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## Spock, Euler, and Madison: Graph Theory in the Classroom

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# **SPOCK, EULER, AND MADISON: GRAPH THEORY IN THE CLASSROOM**

by

**Michael Buhler**

**Thesis submitted in partial fulfillment  
of the requirements for the degree**

of

**DEPARTMENTAL HONORS**

in

**Mathematics Education  
in the Department of Mathematics & Statistics**

**Approved:**

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Logan, UT**

**Spring 2013**

**Abstract**

This work is an attempt to accomplish two main objectives. First is to encourage secondary students to engage in the kinds of mathematical reasoning skills that will be necessary to them when they move to college math classes. My experience in college, along with that of many others is that “school math,” with its obsession with calculations and memorization, is dreadfully insufficient in preparing students for the proof and reasoning-based classes they will face in high school. This is an attempt to integrate some of those reasoning skills into high school courses using graph theory as a vehicle.

The second objective is to give more students greater access to this kind of mathematical thinking. This is accomplished by using several alternative teaching methods and strategies that are designed to engage students in learning as well as providing enough variety to the students that they are able to remain interested in the mathematics they are doing. Graph theory is an excellent vehicle with which to do this principally because it is hardly used in schools today. The thought of being able to do math that none of their colleagues has done before is motivating to students. Furthermore it is empowering to those students who are traditionally portrayed as being “unable to do math.” Putting these students on the cutting edge should have the result of giving these students a second chance to fall in love, or at least fall out of hate, with mathematics and have a higher chance of success in school and college. This same strategy will also have the same result of boosting the skills and performance of those students who are considered to be excellent students, further preparing them in new ways for their college courses.

In the end, it is my goal not only to show how graph theory can help teachers help their student succeed, but also to show them that teaching in this way is simple and something teachers can do without taking up too much of their precious time and patience. I believe that

teachers will find that these resources will be valuable in boosting academic performance for more students and better preparing them for college.

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## Methodology

### A. Introduction

#### a. Overview

This work is an attempt to accomplish two main objectives. First is to encourage secondary students to engage in the kinds of mathematical reasoning skills that will be necessary to them when they move to college math classes. My experience in college, along with that of many others is that “school math,” with its obsession with calculations and memorization, is dreadfully insufficient in preparing students for the proof and reasoning-based classes they will face in high school. This is an attempt to integrate some of those reasoning skills into high school courses using graph theory as a vehicle.

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#### b. Motivation

My motivation for developing this system has been my experience working at a local alternative high school. I had heard from my professors that students become much more engaged and therefore learn more when they are taught using reasoning-based strategies, when they are presented with challenging and perplexing situations, and when teachers recognize their

needs and strive actively to meet them. Conversely, I was told that traditional methods of “plug-and-chug,” memorization, and drilling were not effective in neither a pedagogical nor a motivational aspect. I did well in school, having been taught traditionally so I was inclined not to believe that what I knew about school may have been wrong. However, when I began to work at the alternative school, my perspective began to change.

In my job at the alternative school, where the student population had either been kicked out of, or chose to leave the local public schools, I had a chance to work with many students who, for one reason or another, just didn't get math. In the beginning, working with these students was very challenging. There were multiple motivational problems as most of them had been accustomed to hating math to the point that some would rather not graduate than have to do their math homework. Their attitude about math was about to destroy their future. As I began to work with these students, I began to see that it wasn't just their fault that they hated math. Math had been presented to them for so long in a way that reminded them of the authority and culture that they were trying to rebel from. The math that was being taught to them wasn't relevant to their lives and it wasn't presented to them in ways that piqued their interests.

On one memorable occasion, I was sitting in the tutoring center observing two students having a conversation about professional basketball. The students were discussing who was the best player in the NBA and were citing statistics to back up their claims. They were discussing ratios such as points per game, rebounds per game, and so on, as if they were NBA commentators. These kids knew more about basketball than some of their teachers knew about math. At this moment it hit me: these kids know math, they have simply never been taught math in a way that connected with them. At that point I made it my mission to help students access the innate abilities that students have and channel them in the direction of mathematical

reasoning. I wanted to make mathematics exciting for them and help them make important decisions using these skills.

As I began to work on this project of graph theory lessons, I was discussing it with one of the teachers at the school who I had been working closely with. I told her about a lesson I had developed on map coloring (see lesson plan 4 below). It was the end of the term and didn't have anything planned so she let me present my lesson. The class I taught was one of the most boisterous and loud classes in the entire school. I was worried. I had students discover the relationship of colorability and then I gave them the challenge to draw a map that required five colors. As you will find out when you look at the lesson plan, all contiguous maps can be colored by four colors or less. I knew this but the students didn't. Knowing this and the fact that students always enjoy proving their teachers wrong, I made the following wager to my students: "I will give you all the money in my wallet (which amounted to about \$1.25, being a poor college student) to the first person who can make a map that cannot be colored using four colors or less." The students set out for the final hour of the class to try to prove me wrong. Not a single student was disengaged. They were furiously drawing their maps. When the class came to an end, no one had been able to win my money and so much to my amazement, for the first time in the history of the school, the students asked if they could keep working on this as homework. Using a conceptually simple but pedagogically rich graph theory problem, I was somehow able to get an entire class of "problem" students excited about math. At this moment I became convinced that my previous epiphany with the NBA discussion was true: students really can learn math if it is presented in a reasoning-based, relevant, and interesting way.

c. Framework

The upcoming explanations as well as all the lesson plans are largely based on the work of Jim Cangelosi in his book *Teaching Mathematics in Secondary and Middle School: An Interactive Approach*. This book, an excellent reference for any math teacher, is the theoretical foundation for my work. Most importantly, it provides the basic format of the objectives of each lesson and the focus on higher learning levels such as Construct a Concept, Discover a Relationship, Comprehension and Communication, and Application. (Cangelosi, pp. 176-178)

The rest of this essay will divide my work on the lesson plans into two domains: mathematical thinking elements, or the implicit content I am trying to get across through graph theory, and pedagogical elements, or the strategies I implement to make math relevant, interesting and engaging for my students. I will explain why each is important and how it is employed in my lesson plans.

## B. Mathematical Thinking Elements

### a. Mathematical History

Mathematical history, like any history, provides contextual insights into the why and how behind mathematics. This context, when taught properly can help students see the big picture and make math make sense. For example, many students ask why prime numbers cannot be negative. The answer is simply that when the ancient Greeks were developing number theory, they didn't believe that negative numbers existed. Finally the students have a reason and an answer and can move on to doing something with the math. In these lesson plans there is an entire lesson (lesson 7) devoted to the history of graph theory, specifically Euler's Bridges of Konigsberg problem.



This lesson not only provides the historical underpinnings behind graph theory but also gives students a great opportunity to read a historical mathematical text and interpret it.

#### b. Definitions

In this series of lesson plans, definitions are the basic test. The use of precise terminology and a strong understanding of important mathematical concepts are key to real mathematical thinking. Being able to understand the intricacies of mathematical language is one of the main objectives of this course. However, even more important is that students are able to formulate their own definitions of important concepts. Lessons 1-3, which provide the foundational knowledge for the rest of the course, are built on definitions. In one of the homework problems in lesson three, students are asked to examine a 3-by-3 bipartite graph and decide whether or not it is a planar graph. In so doing students are formulating in a deeper way what a planar graph really is.

#### c. Logic and Proof(Deductive Reasoning)

Deductive reasoning, in the form of logic and proof is the absolute foundation of mathematics. Without it, mathematics would not be stable and upper-level mathematics classes would not function. This knowledge is essential to success in college math. Furthermore, the reasoning skills used in mathematics are useful in all aspects of life. A simple proof about loops is introduced in the first lesson and proofs are essential throughout the entire course and are most fully developed in lesson three. Furthermore, deductive reasoning, combined with inductive reasoning, often leads to the creation of basic algorithms that gives math its practical power. In the map coloring lesson, students work to create an algorithm to most efficiently color a map.

#### d. Inductive Reasoning

Without inductive reasoning and pattern recognition, we would have nothing in math for us to try and deduce and prove. Conjecture forming is the more creative companion of deduction. Separate, mathematics cannot progress, together it knows no bounds. Our treatment of inductive reasoning takes full force in lesson 5 and in it students are asked to discover general rules from specific situations in a variety of ways.

e. Abstraction and Extrapolation from previous knowledge

Helping students activate what they already know in order to make math meaningful is one of the central goals of this project. The constructivist theory of learning basically says that learning is all about reconciling new knowledge with what students already know. This isn't normally done well in many math classrooms. If students have strong conceptual knowledge, teachers can easily help them draw important connections with related concepts, thus cementing both concepts into the mind. The first exercise in lesson eight leads students to use their previous knowledge of graph concepts such as degree, path, and connectivity, add direction, and create the new concepts associated with directed graphs and tournaments.

f. Making Math Relevant to Student Interests and Knowledge

This is the other key objective of this project. One way of achieving this successfully is to integrate examples that students are interested in and creating problems related to those interests. I include a variety of different problems based on construction (lesson 3), aviation (lesson 1), computer science (lesson 1), history (lesson 2), shopping (lesson 2), and sports (lesson 9). Furthermore I provide opportunities, in the form of projects, in lessons eight and ten to connect what they have learned about graph theory to things that are important to each individual student.

g. Data Analysis

The analysis and use of data is arguably the most important and powerful practical aspect of mathematics. Most decisions in life have some mathematical component. However, more and more, data is being interpreted incorrectly because there is so much data out there. Teaching students how to use data correctly and be able to think critically about which data to use when is exceedingly important. The NFL activity in lesson nine gives has students engage in various kinds of data analysis and has them think about which is best in certain situations.

#### h. Creative Expression of Mathematics

In my experience, many students who do not do well in mathematics perform this way because they are more comfortable being creative instead of thinking convergently. Providing creative outlets for these students is key to helping make math meaningful to these students. Lesson ten gives students this opportunity. Furthermore for all students, expressing what they have learned creatively acts a great way to assess how well students understand in a way that keeps the students engaged and provides variety.

### C. Pedagogical Elements

#### a. Open-Ended Questions and Inquiry Based Instruction

On the pedagogical side of things, inquiry-based instruction has been shown time and time again to be the most effective way of producing reasoning-level learning in students. All too often students are asked questions that have fixed answers and thus require them to act as machines and not thinkers. In every lesson plans, I use open-ended questions to allow students to get the conversation going and help them think through the math themselves. This is key to the constructive approach I mentioned above.

#### b. Sharing and Social Learning

Along with the questioning mentioned above, having students learn with each other is key to effective learning. Most research shows that students best construct knowledge socially, known as social constructivist theory. Therefore the majority of the projects, assignments, and activities have been designed for students to do in groups and they guide students to discuss the mathematics together. Working in this manner, students are able to take responsibility for their learning and for helping each other learn, making our job of teaching so much easier.

#### c. Multi-modal Instruction

Just as every fingerprint in the world is unique, so is every student. Thus every student learns in different ways. Taking this into account is the key to effective teaching. By varying the means of instruction, teachers are able to reach the greatest number of students and get them engaged. To this end I have included traditional lecture, group discussion, experimentation, independent research, historical reading, videos, charts, verbal discussions, and even music to help reach out to each student. I also hope that by doing it, I can help students learn to learn in many different ways rather than staying in their own learning comfort zone.

#### d. Perplexity as a motivational tool

I think that the most important pedagogical tool that I have developed in this project is the use of perplexity as a motivational tool. Students have a native curiosity that is piqued when they are perplexed. When they are solving a challenging problem, students are forced to access prior knowledge and use their coping mechanisms to do whatever has to be done to achieve the task and thus their reasoning abilities are enhanced. As we saw in my experience with map coloring, when students are interested in something that perplexes them, they remain engaged and classroom management ceases to be an issue.

### e. Interdisciplinary Learning

Interdisciplinary learning is increasingly becoming a priority in most schools. It provides students with the opportunity to make connections between what they learn in each subject. This makes learning more relevant and thus more engaging for the students. In lesson eight, students are given the opportunity to make connections between directed graphs and the other subjects, such as English, government, history, and chemistry.

### D. Conclusion

Graph theory is the mathematics of connections. Our world is full of connections. Because of this, graph theory is one of the most applicable forms of mathematics. This makes graph theory so attractive to teach in a secondary classroom. Another attraction of graph theory is that it is conceptually simple. Students can easily understand a series of dots and lines. They are easy concepts to graph but at the same time solving basic problems can be exceedingly challenging. This challenge provides opportunities for students to build their mathematical reasoning skills and prepare themselves for college mathematics.

On a different level, these challenges serve to perplex students, which keeps them engaged and thus helps clean up many classroom management issues. Furthermore, when students learn to flex their mathematical muscles with graph theory, students, especially those who normally have trouble with math, they become empowered by learning a “new” type of math that their peers don’t understand. This empowerment can give these students the second chance that they need to succeed in math and in life.

