPDirectionalDerivatives(NT1):

M > alpha1 :=NPS1["alpha"]: beta1:=NPS1["beta"]: epsilon1:=
NPS1["epsilon"]: mu1:=NPS1["mu"]: tau1 := NPS1["tau"]:
M > X := simplify(tau1 - 2*beta1) assuming s::real, x::real,
y::real, u::real:
M > X1 := simplify(conjugate(X)) assuming s::real, x::real,
y::real, u::real:
M > Y := simplify(beta1 - conjugate(alpha1)) assuming
s::real, y::real, x::real, u::real:
M > Z := simplify((epsilon1 - conjugate(epsilon1))*(mu1 -
conjugate(mu1))) assuming s::real, x::real, y::real,
u::real:
M > NPD1["delta"](X1) + Y*X1 - Z;

```

We now construct the electromagnetic field F . We build the solution ϕ to

$$\frac{1}{i}\delta\phi + \tau - 2\beta = 0, \quad \frac{1}{i}\bar{\delta}\phi - \bar{\tau} + 2\bar{\beta} = 0, \quad \frac{1}{i}D\phi - \varepsilon + \bar{\varepsilon} = 0.$$

```

M > NPSvalues := {tau = NPS["tau"], beta = NPS["tau"],
epsilon = NPS["epsilon"]}:
M > Phi := phi(u,v,x,y);
                                Phi := phi(u, v, x, y) (3.2.1.28)

```

```

M > eq1 := eval(1/I*NPD["delta"](Phi) + tau - 2*beta,
NPSvalues);
eq1 := -I \left( \frac{\sqrt{2} \left( \frac{\partial}{\partial x} \phi(u, v, x, y) \right)}{2} + \frac{I\sqrt{2} \left( \frac{\partial}{\partial y} \phi(u, v, x, y) \right)}{2} \right) - \frac{s\sqrt{2}}{2} (3.2.1.29)

```

```

M > eq2 := eval(-1/I*NPD["bardelta"](Phi) + conjugate(tau) -
2*conjugate(beta), NPSvalues) assuming s::real;
eq2 := I \left( \frac{\sqrt{2} \left( \frac{\partial}{\partial x} \phi(u, v, x, y) \right)}{2} - \frac{I\sqrt{2} \left( \frac{\partial}{\partial y} \phi(u, v, x, y) \right)}{2} \right) - \frac{s\sqrt{2}}{2} (3.2.1.30)

```

```

M > eq3 := eval(1/I*NPD["D"](Phi) - epsilon + conjugate
(epsilon), NPSvalues);
eq3 := -4 I e^{s x} s \left( \frac{\partial}{\partial v} \phi(u, v, x, y) \right) (3.2.1.31)

```

```

M > phisol:=pdsolve({eq1, eq2, eq3}, Phi);
phisol := {phi(u, v, x, y) = s y + _FI(u)} (3.2.1.32)

```

The electromagnetic field is obtained from the duality rotation of

$f_{ab} = k_{[a} s_{b]}$ given by ϕ :

$$F_{ab} = \cos(\phi) f_{ab} - \sin(\phi) * f_{ab}$$

```
M > F0 := evalDG(cos(Phi) * convert(f, DGform) - sin(Phi) *
  HodgeStar(g, convert(f, DGform), detmetric=-1));
F0 := 2 cos(phi(u, v, x, y)) e^{sx} s du \wedge dx - 2 sin(phi(u, v, x, y)) e^{sx} s du \wedge dy (3.2.1.33)
```

```
M > F:=eval(F0, phisol);
F := 2 cos(s y + _F1(u)) e^{sx} s du \wedge dx - 2 sin(s y + _F1(u)) e^{sx} s du \wedge dy (3.2.1.34)
```

We verify that (g, F) satisfy the Einstein-Maxwell equations.

The Einstein equations:

```
M > G := EinsteinTensor(g);
G := 4 s^2 e^{2sx} D_v \otimes D_v (3.2.1.35)
```