Thus on both a cognitive and a metacognitive level, students can benefit from the mix of mathematical reasoning and good pedagogy that I have strove to implement through graph

theory. Helping students gain success is the number one goal for teachers and it has been my first goal in creating this course. Who would have thought that a bunch of dots and lines could go so far in helping our students reach new heights?

## **Lesson Plans and Activities**

### **Lesson 1: Graph Theory Basics**

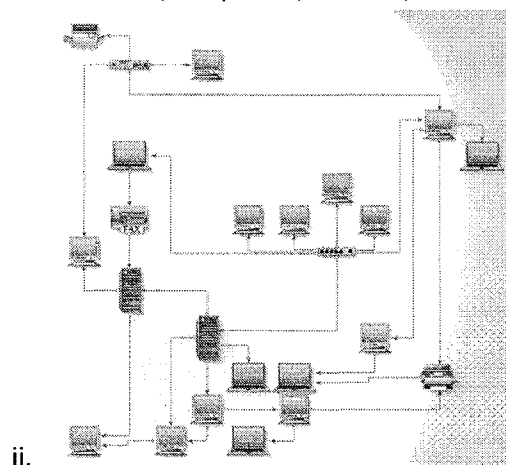
**Lesson Goals:** The goal of this lesson is to introduce the basic concept of a graph as well as its various parts, and the relationships of adjacency and degree through both theoretical representations and real-life use of graphs.

#### **Lesson Objectives:**

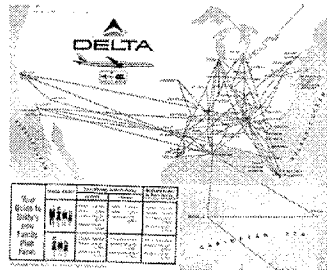
- 1. Students will begin to understand the concept of a graph (Construct a Concept--15%)**
- 2. Students will learn about the basic history of Graph Theory (Simple Knowledge--5%)**
- 3. Students will learn how graphs are used in real-life situations (Application—30%)**
- 4. Students will understand the relationships of degree and adjacency (Discover a Relationship—20%)**
- 5. Students will learn to find adjacencies and degrees in graphs. (Algorithmic Skill—10%)**
- 6. Students will be introduced to proofs and will be comfortable writing and understanding definitions (Comprehension and Communication—20%)**

#### **A. What is a graph?**

- a. Definition: A set of objects where some pairs of the objects are connected by links
- b. Examples
  - i. Computer Networks—How are the different components of a network(Computers, Routers, Servers, Printers, etc.) connected?



- iii. Family Trees—How is everyone in a family related to each other?(i.e. the British Royal Family)
- iv. Air travel—How can you get from one city to another on a certain airline?



- v. What other examples of graphs can you think of?
  - 1. **Homework:** (This homework is designed to get students to solidify the concept of graphs in their minds as well as to consider the relevance of graphs to their lives. After completing the assignment, engage students in a discussion of their graphs.) Find or Draw 5 Different Examples of Graphs and Be Ready to Share Them with Your Classmates

#### B. Parts of a graph:

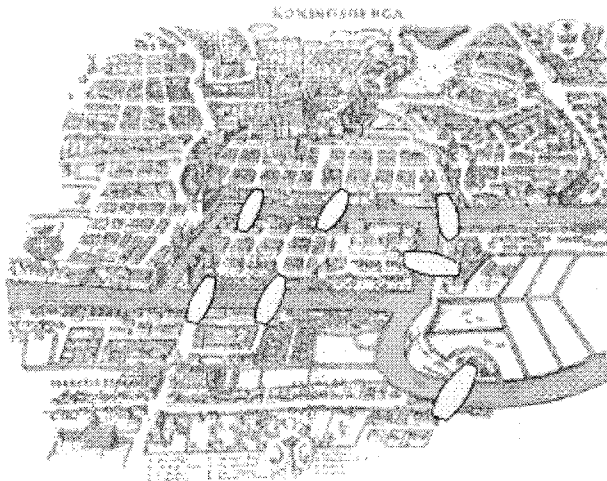
- a. Vertex: The dot that represents an object
  - i. Examples of vertices: a single computer on a network, an individual in a family tree(i.e. Prince Charles in the British Royal Family), a city on an airline map
  - ii. Representation:
    - 1. Graphically—a dot
    - 2. Symbolically—usually a letter or a number
- b. Edge: The link that connects two objects(or an object to itself)
  - i. Examples: an internet connection between two computers, the bloodline connecting two generations, a flight between two cities,
  - ii. Representation:
    - 1. Graphically—a line segment connecting two dots
    - 2. Symbolically—an ordered pair of two letters or numbers that correspond the two vertices being linked by an edge(i.e. (A,C)) or sometimes simply two letters or numbers written side-by-side (i.e. AC)

#### C. Graph Theory History:

- a. Leonhard Euler(1707-1783):



- i. Swiss Mathematician after whom  $e$  is named



- ii.

#### Seven Bridges of Königsberg:

1. Euler was walking in Königsberg and wondered: "Is it possible to walk through the city by crossing each bridge once and only once? Answer: NO (we'll learn why later using graphs 😊)"
2. Euler published a paper about how he answered this question and this led to the invention of Graph Theory.
3. We'll look at what Euler discovered later on in the class

#### D. Adjacency and Degree

- a. Adjacency: Definition—Two vertices are adjacent if they are connected by an edge

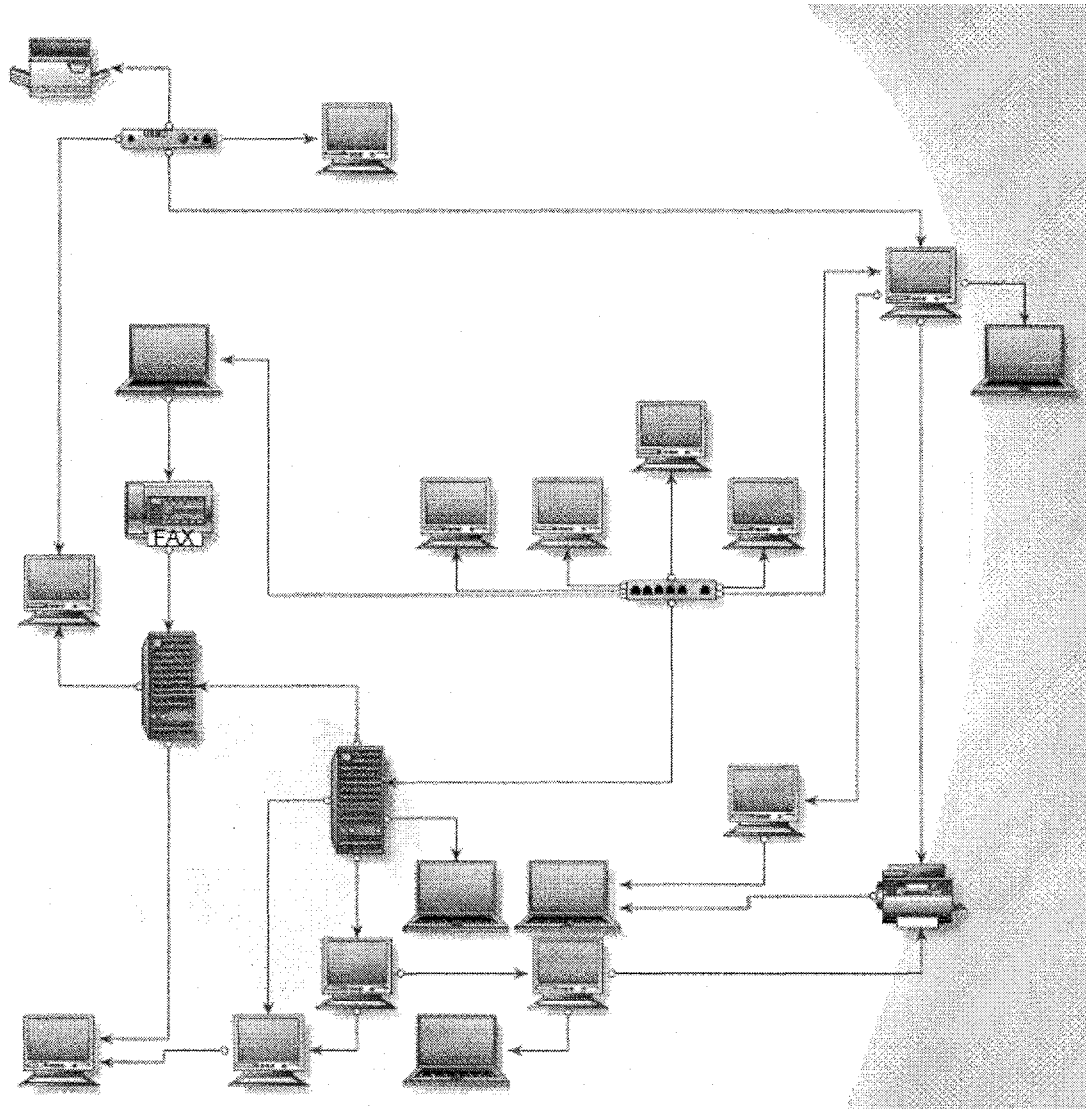
##### i. Examples

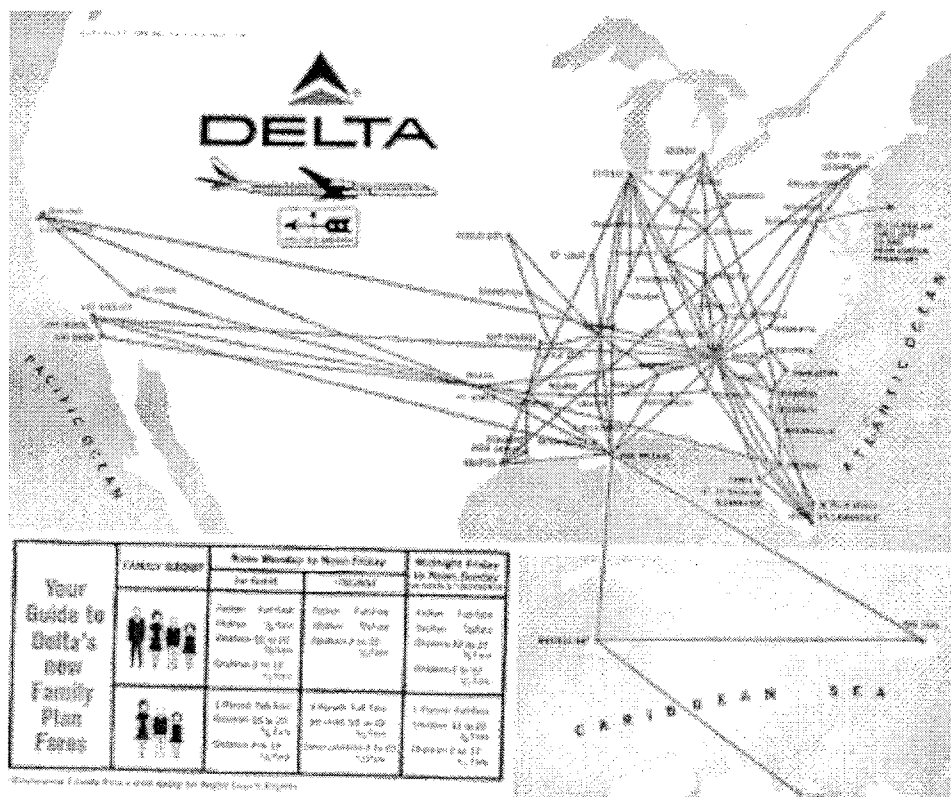
1. In the computer network below, which components are adjacent (connected to) the router?
2. In the Delta Airlines route map below, what two cities are adjacent to Las Vegas? What does it mean for two cities to be adjacent on this map?

##### b. Degree:

- i. Question: In the Delta Airlines route map, how can we tell which airport is the busiest?
  1. Answer: The airport that has the most flights coming in and going out of it.
- ii. Definition: The degree of a vertex is the number of edge ends at that vertex.







### iii. Examples

1. What is the degree of the router of the above diagram of a computer network?
2. What is the degree of Chicago? New Orleans?

### iv. Loops

1. Question: How would we represent on the airline map a sightseeing trip that takes off in New York, flies above the city to see the Statue of Liberty, and then lands at the same airport in New York?
  - a. Answer: This would be a loop.
2. Definition: A loop is a edge that connects a vertex to itself
3. Question: By how much does the degree of a vertex increase when a loop is added?
  - a. Claim: A loop increases the degree of a vertex by 2. Now prove it
  - i. When we prove things in mathematics, all we have to do is give a convincing argument as to why something is true. However, if anyone can show that our reasoning is wrong, then our argument isn't valid and we have no proof. If the argument is valid then what we have proved is known as a theorem.
  - ii. Proofs in math generally start from the basis of definitions, axioms (statements whose truth is

assumed but not proven) and previously proved theorems

b. Proof of our claim:

- i. We know that degree is defined as the number of edge ends at a vertex. Also we know that a loop is an edge that has both ends at the same vertex. We also know that an edge has 2 ends. Therefore a loop adds a degree of two to the vertex it is a part of.
- ii. We successfully proved this claim so we can call it a theorem. ! 😊

c. **Homework for Part D on Adjacency and Degree:** (For part one, the goal is to help students begin to communicate mathematically. Place emphasis on getting students to realize that math is not just computation but also communication, especially writing. Engage students in discussion of their definitions and ask them why they wrote their definitions the way they did. The purpose of part ii is to expand on the goals of the first homework assignment. Engage in similar discussion expanding on the questions asked in the homework prompt.)

- i. Rewrite the definition of Degree using the word adjacency instead of edges.
- ii. Search the internet for an airline route map and print it off to bring to class.

Then answer the following questions:

1. For this airline find a city with degree of at least 5 and say what cities are adjacent to it. Then draw a simple graph showing the adjacencies and also express each adjacency as an ordered pair. Then write a sentence or two about what this all means in real life.
2. Next, pick three cities on the map that you think are the busiest. Then calculate the number of cities that fly into each one of these cities to decide which one is the busiest. Next, use the mathematical terms of adjacency and degree to explain your conclusion.
3. Finally, how would you express on the map a one way flight from one city to another that does not return to the first city? (This will be very important later on in the class.)

## Lesson 2—Paths, Circuits and Connectivity

**Lesson Goal:** Students will begin to understand the concepts of Path, Cycles, Connectivity, and Subgraphs as well as to begin to communicate mathematically and to realize that mathematics can be used in real life.

**Lesson Objectives:**

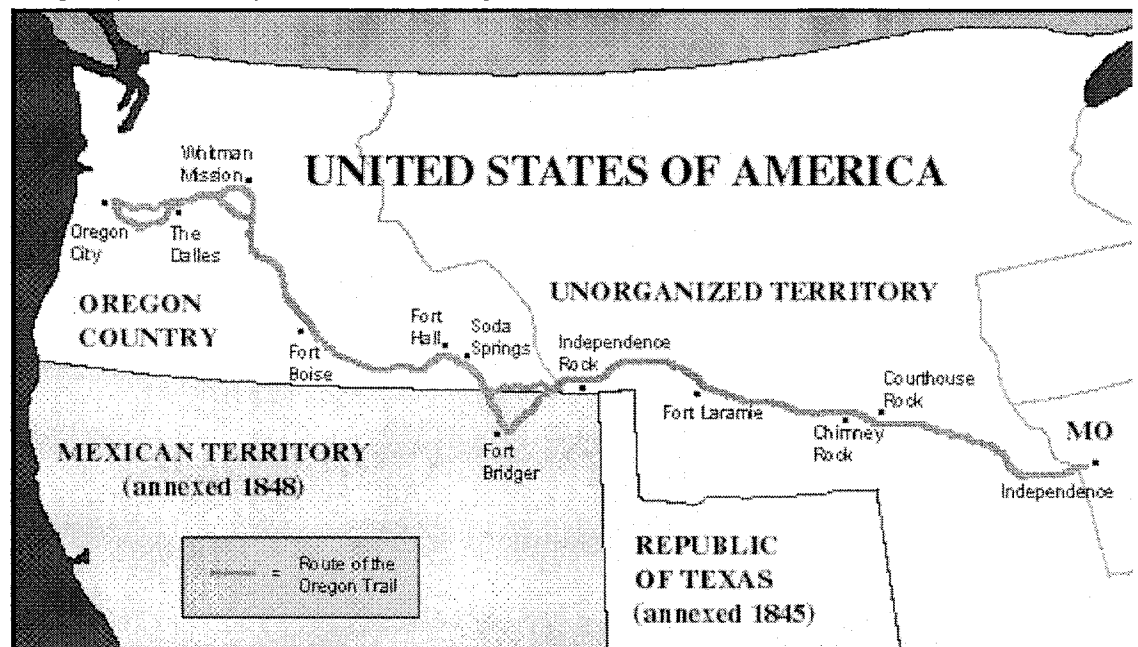
1. Students will understand the concept of Paths (Construct a Concept-20%)

2. Students will be able to identify paths, cycles, and subgraphs in real-life situations (Application--10%)
3. Students will be able to explain the relationship between a path and a cycle (Comprehension and Communication/ Discover a Relationship--5%)
4. Students will be able to calculate the lengths of paths and cycles (Algorithmic Skill--10%)
5. Students will be able to understand the concept of Cycles (Construct a Concept--20%)
6. Students will be able to understand the concept of Subgraphs (Construct a Concept--20%)
7. Students will be able to draw conclusions about and explain clearly paths, cycles, and subgraphs. (Comprehension and Communication—15%)

I. Paths

a. What is a path?

i. Imagine you were a pioneer on the Oregon Trail.

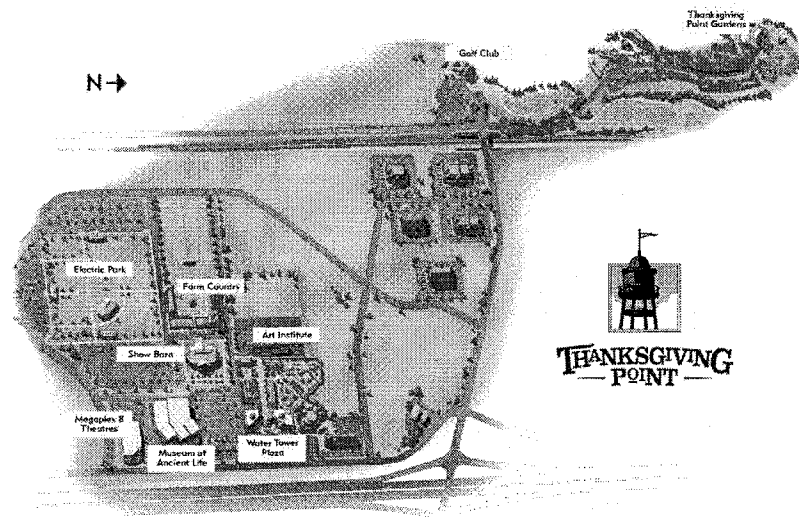


- a. If we made a graph of the points on the Oregon Trail, how would each of the following be represented:
  - a. Stops on the trail—Vertices
  - b. Trail between two stops—Edges
- b. Obviously this map has a length and goes has a starting location and a destination. In graph theory we would call the route that the pioneers took from Independence to Oregon a **path**
  - a. Definition of **path**: A **path** is a connected series of edges
  - b. The first vertex in a path is called a **Start Vertex** and the last is called an **End Vertex**. Together Start and End Vertices are called **Terminal Vertices**.
  - c. A **Simple Path** is a path with no repeated vertices.
  - d. the **length** of a path is the number of edges that the path traverses.

- e.
- c. If the path from Independence to Oregon has a length, what is that length?
  - a. In Graph Theory we often measure length in terms of the number of edges between the terminal vertices.
    - i. Paths are often distinguished from other paths by their length. A path of length 3 is called a 3-path, of length 7 a 7-path and so on. For example, the path between Independence, MO and Ft. Laramie is a 3-path because it traverses 3 edges of the trail.
- d. **Homework: (This homework would be best composed as a work sheet with the map of the Oregon Trail at the top and the questions below, giving ample space for explanation of parts c, dii, diii and div. After completing homework engage students in discussion of their work. Use part div as a lead in to the section about cycles.)**
  - a. What is the length of the graph of the Oregon Trail, according to the map above?
  - b. Please find a 10-Path, a 6-path, and a 3-path on the Oregon Trail so that your three paths all have different terminal vertices.(Express your path as an ordered n-tuple with each city being a "coordinate." For example a 3-path could be (Soda Springs, Fort Hall, Fort Boise))
  - c. Explain why can't we call (Fort Hall, The Dalles, Ft. Laramie) a path?
  - d. Create a graph of your route from the front door of the school to each of the rooms you have been to during the course of a typical Monday.
    - i. What is the length of the path you took?
    - ii. Is it a simple graph or not? Why or why not?
    - iii. Describe your path from lunch until you leave school for the day. (Length, simple/not simple)
    - iv. What could you say about your path if you went in the same place in the morning as you left in the afternoon to go home?

## II. Cycles/Circuits

- a. Imagine that we have gone on a field trip to Thanksgiving Point

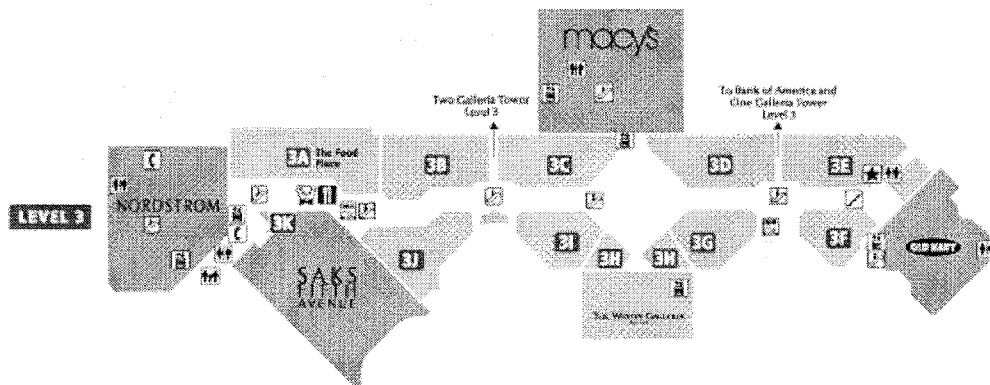
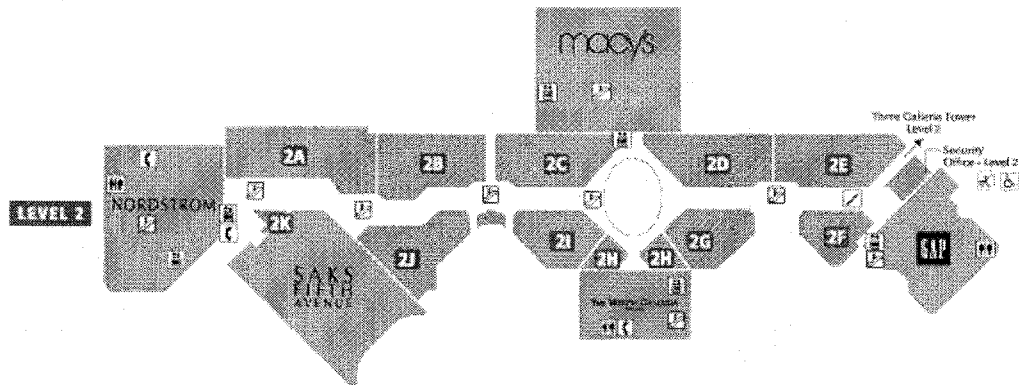
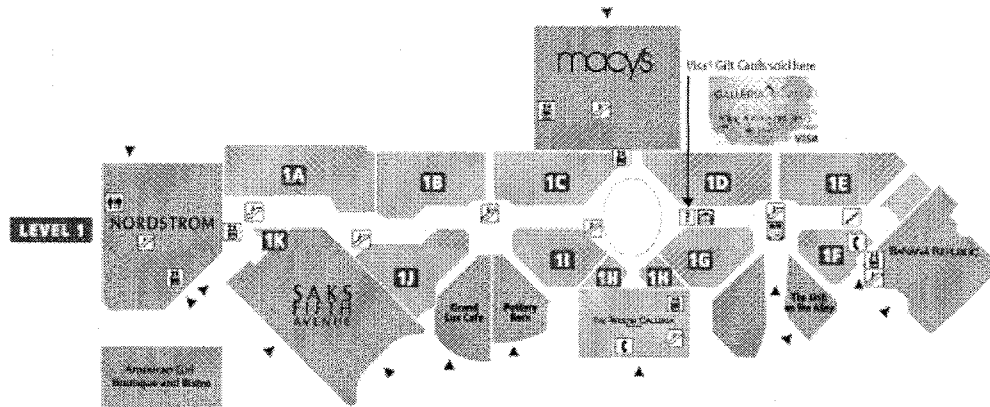
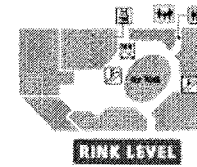


- b. Let's say that we want to go on a walking tour of the grounds and we park our car at the Movie Theater.
  - i. Where will we most likely have the start vertex and end vertex of our graph if we graph every place we go from the time we leave the car until the time we get back in? (Thank back to the last question on the homework)
  - ii. What does it mean when our starting and ending point is the same?
    - a. In Graph Theory we call this a circuit or a cycle (terms are interchangeable)
  - iii. Definition of **Circuit** or **Cycle**
    - a. A **Circuit** or a **Cycle** is a path whose start vertex is the same as its end vertex.
    - b. Cycles are measured and described in the same way as paths (length, simple, etc.) because they are paths.
- c. Applications of cycles or circuits
  - i. Electricity
    - a. Cannot flow if there is not a circuit (If part of an electrical circuit malfunctions and the cycle is broken, electricity cannot flow)
  - ii. Geometry
    - a. All polygons are cycles
  - iii. Travel
    - a. Round trip airplane trips are cycles
- d. **Homework: (Engage in discussion similar to that of the previous homework. Discuss applications of cycles with class. Use iiic as a lead into the section on connectivity and subgraphs)**
  - i. Please explain convincingly (or in other words, prove) why a loop is a 1-cycle.