```
M > T := EnergyMomentumTensor("Electromagnetic", g, F);
T := 4 s^2 e^{2sx} D_v \otimes D_v (3.2.1.36)
```

```
M > evalDG(G - T);
0 (3.2.1.37)
```

The Maxwell equations:

```
M > MatterFieldEquations("Electromagnetic", g, F);
theta D_u, theta du \wedge dv \wedge dx (3.2.1.38)
```

- Notice that the electromagnetic field involves an arbitrary function of u , which labels the null hypersurfaces generated by k^a . This is a general feature of null electrovac spacetimes when k^a is twist-free. In the twisting case the electromagnetic field involves only one free parameter (specifying a global duality rotation).

- All homogeneous, pure radiation spacetimes are electrovacua.

- Notice also that this electromagnetic field does not inherit the symmetries of the metric.

```
M > KV := KillingVectors(g);
KV := [-y D_v + u D_y, D_y, -s u D_u + s v D_v + D_x, D_u, D_v] (3.2.1.39)
```

```
M > LieDerivative(D_y, g);
(3.2.1.40)
```

$$0 \, du \otimes du \quad (3.2.1.40)$$

```
M > LieDerivative(D_y, F);
```

$$-2 \sin(sy + _FI(u)) e^{sx} s^2 du \wedge dx - 2 \cos(sy + _FI(u)) e^{sx} s^2 du \wedge dy \quad (3.2.1.41)$$

```
M > g;
```

$$-2 e^{2sx} du \otimes du + du \otimes dv + dv \otimes du + dx \otimes dx + dy \otimes dy \quad (3.2.1.42)$$

```
M > F;
```

$$2 \cos(sy + _FI(u)) e^{sx} s du \wedge dx - 2 \sin(sy + _FI(u)) e^{sx} s du \wedge dy \quad (3.2.1.43)$$

Example 2: A pure radiation Robinson-Trautman spacetime

```
> restart;
```

```
> with(DifferentialGeometry): with(Tensor): with(Tools):
```

Here we consider a class of pure radiation solutions of Robinson-Trautman type (See Theorem 28.6 of [2]). We determine under what conditions these solutions are in fact solutions to the Einstein-Maxwell equations.

Initialize the manifold, define the metric, and check that it is of pure radiation type.

```
> DGsetup([u, r, x, y], N);
```

```
N > g := evalDG(2/r*(-f(x)*diff(f(x), x, x)*r + diff(f(x),
```

```
    x)^2*r + m(u))*du &t du - 2*du &s dr + r^2/2/f(x)^2*(dx
```

```
    &t dx + dy &t dy));
```

```
N > Ricci := RicciTensor(g);
```

```
N > TensorInnerProduct(g, Ricci, Ricci, tensorindices = [1])
```

```
;
```

This is a pure radiation spacetime only when the u-u component of Ricci is positive, $R_{uu} > 0$,

which we shall assume henceforth. Define $R_{uu} = \frac{1}{4}\psi^2$.

```
N > psi := 2*sqrt(Hook([D_u, D_u], Ricci));
```

We can easily read off the covariant form of the preferred null vector field k .

```
N > Kdown := evalDG(psi*du);
```

```
N > evalDG(Ricci - 1/4*Kdown &t Kdown);
```

The contravariant form of k is as follows.

```
N > Kup := RaiseLowerIndices(InverseMetric(g), Kdown, [1]);
```

We now build a null tetrad NT adapted to k and compute its Newman-Penrose spin coefficients and directional derivatives.

```
N > OT0 := DGGramSchmidt([D_u, D_r, D_x, D_y], g, signature
```

```

= [[-1, 1], 1, 1]);
N > NT0 := simplify(NullTetrad([OT0[1], OT0[3], OT0[4], -OT0
[2]]), symbolic);
N > NT := NullTetradTransformation(NT0, "boost", psi);
N > Spin := NPSpinCoefficients(NT):
N > NPdiff := NPDirectionalDerivatives(NT):

```

We check that the congruence is indeed geodesic and shearfree: $\kappa = 0$ and $\sigma = 0$. In addition, the spin coefficient ρ is real, indicating the congruence tangent to k is twist-free.