- ii. Think of at least three real-life applications of cycles or circuits and explain why it is important that they are cycles.
  - iii. Please create a graph of a walking tour of Thanksgiving point that begins at the Movie Theater and visits every site on the map without repeating.
    - a. What length is this cycle?
    - b. Please find a 7-cycle, a 3-cycle, and a 4-cycle on this graph.
    - c. Is it possible to divide the graph of Thanksgiving Point into two separate cycles that have no common adjacencies? How would you do this?
- III. Connectivity and Subgraphs.
- a. Imagine that we are now in a 3-story mall like the one pictured below
  - b. Say that you wanted to visit every store on the first floor. How would that graph look?(Draw several possible way to do it: cycle, ziz-zag pattern, random pattern, etc.)
    - i. What do all these ways of drawing the graph have in common?
    - ii. Definition of **Connected Graph**: A **graph** is **connected** if there is a path linking every pair of vertices.
      - a. Connected graphs can be cycles, or paths
  - c. Now pretend that you take an elevator to the next level and do the same thing and same on level 3. Now not counting the trip on the elevator, how would you characterize the union of the levels in terms of subgraphs?

## LEGEND

- |                       |                            |                       |
|-----------------------|----------------------------|-----------------------|
| ATM                   | Food Place                 | Street Level Entrance |
| Children's Play Place | Galleria Management Office | Telephone             |
| Elevators             | Guest Reception Center     | Text telephone        |
| Escalators            | Restrooms                  |                       |
| Family Restrooms      | Stairs                     |                       |
| Wheelchairs           | Strollers                  |                       |





- i. If we don't count the elevators, are the graphs of each floor connected? Are they all connected graphs? What is different about each graph? What is similar?
  - a. If we think of the whole mall as one graph, we can think of each floor as its own separate mini-mall. In graph theory we call this a component.
  - b. Definition of **Subgraph**: A **subgraph** is a *graph* whose *vertices* and *edges* are *subsets* of another *graph*.
  - c. Definition of a **Component**: A **component** is a separate connected subgraph.
- ii. **Group Activity**: (Divide into groups of 5 students. Give each group a map of the United States, a set of 5 pushpins per person and a piece of string for each person. Have each student think of a 5-city dream vacation, they would like to take, that begins and ends in the town they want to live in when they grow up. Have the students, one-by-one make put a pushpin in every city on their trip and then tie the string around the pushpins in a way that the cycle of each trip is easily seen. Once every student has done this, have them answer the following questions about their trips. Allocate about 20 minutes for work on this activity and then have each group report on their findings and lead whole class in a discussion of the differences and similarities of the graphs as it pertains to this lesson's objectives)
  - a. How many subgraphs do you have?
  - b. How many components are there in your graph?
  - c. Are there any overlapping cities in any of the trips? What does that do to the length of the cycle created by combining the trips with overlapping cities.
  - d. What is the longest cycle? The shortest?
  - e. Why do these need to be cycles?
  - f. What else do you notice about the graphs?
  - g. Will any of you cross paths on your trips?
  - h. Is there any way to reroute your trips so that you don't cross paths?

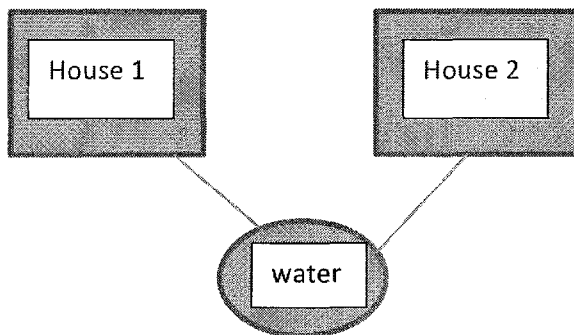
### Lesson 3—Planar Graphs and Contrapositive Proofs

**Lesson Goal:** The goal of this lesson is to understand the concept of Planar Graphs and use that understanding to introduce logic, the equivalence of a statement and its contrapositive, and to begin using formal proof techniques to prove theorems.

**Lesson Objectives:**

1. Students will be able to distinguish planar graphs from non-planar graphs. (Construct a Concept—10%)
2. Students will be able to apply planarity of graphs to real-world situations. (Application—10%)
3. Students will learn the basics of If, Then statements. (Construct a Concept—20%)

4. Students will learn the relationship between a statement and its contrapositive. (Discover a Relationship—20%)
5. Students will use their knowledge of logic, planarity and given theorems to give a formal proof of a theorem about planarity of a graph. (Comprehension and Communication—40%)
  - I. Introduction: Suppose you and your neighbor each own a house that you need to connect to the city water line but there is a city ordinance saying that the projection on paper of no two pipes or wires can cross each other. Is it possible to do this? Let's use a graph to find out.



By looking at this graph, we can see that the pipes do not cross each other so the law is not broken.

This kind of graph is called a **Planar Graph**

- a. Definition of Planar Graph— A graph is planar if it can be drawn so that the edges connect only at the vertices.
  - i. Note: It does not matter how the graph is drawn, as long as the graph CAN be drawn so that the edges don't cross.
  - ii. Example:

This is the same graph as the one above except that the edges cross. Since graph has already been drawn without intersections, this graph is planar.

```
graph TD; H1[House 1] --- W((water)); H2[House 2] --- W;
```

## II. Planarity of Bipartite Graphs

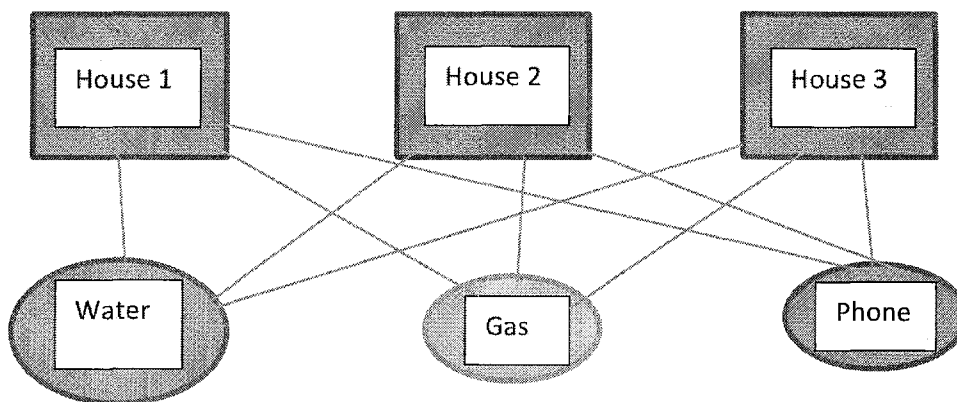
**Group Activity:** (Break students into groups of 3 or 4. Have group members discuss the following questions and record their work on paper. This should take about 5 minutes. Then lead the entire class in a discussion of the groups' conclusions)

1. Draw a Graph that represents the situation so that there are two houses that are each connected to both a water line and a gas line. Is this graph planar? Why?
2. Draw a Graph that represents the situation that there are three houses connected to a water line, a gas line, and a phone line. Is this graph planar? Why?

(Use group discussion as a lead in to the following discussion:)

Now let's say that you have a third neighbor and you want to each connect to the water line, gas line and and underground phone line. Is it possible to connect the three houses to the three utilities without breaking the "no-crossing" law?

Let's find out by drawing the graph:



This is just one way to draw this graph and, as you can see, the graph is not planar. However that doesn't mean that it can't be planar.

**Homework:** Try at least 5 different ways of drawing this graph to determine whether this graph is planar. After your attempt, think about why this graph is or isn't planar and write a paragraph giving your argument as to why or why not you think this graph is planar. (The goal of this assignment to lead students to conclude empirically that this graph is not planar. This conclusion will be made inductively. Our next step will be to help students deductively prove that their conclusion is true. After completing the homework, engage the students in a discussion of their results and discuss with them the importance of proving their observations to be correct, i.e. explain that their 5 graphs are not conclusive proof that there is no way to draw this graph planarly)

Our house and utilities graph is called a complete bipartite\* graph  $K_{3,3}$  because every house is connected to all three utilities but houses aren't connected to each other, and neither are utilities. (The  $K_{3,3}$  notation means that the two "sides" of the graph have three vertices each. Thus a  $K_{4,2}$  graph, for example would have one group of 4 vertices connected to another group of 2 vertices.)

**\* Note: Definition of Bipartite Graph:** A graph is bipartite if its vertices can be divided into two distinct sets so that each edge connects a vertex from one set to a vertex in the other.

**Definition of a Complete Bipartite Graph:** A complete bipartite graph is one in which every vertex in one set is connected to every vertex in the other.

**Notation for Bipartite Graphs:** A bipartite graph is denoted as  $K_{n,m}$  where  $n$  is the number of vertices in one of the distinct sets of vertices and  $m$  is the number of vertices in the other.

Since we know that this is a  $K_{3,3}$  bipartite graph, we can use the following theorem, which we will accept as true (although we never proved it) to decide whether the graph is planar.:

**Bipartite Planarity Theorem:** *Let  $G$  be a connected planar simple graph with  $n$  vertices and  $m$  edges, and no triangles. Then  $m \leq 2n - 4$ .*

In order to use this theorem we must divide the theorem into two parts. The first part is called the hypothesis. The hypothesis tells us what is required for the theorem to work. The first sentence in this theorem is the hypothesis: “Let  $G$  be a connected planar simple graph with  $n$  vertices and  $m$  edges, and 3-cycles.” Let’s look at each part of the hypothesis to see if we have it:

- a. Connected- As we learned earlier, a graph is connected if there is a path connecting every set of vertices
  - i. Group activity: Find and describe a path that connects every set of vertices in the graph.
- b. Planar- Since we want to show that the graph is planar, let’s hold off on this for now.
- c. Simple – A simple graph has no loops and no repeated edges.
  - i. Group activity: List every edge in the graph to show that there are no repeated edges or loops.
- d.  $n$  vertices- This graph has 6 vertices so  $n=6$
- e.  $m$  edges- This graph has 9 edges so  $m=9$
- f. No 3-cycles- In this case, a triangle refers to a 3-cycle. By looking at the graph, there is no 3-cycle, so we are good.

It seems as if we have the hypothesis, Now let’s look at the second part of the theorem, called the “conclusion”: then *Then  $m \leq 2n - 4$ .*

This means that if our hypothesis is true, then the number of edges of our graph is less than or equal to twice the number of vertices minus 4.

There is one more piece to the puzzle that we need to solve our puzzle. That piece is called logic.

Every theorem has the basic form of:

**If** the hypothesis is true, **Then** the conclusion is true.

This is known as and “If, Then” statement. If then statements are normally written like this:

$$P \rightarrow Q.$$

In this case P is the hypothesis and Q is the conclusion. The arrow means implies, or in other words, P, the hypothesis, implies(or means that), Q the conclusion is true.

For example we could say that If a fruit is a lemon, then the fruit is sour. In symbols we could say:

$$\text{Lemon} \rightarrow \text{Sour}.$$

However there is another way of looking at this. Since a lemon is sour, it is safe to say that if a fruit isn't sour, then it is not a lemon, or in other words:

$$\text{Not Sour} \rightarrow \text{Not Lemon}.$$

Reversing the order of the hypothesis and conclusion and making both not true is called the **Contrapositive**.

The contrapositive of a statement is logically equivalent to or says the same thing as the statement itself.

**Homework:** Write down 10 “If, Then” statements in sentence form and in symbolic form. Then write the contrapositive of each in sentence form and symbolic form. Is the contrapositive the same as the original “If, Then” statement?(**The goal of this assignment is to get students comfortable with If, Then statements and their contrapositive as well as for them to discover that the contrapositive of a statement is equivalent to the original statement. After completing the homework, discuss several samples of the students' statements and their contrapositives. Discuss with students how the contrapositive and the statement are actually equivalent.**)

Now let's use the theorem to decide if our graph is planar or not.

First we should use the inequality we were given by the Bipartite Planarity Theorem to find out what is going on.

$$m \leq 2n - 4$$

We know that  $m=9$  and  $n=6$ . That means that  $9 \leq 12-4$ , so  $9 \leq 8$ . We know that this isn't true. That means that our conclusion is false. Therefore we should use the contrapositive to prove that this graph isn't planar.

Lets start with  $\text{Not } Q \rightarrow \text{Not } P$

Now we can replace “Not Q” with “ $m \leq 2n - 4$  is not true”.

Therefore " $m \leq 2n - 4$  is not true" implies that " $G$  is a connected planar simple graph with  $n$  vertices and  $m$  edges, and no triangles." Is not true.

Since we know that the graph is connected, and simple, with  $n$  vertices,  $m$  edges, and no triangles, the only thing that could make this statement untrue would be that this graph is planar. Therefore we know that this graph is not planar.

**We have just proven that our house-utility graph is not a planar graph!**

### **Group Final Project for Unit 1:**

Use the following theorem and the techniques that we used in class to, as a group, prove that any complete, simple graph with 5 vertices,  $K_5$ , is not planar.

**Complete Planar Theorem:** *Let  $G$  be a connected planar simple graph with  $n$  vertices, where  $n \geq 3$  and  $m$  edges. Then  $m \leq 3n - 6$ .*

(Have students break up each part of the hypothesis and conclusion to make sure that the situation applies, as we did in the example. Then have them follow the same technique as we used in the example to prove the theorem. Emphasis should be put clear mathematical communication. Have the students write out the process in their own words so that you can see the process unfold in their minds. Flawless logic is not as important as clear communication in this early stage. If we can get them to express themselves mathematically, we can use that as a foundation upon which we can build their mathematical reasoning.)

### **Lesson 4—Map Coloring and Inductive Reasoning**

#### **Objectives:**

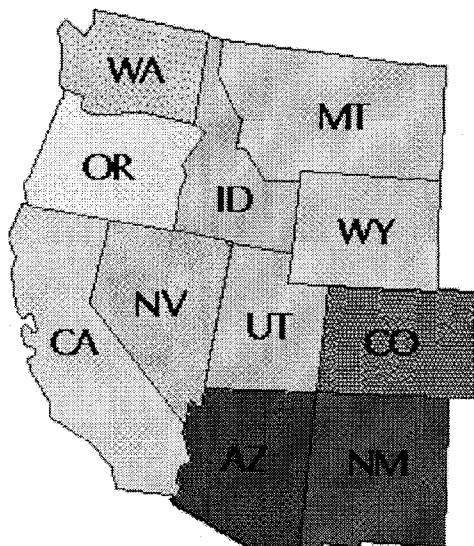
1. Students will begin to understand the concept of map coloring. (Construct a Concept)
  2. Students will attempt to develop their own algorithm for optimally coloring graphs. (Discover a Relationship/Willingness to try)
  3. Students will inductively discover (but not prove) the relationships described in the following fact: "Every map formed by contiguous states can be transformed into planar graphs." (Discover a relationship)
- I. Graph Coloring Activity: Engage in Activity 13 from "Computer Science Unplugged ([http://csunplugged.org/sites/default/files/activity\\_pdfs\\_full/unplugged-13-graph\\_colouring\\_0.pdf](http://csunplugged.org/sites/default/files/activity_pdfs_full/unplugged-13-graph_colouring_0.pdf))" from page 129-133. This activity will get students thinking about map coloring
  - II. Conversion from maps to graphs:
    - a. Show students one of the maps on page 139 and ask them the question: "How could we represent this map as a graph? Guide students in an inquiry discussion that includes the following points: Each "country" can be represented as a vertex; each border can be

represented as an edge; countries that share only a corner boarder(i.e. such as exists at the borders of Utah, Colorado, New Mexico and Arizona) are not considered adjacent.

- b. After the discussion has taken place and students are clear on how to draw a graph from a map, have each student individually create graphs for each of the maps they colored in Activity 13. Then instruct students to color their graphs so that the color of each vertex is the same as the country's color.

### III. Coloring Optimization

- a. Remind students of the "has-to-be rule" from Activity 13 and ask students ask them how that rule should be adapted to a map where there has to be more than two colors.
  - i. Display the following graph of the Western United States:



- ii. ASK students "how many colors are needed to color California, Nevada and Arizona? Why?" Give students 30 seconds or so to think about it and another 2 minutes to discuss it with a neighbor. Then lead the class in a discussion of why three colors are needed. Then have each pair of students find all the combinations of states that need three colors(i.e. combinations such as Idaho, Montana and Wyoming that mutually border each other.)

Homework: Give students a copy of the above map of the Western US and the following assignment:

1. Create a graph of this map using each state as a vertex and edges that represent a common border between two states.
2. In a paragraph, answer the following question: Is this graph a planar graph? Why or why not? Be prepared to share this with the class.
3. Consider the following statement: Every map formed by contiguous (i.e. Lower 48 states are contiguous but the US is not contiguous if you include Alaska and Hawaii) "states" can be transformed into a planar graph.
  - a. In a paragraph, state why you think this statement is or isn't true. Give three specific reasons to back up your opinion. Also use a map from a

separate region of the world(such as Southern US, Western Europe, West Africa, etc.) to back you up. Be prepared to share this with the class.

4. What is the minimum number of colors you can use to color this map
5. Try to come up with a systematic way (an algorithm) to color the graph of this map of the Western US so that you use the smallest number of colors possible. Break this process into a list of steps that you could teach to your classmates so that they can easily complete the task. Be sure to include what vertex to start with and how to determine which vertex should be what color and also how to make sure there is not a simpler way. Be prepared to share your algorithm with a small group and then discuss it with the class.

The following day in class, break students up into groups of 4 or 5 and have them share their graphs and discuss their answers for number 2 and 3. Allow them to discuss for 5 to 10 minutes and then have each group come to a conclusion. Then have each group share their conclusions and reasoning and then from this lead the class in a discussion that helps the students realize, based on the evidence that the fact in statement 3 is correct. Follow a similar process in discussing numbers 4 and 5. This discussion should lead to the class developing an effective algorithm for coloring a graph that minimizes the number of colors used. This discussion will lead into the next lesson on the Four Color Theorem and Grotzsch's Theorem.

### **Lesson 5 Inductive Reasoning, the Four Color Theorem and Grotzsch's Theorem.**

#### **Objective:**

**Students will inductively discover(but not prove) the relationships described in the following theorems: The Four Color Theorem, Grotzsch's Theorem(Discover a Relationship)**

**Students will use inductive reasoning to recognize mathematical patterns(Discover a Relationship)**

#### **Warm-up Activity:**

Break up the class into groups of 4 or 5 and give each group a worksheet with the following:

1. Read the following passage from

[http://www.csun.edu/science/ref/reasoning/inductive\\_reasoning/inductive\\_reasoning.html](http://www.csun.edu/science/ref/reasoning/inductive_reasoning/inductive_reasoning.html):

"Induction is a major kind of reasoning process in which a conclusion is drawn from particular cases. It is usually contrasted with deduction, the reasoning process in which the conclusion logically follows from the premises, and in which the conclusion has to be true if the premises are true. In inductive reasoning, on the contrary, there is no logical movement from premises to conclusion. The premises constitute good reasons for accepting the conclusion. The premises in inductive reasoning are usually based on facts or observations. There is always a possibility, though, that the premises may be true while the conclusion is false, since there is not necessarily a logical relationship between premises and conclusion." From: Grolier's 1994 Multimedia Encyclopedia



Inductive reasoning is used when generating hypotheses, formulating theories and discovering relationships, and is essential for scientific discovery.

2. Now that you have read the definition of Inductive reasoning, read each of the following statements (from [http://www.education.com/study-help/article/inductive-reasoning-part\\_answer/](http://www.education.com/study-help/article/inductive-reasoning-part_answer/)), as a group decide on the correct answer and write a paragraph explaining why you think your answer is correct:

Now it is your turn to play detective and use your reasoning skills to draw logical inferences. Read carefully the information you are given (the premises) and consider what would be the most logical conclusion to draw from that evidence.

1. Every summer, when the cotton wood trees bloom, you start to sneeze and your eyes water and itch. This morning, you saw a few cotton wood blossoms floating by. You can therefore logically conclude that:
  - a. summer has just officially started
  - b. you had better go out and buy some tissues
  - c. there are more cotton wood trees in the area than any other species
2. Each time you go to your city's biggest department store, you end up buying far more than you had planned on. You carefully make a list and plan to spend \$25, but usually spend twice that amount. You can therefore logically conclude that:
  - a. the department store charges too much for its products
  - b. you need more items than you had originally thought
  - c. a need for immediate gratification is one of your main traits
3. For the past six months, you have gotten a massage. The therapist advises you to drink a big glass of water after each session, but you never do. Following each massage, you get a terrible headache. You can therefore logically conclude that:
  - a. massages cause massive dehydration
  - b. not drinking water is a serious health hazard
  - c. massages may very likely give you headaches

4. Your older brother gave you a coffee pot for Christmas. Every time you brew a pot of coffee, it tastes scalded. You have tried several brands of coffee but it doesn't make any difference.

You can therefore logically conclude that:

- a. there is a malfunction with the coffee pot
- b. it is time for you to stop drinking coffee
- c. the type of coffee makes a big difference

5. Every June, your Aunt Patricia sends you a birthday check in the mail. Every July, your Uncle Ian sends you a check with a belated birthday card. When you go out to the mailbox on June 16, there is an envelope with your name on it waiting inside. You can therefore logically conclude that you are getting:

- a. a check early from your Uncle Ian
- b. Aunt Patricia's card
- c. a letter from a friend

3. Now that you have come up with an answer to each question, read the answers given by the author and then after each answer write a paragraph stating why you agree or disagree with the author's conclusion. (Remember that the author's conclusion isn't always the only correct conclusion and even logical conclusions can be false) Be prepared to have someone in your group share your responses in a class discussion.

### Answers

1. b. Cotton woods bloom in the summer, but there is not enough information here to indicate that when they bloom, the season begins. There is also nothing to indicate that there are more cotton wood trees than anything else, just that they are in the area. Since the blooming causes you to sneeze, you will soon need tissues.

2. c. There is not enough information here to indicate there is anything wrong with the store's prices. Since you carefully make a list of items needed, it is unlikely that you need additional items on every trip. Much more likely is that once there, you keep purchasing things not on the list because you see them and want them right now.
3. c. Although your therapist recommends that you drink water, that is not enough information to indicate massages dehydrate a person. (Water is needed to help flush toxins from the muscles.) Being dehydrated is not a serious health hazard, even if it causes a headache, so option b is incorrect.
4. a. There is no reason for you to stop drinking coffee here and the problem already stated that you had tried different brands of coffee, so option c does not make sense. Instead, it is apparently a problem with the machine itself.
5. b. Your check from Uncle Ian traditionally does not come until July, so there is no reason to suspect it came early this time. There is no indication that you would be getting a letter from a friend.

Now engage the class in a discussion of the activity in which students discuss and debate their conclusions. Ask students how inductive reasoning can be useful in making mathematical discoveries.

Now in their groups, have students consider a circle with two points on it. Then answer the following questions: How many chords are there in the circle? How many intersections of chords are there in the circle. Into how many regions has the circle been divided. Now add a point on the circle and answer the same questions. Do this for up to 7 points and fill out the following table.

Points	Chords	# Intersections	# Regions
--------	--------	-----------------	-----------

---

2
3
4
5
6(conjecture)
6(actual)
7(conjecture)
7(actual)

---

After filling out the table for 2,3,4, and 5 points, have students conjecture how many chords, intersections and regions there will be for 6 points. Then have students answer the following questions: How did your conjectures agree or disagree with the actual data? Why could this be? What is it about the division of circles that could cause this to be so? Come up with at least one observation from the table and attempt to prove or disprove that this observation is true. Repeat this process for 7 points.

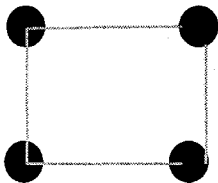
Now engage in a similar process with a rectangular plane(i.e. a piece of paper). Divide a piece of paper and observe the relationship between the number of lines used to divide it, how many regions the paper is divided into, and the number of intersections between the lines.

Bring up the Goldbach Conjecture to students: "Any even integer greater than two can be expressed as the sum of two primes." Have students give evidence about why they think this is true. After several examples, inform students that they have engaged in inductive reasoning as to the truth of this conjecture. Remind students that even though we have great amounts of empirical evidence that they still haven't given a definitive proof because there are still infinitely many cases to check. However there are some cases in which ideas such as Goldbach's conjecture cannot, or at least haven't currently been proven deductively so we often accept the inductive reasoning.

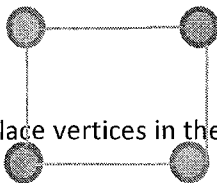
Now write "The Four Color Theorem: Every planar graph can be colored using at most four colors." on the board. Tell students that for 110 years, until the invention of a computer that could crunch enough data to prove the theorem, this statement was treated much like Goldbach's conjecture, accepted but not proven.

Engage the class in the following large-group discovery and discussion activity:

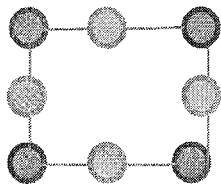
1. Draw the following square graph on the board:



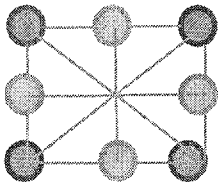
Now ask the students how to color this graph using the minimum number of colors. They should get something similar to this:



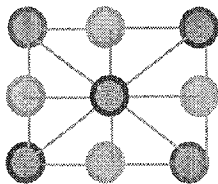
Now place vertices in the "midpoints" of each edge like so:



So far we have 3 colors. Now let's connect each vertex to its opposite vertex as follows:

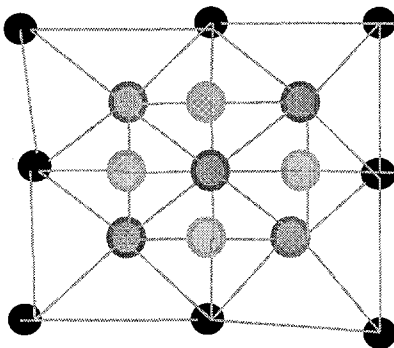


Now the statement says that we have to have a planar graph so where can we put a vertex so that the graph remains planar (ask students for suggestions, eventually leading them to the intersection of the inside edges.)