```

N > Spin["kappa"];
N > Spin["sigma"];
N > Spin["rho"];

```

We now compute and solve the geometric condition on the spacetime (in the twist-free case), denoted `int_cond`, which indicates whether the spacetime admits an electromagnetic source.

```

N > X := simplify(Spin["tau"] - 2*Spin["beta"]):
Y := simplify(Spin["epsilon"] - DGconjugate(Spin
["epsilon"])):
Z := simplify(Spin["mu"] - DGconjugate(Spin["mu"])):
W := simplify(DGconjugate(Spin["beta"]) - Spin["alpha"]):
:
eq:=NPdiff["bardelta"](X) + W*X - 1/2*Y*Z:
int_cond := DGRe(eq);

```

We solve for functions f and m such that the integrability condition holds. This command may take a few minutes to complete.

```

N > PDETools:-Solve(int_cond);

```

Evidently, for generic f this pure-radiation Robinson-Trautman spacetime does not admit an electromagnetic source. There exists an electromagnetic source for the choice $f = a e^{bx}$, with $m(u)$ remaining arbitrary. Here is the metric and Ricci tensor in this case.

```

N > g2 := eval(g, f(x) = a*exp(b*x));
N > RicciTensor(g2);

```

We see that this is a pure radiation solution provided $\frac{dm}{du} < 0$. We now construct the corresponding electromagnetic field from a duality rotation determined by the equations

$$\frac{1}{i}\delta\phi + \tau - 2\beta = 0, \quad \frac{1}{i}\bar{\delta}\phi - \bar{\tau} + 2\bar{\beta} = 0, \quad \frac{1}{i}D\phi - \epsilon + \bar{\epsilon} = 0.$$

```

N > X := simplify(Spin["tau"] - 2*Spin["beta"]):

```

```

Y := simplify(DGconjugate(Spin["tau"]) - 2*DGconjugate
(Spin["beta"]));
Z := simplify(Spin["epsilon"] - DGconjugate(Spin
["epsilon"]) - 1/2*(Spin["rho"] - DGconjugate(Spin
["rho"]))):

```

```

N > solutions := {f(x) = a*exp(b*x)}:
NP1:=simplify(1/I*NPdiff["delta"](phi(u,r,x,y)) + X):
NP2:=simplify(-1/I*NPdiff["bardelta"](phi(u,r,x,y)) + Y)
:
NP3:=simplify(1/I*NPdiff["D"](phi(u,r,x,y)) - Z):
simplify(eval({NP1, NP2, NP3}, solutions)):
phisol := pdsolve(simplify(eval({NP1, NP2, NP3},
solutions)), {phi(u,r,x,y), m(u)});

```

The only non-trivial solution is therefore $\phi = -by + h(u)$, with $m(u)$ remaining arbitrary. The electromagnetic field is constructed as follows.

```

N > phi := -b*y + h(u):
k := eval(RaiseLowerIndices(g, NT[1], [1]), solutions):
S := eval(eval(evalDG(1/sqrt(2) * (exp(I*phi) * NT[3] +
exp(-I*phi) * NT[4])), solutions):
s := eval(RaiseLowerIndices(g, S, [1]), solutions):
F := convert(evalDG(1/2*k &w s), DGform) assuming r > 0;

```

We verify that g_2 and F do define a solution to the Einstein-Maxwell equations.

```

N > MatterFieldEquations("Electromagnetic", g2, F);
N > evalDG(EinsteinTensor(g2) - EnergyMomentumTensor
("Electromagnetic", g2, F));

```

This solution is (isometric to) the general form of the Petrov type D, Robinson-Trautman null electrovacuum [2].