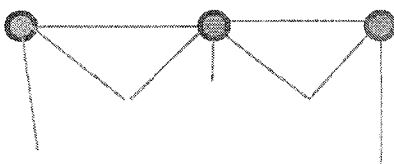


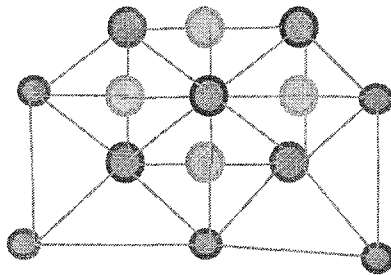
Now we have a planar graph that is colored by a minimum of 4 colors, let's see if we can make one with 5 colors that doesn't break the planar rule.

First let's try making the graph bigger by making each of the corner vertices the midpoints of a bigger square:

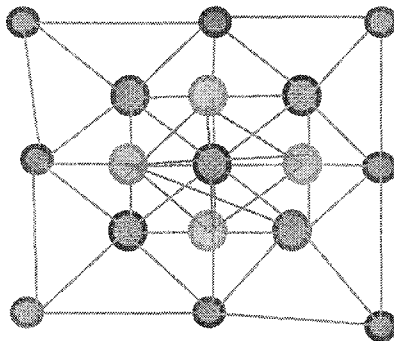


There are two questions we need to ask ourselves about this graph. First, is it planar. (Have students explain why this graph is planar) Second, is there a way to color this graph with 4 colors or less? (Help students come to the conclusion that it can with a graph similar to the one below:





Now let's consider connecting any two vertices of the same color that are adjacent to the other 3 colors, for example any of the green vertices.



The red edges represent all the different ways to connect two green vertices. If we can do this then we know that we would have to have 5 colors. But wait! Is the graph planar? If we look at each possible red edge, we see that it crosses another edge, which breaks the planar rule. So it seems as if this strategy shows that once again the statement seems to be true.

Now continue to field students' suggestions about how to find ways to make a 5 color planar graph until students are convinced that all planar graphs have to be 4-colorable.

#### Final Project: Grotzsch's Theorem:

Divide the class into four groups. Each group will be assigned one of the following projects on which to give a 15-minute class presentation to the class:

1. Using what you know about the 4-color theorem and how we constructed evidence to demonstrate that it was true, use inductive reasoning to show us convincing evidence that Grotzsch's Theorem is true. Be sure to explain how your evidence is different from a formal deductive proof of the theorem.
2. Research the actual formal proof of Grotzsch's Theorem. Present as much about the theorem and its proof as you can with your current knowledge of mathematics. Be sure to explain how this formal deductive proof is different from the inductive evidence we looked at for the Four-Color Theorem.
3. Research the life and career of Herbert Grotzsch. Answer such questions in your presentation as: Why did he decide to become a mathematician? What were his most important accomplishments? How did he come up with his theorem of 3-colors in a triangle

free graph(Grotzsch's Theorem)? Who did he work with in his career and what important accomplishments did they attain. How was Grotzsch important in the development of Graph Theory and Map Coloring?

4. What is the difference between Deductive Reasoning and Inductive Reasoning? How are each used in Mathematics? What is the Principle of Mathematical Induction? Is it inductive, deductive, or both? How is it used in mathematics.

Spend a class period or two in the library to allow students to work on their presentation. Grade their presentation according to the following criteria:

1. Each student actively participated in the presentation.
2. Each student actively contributed to the development of the project
3. The project effectively presented the information asked for in the prompt, although it is not completely necessary to have a complete understanding of the difficult aspects of the advanced mathematics associated with the project.
4. Each student is actively engaged in learning from the other groups' projects.

#### Lesson 6: Eulerian Paths and Cycles

Objective: Students will conceptualize, through empirical observations, Eulerian Paths and Cycles.  
(Construct a Concept)

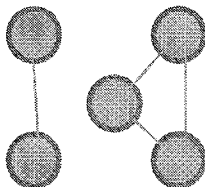
Students will be arranged into groups and will be given a worksheet with the following components:

Definition: An Eulerian Path is a path in a connected graph that uses every edge exactly once.

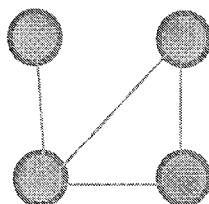
Definitions: An Eulerian Cycle is a Eulerian Path that is a cycle.

Now, based on the definitions given, do the following graphs contain Eulerian Paths? Write a paragraph explaining your reasoning for each graph.

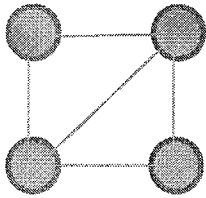
1.



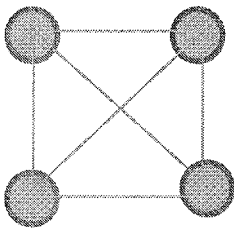
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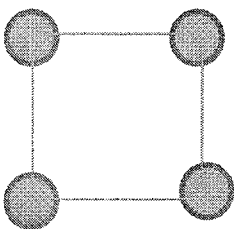
3.



4.



5.



Now answer the following questions:

1. Of the paths that you determined to be Eulerian, did the vertices which were not endpoints have even degree, odd degree, or both?



2. For those same paths, which is/are also cycle(s)?
3. If the Eulerian path is a cycle, what degree do the endpoints have?
4. If the Eulerian path is not a cycle, what degree do the endpoints have?

Now make a conjecture about the degrees of the vertices in an Eulerian Cycle, then in a paragraph, explain why you are correct:

Now make a conjecture about the degrees of the vertices in an Eulerian Path that is not a cycle, then, in a paragraph, explain why you are correct:

(End of Worksheet)

After completing the worksheet as a group, discuss with the class the observations and conjectures they made. Then in the discussion lead the students to formulate the following two conjectures:

A connected graph has an Eulerian Path if, and only if, it has exactly 2 vertices with odd degree.

A connected graph has an Eulerian Cycle if, and only if, all vertices have even degree.

Then explain to students the meaning of “if, and only if” and remind them that they must examine and prove the statement in both directions. Discuss with students why both statements work in both directions.

**Homework:** Give the students a copy of the following article and have them read it as homework in preparation for the next class in which we will go through the article piece by piece. The goal of their homework reading is not to completely comprehend but to become comfortable with the article in order to aid an in-depth discussion in class.

Barnett, Janet Heine. “Early Writings on Graph Theory: Euler Circuits and The Konigsberg Bridge Problem.” <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.89.4462>. (2005)

### **Lesson 7: The Bridges of Konigsberg**

**Objectives:**

**Students will understand the historical importance of the “Bridges of Konigsberg” problem.  
(Appreciation)**

**Students will attempt to read and understand mathematical literature through reading the article  
“Barnett, Janet Heine. “Early Writings on Graph Theory: Euler Circuits and The Konigsberg**

**Bridge Problem.”** <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.89.4462>. (2005)  
(Willingness to Try)

**Students will begin to understand the theorems and processes used by Euler in his article about graph theory. (Comprehension and Communication)**

**Warm-up Activity:**

As a class view the video “The Seven Bridges of Königsberg” on YouTube ([http://www.youtube.com/watch?v= OiZrmnni9Y](http://www.youtube.com/watch?v=OiZrmnni9Y)).

After watching the videos have students recall their conjecture about Eulerian Paths from the last lesson. Have the students assume, for the time, that their conjecture has been proven true and thus is a theorem. Based on this assumption, engage the students in a discussion of how they could use the theorem to explain why the trip described in the video is impossible(i.e. the path is not Eulerian).

**Main Activity:**

Have each student take out the article by Janet Barnett that was assigned for homework. Give the students a worksheet that contains the following components that you will fill out together in class as you read:

1. Based on the introduction: What three mathematicians were the first 3 to investigate what we now know as Graph Theory?
2. Besides Graph Theory, on what other important branch of mathematics did both Leibnitz and Euler predominantly work?
3. What mathematical problem is this article about?
4. What important mathematical question arose because of the Four-Color Theorem?
5. Euler’s 1<sup>st</sup> Paragraph:
  - a. What is Leibnitz’s name for Graph Theory?
  - b. How is Graph Theory different from traditional geometry
6. Euler’s 2<sup>nd</sup> Paragraph
  - a. How does Euler label the land masses(vertices)? How does he label the bridges(edges)
  - b. What problem is Euler trying to solve?
7. How did do we simplify problems in Graph Theory?
8. Engage in Activity 1
9. Euler’s 3<sup>rd</sup> Paragraph
  - a. What different methods of solving this problem did Euler consider as he started to work on the problem?
  - b. What method did he choose, and why did he choose it?
10. Euler’s 4<sup>th</sup> Paragraph
  - a. What notation does Euler use to represent bridge crossings?
  - b. What concept in modern graph theory is he representing with this notation?
11. Euler’s 5<sup>th</sup> Paragraph
  - a. What rule does Euler give for the relationship between the number of bridges and the the number of land masses used?
  - b. What is this relationship in terms of edges and vertices?
12. Engage in activity 2
13. Euler’s 7<sup>th</sup> Paragraph
  - a. What question does Euler propose to solve in this paragraph.

14. Euler's 8<sup>th</sup> Paragraph

- a. What is Euler's rule for the amount of times a vertex will appear if an odd number of bridges lead to it?

## 15. Engage in activity 3

16. Euler's 9<sup>th</sup> Paragraph

- a. What is Euler's justification for the impossibility of the Konigsberg journey?
- b. What does this mean in terms of the path of the graph in modern terms?

17. Euler's 10<sup>th</sup> Paragraph

- a. What is Euler's rule for determining whether or not a journey is possible?
- b. What is this rule in modern terms?

18. Euler's 11<sup>th</sup> Paragraph

- a. What is Euler's rule when the number of bridges leading to a place is even?
- b. What does this mean in modern terms?
- c. Why does it matter whether or not we start in A or don't start in A?

19. Euler's 12<sup>th</sup> Paragraph

- a. What rule does Euler make if the number of bridges is odd?
  - i. When you start in A
  - ii. When you start somewhere else

20. Euler's 13<sup>th</sup> Paragraph

- a. What two rules does Euler establish in this paragraph?

## 21. Engage in Activity 4

- a. What is the modern day equivalent of Euler's definition from the previous paragraph:

22. Euler's 14<sup>th</sup> Paragraph

- a. Describe briefly the process(or algorithm) of how to easily determine if a path is Eulerian

23. Euler's 15<sup>th</sup> Paragraph

- a. Why does Euler say that the journey has to start at either D or E?

## 24. Engage in Activity 5

25. Euler's 16<sup>th</sup> Paragraph

- a. According to Euler, what is the relationship between the sum of the degrees of the vertices and the number of edges

26. Euler's 17<sup>th</sup> Paragraph

- a. According to Euler, what do we know about the number of vertices with odd degree?

Skip Activities 6 & 727. Euler's 18<sup>th</sup>, 19<sup>th</sup> and 20<sup>th</sup> Paragraphs

- a. What three rules does Euler give for determining if a path is Eulerian?
  - i. What kind of path is described in rule 1?
  - ii. What kind of path is described in rule 2?
  - iii. What kind of path is described in rule 3?

## 28. Engage in Activity 9

## Lesson 8 Directed Graphs

### Objectives:

1. Students will construct the concept of Directed Graphs using their previous knowledge of “undirected” graphs. (Construct a Concept)
2. Students will learn to make connections between directed graph theory and their other school classes.(Comprehension and Communication)

### Directed Graphs

Pose the following situation and questions to the whole class to introduce the concept of Directed Graphs:

Imagine you live a part of town which has only one way streets. Your friend, from out of town, needs you to draw a map that will take him to your house.

First, ignoring the one-wayness of the streets, how could you draw the road map as a graph?(Lead students to create a graph with each street corner as a vertex and each street as an edge. )

Second, how can you adjust the graph to account for the the fact that the roads only go one way? (Have students agree to the convention of changing edges to arrows in the direction of travel.)

Third, how would you account for a two-way street?(Have students agree to the convention of representing a two-way road as two arrows going in opposite directions between the same two vertices)

Fourth, how do we adjust the following concepts of the “indirected” graphs we learned about before to “directed” graphs?

- Degree
  - (Ask students whether or not an arrow going “into” a vertex is the same thing as an arrow going “out” of a vertex. Then have them come up with definitions and notation for indegree and out degree)
- Paths and Cycles
  - (Help students discover that paths now have to follow the directions of their arrows, for example a car must obey the direction of a one-way street and therefore must make adjustments to get to its destination)
- Connectivity
  - (Have students first consider the “undirected” structure of the graph. If the “undirected” graph is connected, call this graph weakly connected)
  - (Have students the consider the directed structure of the graph. If there is a directed path to and from every pair of vertices, call this graph strongly connected.)
- Planar Graphs

- (Help students to realize that there is no difference in planariness in directed graphs.)

**Activity: Directed Graphs and Real Life Research Project**

Based on their interests, divide students into groups to study the application of directed graphs in the following subjects:

- History and Government
- English
- Chemistry
- Physics
- Biology
- Music
- Art
- Business and Computers
- Foreign Languages
- Sports and Physical Education
- Technology and Agriculture

Have each group read and discuss the article: “Vertex-Edge Graphs: An Essential Topic in High School Geometry” by Eric W. Hart focusing on how graph theory is used in real-life applications.

Now have students brainstorm at least 10 ways in which directed graphs can be used in their assigned field. Suggest to students that pretty much any diagram that has arrows in it can be represented by a directed graph. (Some examples include: The Checks and Balances relationship in the US Government, The Relationship between nouns and adjectives in English, Chemical reactions, Genetics charts, play diagrams in sports, computer networking.

Have groups choose one of their brainstormed applications of directed graphs and using the school’s library and computer resources to research, and write a minimum of 5 page paper that addresses the following:

- A. What does each vertex of the graph represent?
- B. What relationship(s) does each arrow in the directed graph represent?
- C. Why is a directed graph better than a non-directed graph for this representation?
- D. Make an observation about some property of this graph that you will attempt to convince us mathematically(prove) is true
- E. Make your best attempt to prove your observation
- F. Answer the question: Based on your work on this project, why is the study of graph theory important to me and my life and the world I live in?

If possible have students work with their English teachers on the paper writing as well as their teachers in their assigned subjects to make this a co-curricular project and to provide guidance and insights.

After completing the paper, hold a “Press Conference” in which each group will prepare a 5-minute presentation about their paper and then be prepared for a 5-minute question/answer session about their work in which students and the teacher will ask questions.

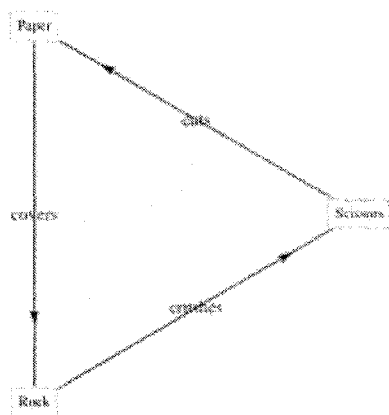
### Lesson 9 Tournaments, Queens and Emperors

#### Objectives:

1. Students will construct the concept of a Tournament(Construct a concept)
2. Students will discover the relationships of Queens and Emperors in Tournaments(Discover a Relationship)
3. Students will use knowledge of graph theory and other data to make decisions(Application)

#### Tournaments:

Have the students consider the game Rock, Paper, Scissors. Pair up students and have them play several games to refamiliarize themselves with the game. Now, reconvening as a whole group, ask students how we could draw the winning/losing combinations as a directed graph. (If needed suggest that an arrow represents the “beats” relations i.e. an arrow from a to b means that a beats b.) Now on the board, draw the following graph:



Next have students consider the case where both players “shoot” the same object(i.e. rock vs rock). Since this results in a tie thus another round of “shooting” is it necessary to represent this on a graph? (Hint, arrows represent the “beats” relation, and there is no “beating” in a tie)

Now break up students into groups of 4 or 5 and have each group consider the structure of the rock-paper-scissors graph. Give students 3-5 minutes to brainstorm and list any properties or qualities that the graph has.

Reconvene as a large group and have students share their lists. If listed focus in on the completeness property of the graph, that is the fact that there is one, and only one, arrow between each pair of

vertices. Through discussion, help the students articulate this definition for themselves and come up with a definition that the entire class can agree on. Inform students that graph theorists call this structure a Tournament.

Now have students watch a clip from *The Big Bang Theory*

(<http://www.youtube.com/watch?v=Kov2G0GouBw>) and then construct the graph for Rock-Paper-Scissors-Lizard-Spock:



Is Rock-Paper-Scissors-Lizard-Spock a Tournament? Why?

Now tell students that you have invented a game that is Rock-Paper-Scissors-Lizard-Spock-Euler(whose symbol is made by making an “e” shape with your hand)- Shark(whose symbol is made by holding the hand flat and vertical as if giving a handshake). Have them try to make a tournament with these objects. Ask the students if their creation is a tournament and if it has the same properties of the previous games.

**Queens and Emperors:**

Now, back in the whole-class group, ask the students if there is a way of determining which is the “best player” in a tournament. Explain to students that graph theorists have come up with two ways of determining the “best player.”

Have the students read the following paragraph from “A Model of Modeling: Queens in Tournaments” by Dave Brown:

Consider a set of individuals who are in competition with one another. We wish to model this competition so that we may construct the social hierarchy this competition creates. Keep in mind that the individuals are certainly aware of their interactions and competitions with one another, but are most likely unable to comprehend the overall structure of the entire set's interactions and how they influence the set as a whole. That is, there is no organization behind the individuals' competitions, no entity governing who competes with whom and hence no entity to report the results to the set of individuals. In accord with this we will assume that every individual competes with every other individual and, for now, we'll assume no ties or draws result in any competition. An actual example of this is that of a barnyard of chickens: in every pair of chickens, one is dominant over the other, and the dominance can be observed by noting the chicken which pecks another on the head and neck. Whence the expression “pecking order.” The goal is to model this in such a way that we can discern a most dominant individual and perhaps even a ranking of the individuals.

The first way to choose the “best player” is called an Emperor. An emperor is vertex on a tournament that beats all others. Now break again into groups and have students decide if there is an emperor in any of the Rock-Paper-Scissors variations we discussed above. Then have each group construct a tournament that has an emperor. Return to the large group and have discuss the question and share the created tournaments and discuss why there is an emperor.

Now ask the students if how likely is that there will be an emperor in any given tournament. (Guide students to understand that as the number of vertices in a tournament increases, the probability of having an emperor decreases)

Once students have discovered that many tournaments don't have emperors, introduce to students another way to determine the “best player”: Queens. Read the following excerpt from Brown to get the definition of a queen:

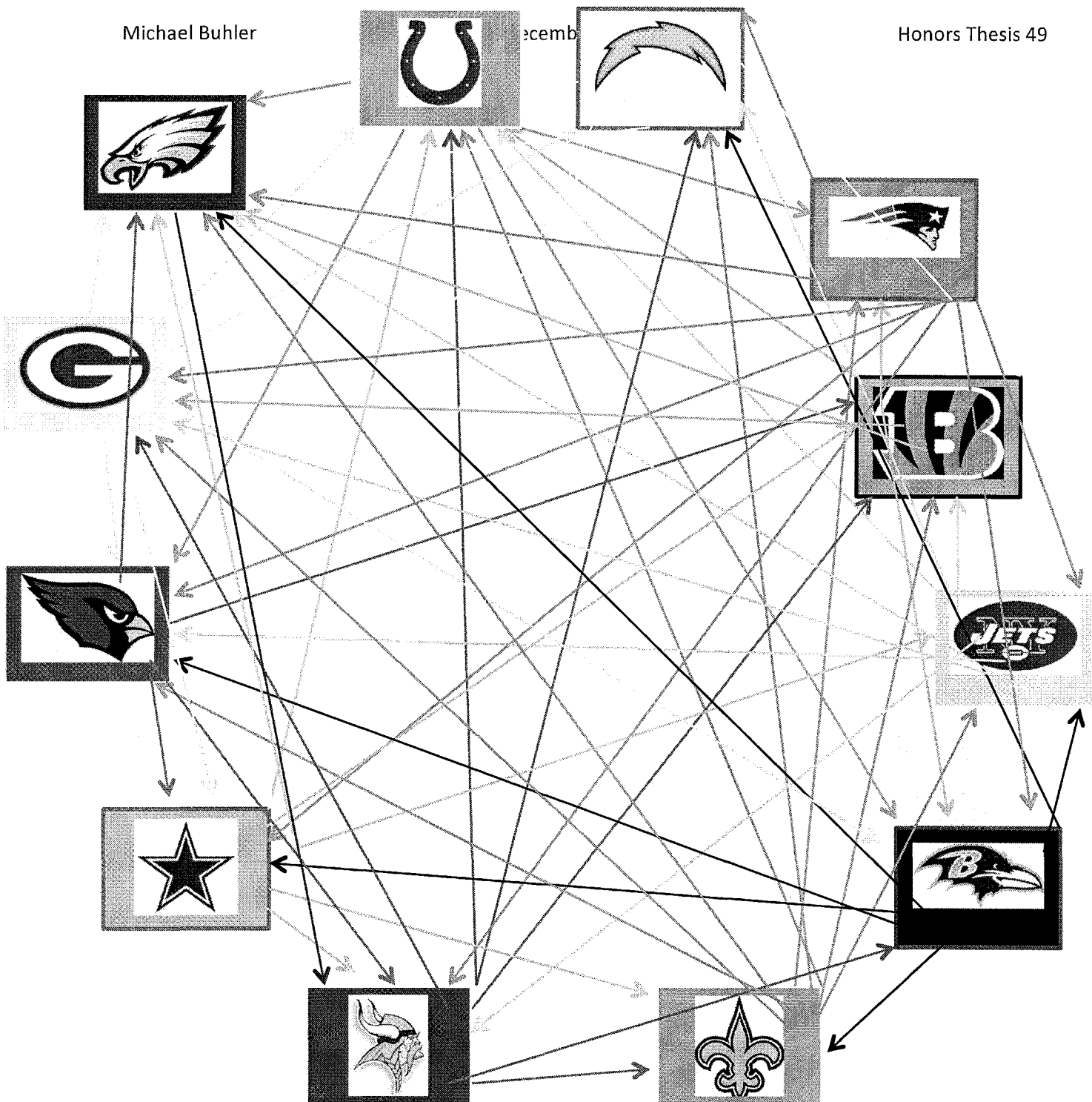
In the early 1950's H. G. Landau wrote several papers on the problem we are considering. He identified the following notion for dominance (although he didn't use the same terminology): A queen of  $F$  has the property that for every other  $x$  in  $F$ ,  $qPx$  or there is a  $y$  in  $F$  with  $qPy$  and  $yPx$ . In other words, we have a member of  $F$  which either pecks every other member or, if it doesn't peck a certain member  $x$  then it pecks another  $y$  that does peck  $x$ . Note that an emperor is necessarily a queen, but not conversely; so the notion of a queen is a generalization of that of an emperor.

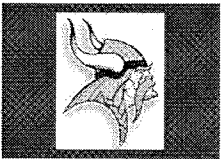
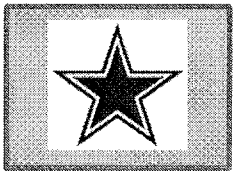
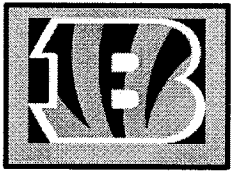
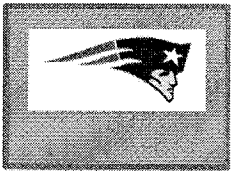
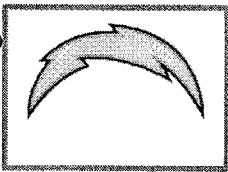
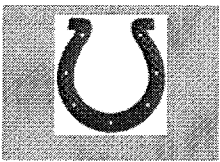
Discuss and clarify the definition as befits the students' needs. Now break students into groups again and have them discuss if there is a queen in each of the Rock-Paper-Scissors variations and identify each queen and how many queens there are. Have each group try to construct a tournament with 1 queen, 2 queens, 3 queens, and 4 queens.

Bring the class back together and discuss what was discovered by the groups. Then break into groups one more time and have students engage in the Predicting the Super Bowl activity. After completing the activity, engage the class in a discussion of the activity, and in particular, the last set of questions.









**Predicting the Super Bowl:**

During the 2009-2010 NFL Season, the New Orleans Saints, led by Drew Brees won the Super Bowl. However were they really the best team in the NFL? Was there a better team? What make a team, a better team? Let's use several types of math to determine who was the best team of the 2009-2010 season.

**1. The traditional Playoff Method:**

The NFL uses a 12-team playoff to determine its champion. The 4 regular-season division leaders and the 2 best non-division-winning (Wild Card) teams make the playoffs from each conference. The two lowest ranking division leaders and the two wild card teams from each conference play. Then the winners from this game face the 2 highest ranking division winners in the 2<sup>nd</sup> round. Then the winners from round 2 face each other in the conference championship games. Then the two conference champions face in the Super Bowl. In the 2009-2010 season, the New Orleans Saints beat the Indianapolis Colts to win the Super Bowl by this method.

#1 Champion: Saints

#2 Runner-up: Colts

#3-4 Semi-Finalists: Vikings & Jets

#5-8 Quarter-Finalists: Cowboys, Cardinals, Chargers, Ravens

#9-12 Wild Card Round: Eagles, Packers, Bengals, Patriots

## 2. The Winning Percentage Method:

Another method to determine the best team is to find the number of wins and losses of each team and then compute the winning percentage by dividing the number of wins by the total number of games.

Use the playoff spreadsheet to complete the following chart to determine which team was the best among playoff teams in 2009-2010.

Team	Wins	Losses	Winning %	Ranking
Colts				
Chargers				
Patriots				
Bengals				
Jets				
Ravens				
Saints				
Vikings				
Cowboys				
Cardinals				
Packers				
Eagles				

- a. What differences do you note between the results of the playoffs and our analysis of Winning Percentage?
- b. How are these results different from the final regular season standings for the 2009-2010 season?
- c. How do you explain the fact that Colts made it to the Super Bowl and not the Patriots?
- d. What does this say about the basic difference between a playoff system and determining a champion simply by winning percentage?

**3. The Graph Theory Method:**

- a. Use colored pencils and the playoff spreadsheet to fill in the graph to create a tournament .
- b. Use the tournament to fill in the following table:

Team	# of Teams Directly Defeated	# Teams Defeated in 2 Steps	# Teams Defeated in 1 or 2 Steps
Colts			
Chargers			
Patriots			
Bengals			
Jets			
Ravens			
Saints			
Vikings			
Cowboys			
Cardinals			
Packers			
Eagles			

- c. Is there an emperor?
- d. Which teams are “queens”, which are not?
- e. Is the queen’s method a good way to determine who will win the Super Bowl?  
Why or why not?
- f. In what other ways could you use graph theory to find a more definitive result?
- g. Is it safe to say the the team with the highest winning percentage will automatically be a queen?

- h. How would the Raven's queenliness be different if the Patriots would have beaten the Saints? What does this say about this system of choosing a champion?
- i. Why do you think there are so many queens?
- j. What would happen to the number of queens if we included all 32 teams in our tournament?
- k. How does what you learned about tournaments and queens fit with the following statement: The probability that every vertex of a random  $n$ -tournament is a queen approaches 1 as  $n$  grows large.

#### Lesson 10 Graph Theory Wrap-up and Interpretation Project

##### Objectives:

1. Students will demonstrate their comprehension of some aspect of graph theory that is meaningful to them (Comprehension and Communication)
2. Students will connect said aspect of graph theory to something that is important to them and express that connection as an original creative presentation(i.e. Video, music video, poem anthology, story book, game, song, puzzle, drama, cartoon, mathematical biography, visual artwork, etc.)(Willingness to Try).

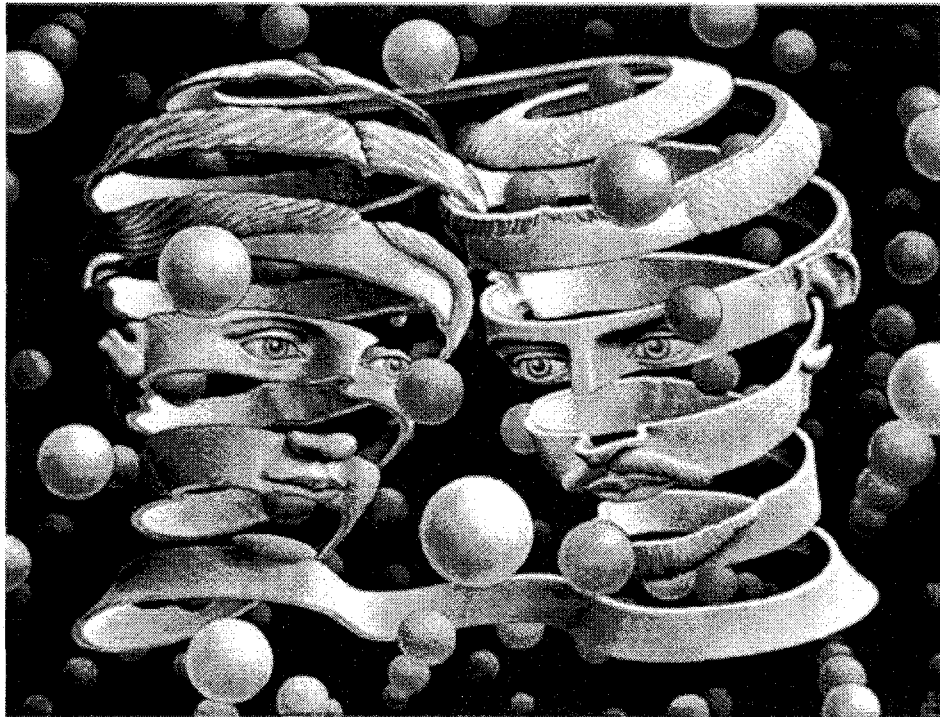
As an introduction to the project, show students the following:

- a. *Teach Me How to Factor*: <http://www.youtube.com/watch?v=OFSrINhfNsQ>
- b. "Fermat" by Jonathan P. Dowling: Mssr. Fermat -- what have you done?  
Your simple conjecture has everyone  
Churning out proofs,  
Which are nothing but goofs!  
Could it be that your statement's an erudite spoof?  
A marginal hoax  
That you've played on us folks?  
But then you're really not known for your practical jokes.  
Or is it then true

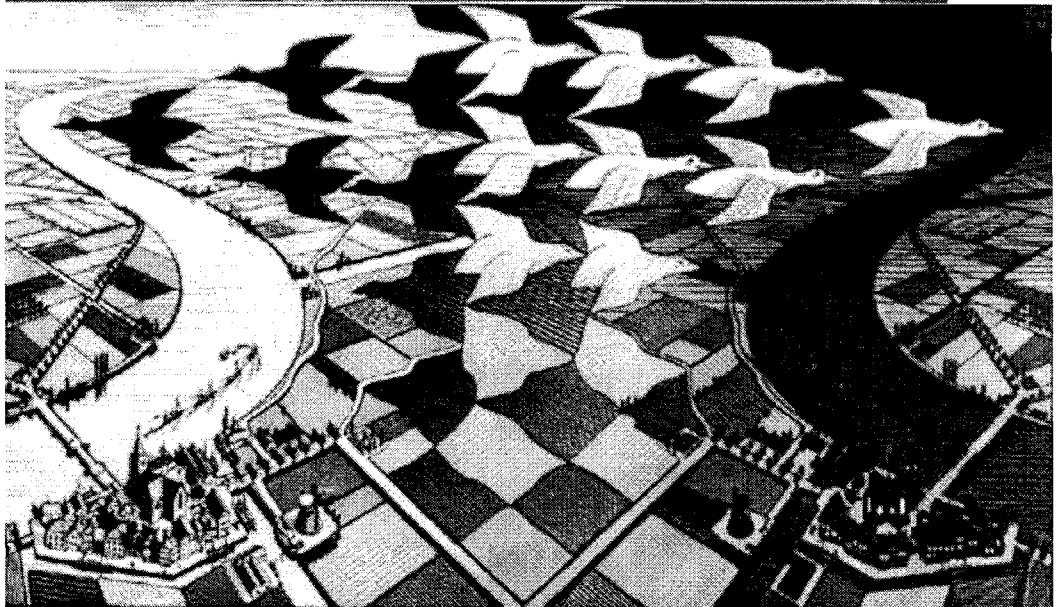


That you knew what to do  
When  $n$  was an integer greater than two?  
Oh then why can't we find  
That same proof...are we blind?  
You must be reproved, for I'm losing my mind.

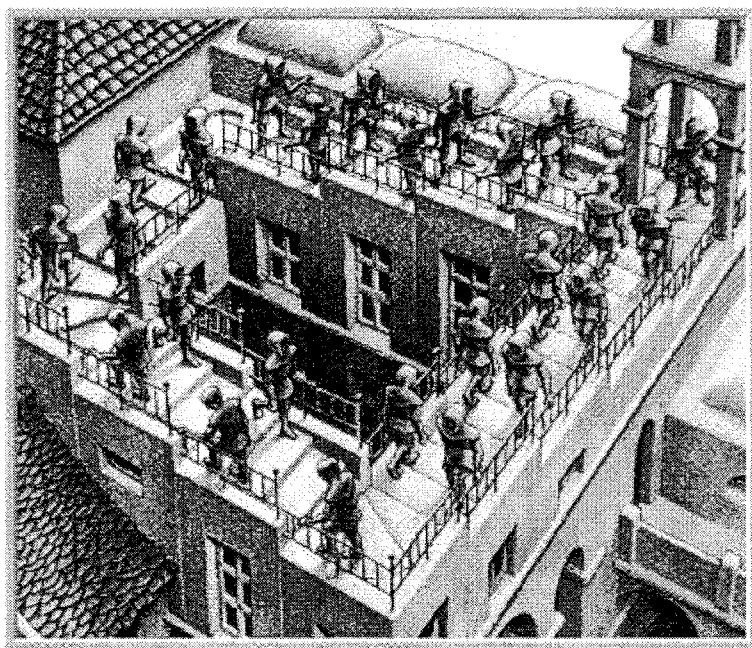
- c. An elementary math story book such as *How Do Octopi Eat Pizza Pie?: Pizza Math* by Neil Kagan
- d. Artwork by M.C. Escher:



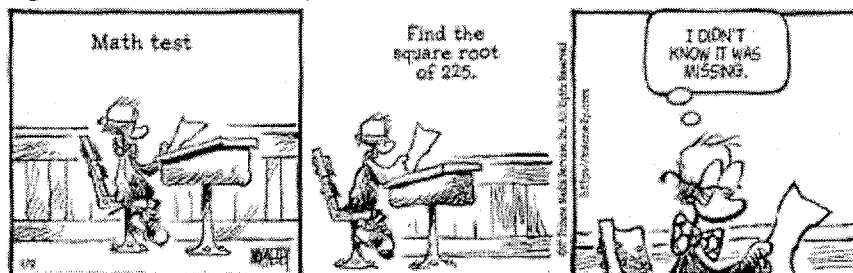
i.



ii.



- iii.
- e. The board games *Ticket to Ride* or *Pandemic* (Two map-based games that are covered in graphs representing routes between cities)



- f. A math cartoon:
- g. Snake Doodling video: <http://www.youtube.com/watch?v=heKK95DAKms>

After examining these mathematical creations, have students discuss what all of these things have in common. Lead students to understand that each are original creative expressions of mathematical ideas (whether or not the creator even knew they were expressing mathematical ideas). After students understand this, tell them that you want them to do the same.

Inform the students that for their final project on graph theory they will be using any medium they want to creatively show their comprehension of something they learned in graph theory. They may use a format similar to those shown above or come up with their own idea.

They will follow the following guidelines in creating their project:

1. The project must have some clear link to something they learned our investigation of graph theory.
2. The project must be both original and creative.
3. The project must show that the student(s) have a firm grasp of the mathematical idea expressed in the project.
4. The project must show that significant amount of effort was exerted in developing the project
  - a. As a rule of thumb, the project should have taken 1.5 times the amount of classtime used in class to work on the project; for example if a full 5-day week of one-hour classes

are given to students to work on this project, then the project should exhibit that at least 7.5 hours(at least 5 hours of class time and at least 2.5 hours outside of class) were used to develop and create the project.

- b. If students wish to work on the projects in collaborative groups, each student should be accountable for the time spent on part of the project (i.e. one student should not do all the work, and each student should do approximately the same amount of work and be able to show what work each individual carried out).
5. By the second day of work on the project, each student or group should submit a proposal to the teacher detailing:
    - a. The mathematical idea being expressed
    - b. The medium or media through which the idea is being expressed
    - c. How time will be used to work on the project
      - i. For groups outline the responsibilities of each group member
    - d. What the class should get out of the presentation
    - e. How the work will be presented to the class
    - f. What the creators expect to get out of the project.

Upon turning in the proposals, the teacher will read through them and then suggest any changes or improvements that need to be made in the projects.

6. Each project should be presented and explained to the class in between 5 and 10 minutes and be prepared to answer any questions the teacher or class may have.
7. During the presentations, each student is to keep a journal of all the presentations and be able to write at least one paragraph about each project presented.
8. After all projects have been presented, the class will engage in a discussion of the overall experience and discuss what has been learned and how this assignment could be improved for the next class that engages in it.

## Biography

Michael Buhler, raised in Idaho Falls, Idaho, graduated from Bonneville High School in 2007. A Presidential Scholar, he entered Utah State University that fall as Mathematics Education Major with Minors in Political Science Teaching and Spanish Teaching. After one year of classes, he took two years off from school to serve a mission to Lima, Peru, where he became fluent in Spanish. During his years Michael worked in various teaching jobs such as recitation leader, tutor, and teaching assistant at Utah State as well as a tutor and substitute teacher at Fast Forward Charter High School. For several years he also worked as a writer for the Utah Statesman and officiated high school football and baseball games.

After graduating in 2013, Michael plans on perusing a teaching career for a few years and then attending graduate school. His goals are to be a college mathematics professor and being an official in the National Football League.

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